

On Estimates in Hardy Spaces for the Stokes Flow in a Half Space

YOSHIKAZU GIGA, SHIN'YA MATSUI†, YASUYUKI SHIMIZU

儀我 美一・松井 伸哉・清水 康之

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

†Faculty of Business Administration and Information Science,
Hokkaido Information University, Nishi-Nopporo, Ebetsu 069, Japan

We consider the Stokes equation

$$(1) \quad \begin{aligned} u_t - \Delta u + \nabla p &= 0, \operatorname{div} u = 0 \text{ in } \Omega \times (0, \infty), \\ u &= u_0 \text{ at } t = 0, \\ u &= 0 \text{ on } \partial\Omega \times (0, \infty) \end{aligned}$$

in a domain Ω in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary. Here $u = (u^1, \dots, u^n)$ is unknown velocity field and p is unknown pressure field. Initial data u_0 is assumed to satisfy a *compatibility condition*: $\operatorname{div} u_0 = 0$ in Ω and the normal component of u_0 equals zero on $\partial\Omega$. This system is a typical parabolic equation and it has several properties resembling to the heat equation.

If $\Omega = \mathbb{R}^n$, u is reduced to a solution of the heat equation with initial data u_0 because there is no boundary condition. For example regularity-decay estimate

$$(2) \quad \|\nabla u(t)\|_p \leq C t^{-1/2} \|u_0\|_p \text{ for } t > 0$$

holds for all $1 \leq p \leq \infty$ with C independent of t and u_0 , where $\|f(t)\|_p := (\int_{\Omega} |f(t, x)|^p dx)^{1/p}$ and ∇ denotes the gradient in space variables. If $p = 2$, the estimate (2) is still valid for any domain. Indeed, since the Stokes operator A is self-adjoint and nonnegative, the operator A generates an analytic semigroup e^{-tA} . This yields

$$\|A^{1/2} e^{-tA} u_0\|_2 \leq C t^{-1/2} \|u_0\|_2.$$

Since $u = e^{-tA} u_0$ and $\|A^{1/2} u\|_2 = \|\nabla u\|_2$, (2) follows for $p = 2$. (See Borchers and Miyakawa [3] for applications.) For $1 < p < \infty$, (2) is valid for bounded domains (Giga [7]) and for a half space (Ukai [13]). The estimate (2) is also valid for exterior domain with $n \geq 3$, with extra restriction $1 < p < n$. (See Borchers and Miyakawa [2], Giga and Sohr [8], Iwashita [10].)

However, there was no result for $p = 1$ or $p = \infty$ where the boundary of Ω is not empty. The main difficulty lies in the fact that the projection associated with the Helmholtz decomposition is not bounded in L^1 type spaces, because it involves the singular integral operator such as Riesz operators. Nevertheless, we prove (2) for $p = 1$ where Ω is a half space $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n); x_n > 0\}$.

Theorem 1. *Let u be the solution of the Stokes equation (1) in $\Omega = \mathbb{R}_+^n$ with initial data $u_0 \in L^1(\mathbb{R}^n)$, which satisfies the compatibility condition. Then there is a constant C independent of u_0 such that*

$$(3) \quad \|\nabla u(t)\|_1 \leq Ct^{-1/2}\|u_0\|_1$$

for all $t > 0$.

This is rather surprising since we do not expect $\|u(t)\|_1 \leq C\|u_0\|_1$ for $\Omega = \mathbb{R}_+^n$. Actually, the estimate (3) follows from a stronger estimate:

Theorem 2. *Under the same hypothesis of the Theorem 1, there is a constant C' independent of u_0 such that*

$$(4) \quad \|\nabla u(t)\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \leq C't^{-1/2}\|u_0\|_1$$

for all $t > 0$.

Here

$$\|f\|_{\mathcal{H}^1(\mathbb{R}_+^n)} = \inf\{\|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)}; \tilde{f} \in \mathcal{H}^1(\mathbb{R}^n), \tilde{f}|_{\mathbb{R}_+^n} \equiv f\},$$

where $\mathcal{H}^1(\mathbb{R}^n)$ is the Hardy space in $\mathbb{R}^n$ with a norm

$$\|f\|_{\mathcal{H}^1} = \|f^*\|_{L^1(\mathbb{R}^n)} = \left\| \sup_{s>0} |f * G_s| \right\|_{L^1(\mathbb{R}^n)}.$$

Here G_s is the Gauss kernel.

To show (4), we recall the solution formula obtained by Ukai [13]. The solution is represented by the Gauss kernel and various Riesz operators. It is known by Carpio [4] that the solution $u = G_t * u_0$ of the heat equation with initial data $u_0 \in L^1(\mathbb{R}^n)$ enjoys

$$(5) \quad \|\nabla u(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_1 t^{-1/2}\|u_0\|_1$$

If the solution of (1) were represented only by G_t and a Riesz operator in $\mathbb{R}^n$, (6) could yield (4) since the Riesz operator is bounded in $\mathcal{H}^1$. Unfortunately, the formula contains the Riesz operator in tangential variables $x' = (x_1, \dots, x_{n-1})$ to $\partial\mathbb{R}_+^n$, it is not clear that such operators are bounded in $\mathcal{H}^1(\mathbb{R}^n)$. To overcome this difficulty, we rewrite Ukai's formula so that ∇u does not have tangential Riesz operators with use of the operator Λ whose symbol equals $|\xi'|$, where $(\xi', \xi_n) = \xi \in \mathbb{R}^n$. Because of this, we need to prove

$$(6) \quad \|\Lambda u(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C_2 t^{-1/2}\|u_0\|_1$$

in addition to (5). Although there are several extra technical difficulty, because of the formula, this is a rough idea for the proof of (4).

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