

A Note on Essential Self-adjointness of Dirac Operator with a Monopole

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Abstract

The purpose of this paper is to analyse the essential self-adjointness of Dirac operator $H = H_0 + V = c\boldsymbol{\alpha} \cdot (-i\nabla + i\mathbf{A}) + \beta m_0 c^2 + V$, where $\mathbf{A}$ is the vector potential induced by a monopole. The potential V is assumed to be spherically symmetric and of the form $V = u(r)I_4 + v(r)\beta + iw(r)\beta(\boldsymbol{\alpha} \cdot \mathbf{e}_r)$. It is shown that H is essentially self-adjoint under some conditions on the behavior of u, v and w in a neighbourhood of the origin.

Key words. Dirac operator, essential self-adjointness, monopole, complex line bundle, section

§1 Introduction

Since 1976 several authors have investigated the Schrödinger operator with a magnetic field induced by a magnetic monopole (simply called a monopole) [7, 8, 15, 17]. It seems worth-while to throw light upon Dirac operator in such a case [15, 17].

Mathematically, a wave function is described as a section of a vector bundle [3] and a vector potential is represented by a connection form of the principal fibre bundle associated with the vector bundle. In this paper we construct the Hilbert space on which Dirac operator H with a monopole operates and study the essential self-adjointness of H . In the sequel, we use the quantity $q := \frac{eg}{c\hbar}$ (e : electric charge, g : magnetic charge) as a monopole parameter on the basis of Dirac's quantization condition $2q$ should be an integer [1].

In §2 we build up a line bundle $D^{(q)}$ over $\mathbb{R}^3 \setminus \{0\}$ and another one $E^{(q)}$ over the sphere S^2 with the same structure group $U(1)$. Then we make the Hilbert space $\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ on which H operates and the corresponding one $\tilde{\Gamma}(S^2, E^{(q)})^4$. Subsequently we define the vector potential $\mathbf{A}$ explicitly. Since we assume that the potential V in H is spherically symmetric, we rewrite the unperturbed part H_0 of H so that it may contain radial terms and a generalized spin-orbit coupling operator K (Eq.(2.11)).

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^{†)} c is the speed of light and $\hbar$ is the Planck constant.

In §3, using Wu-Yang's monopole harmonic sections $Y_{l,m}^q(\theta, \varphi)$ [7] which form an orthonormal basis for $\tilde{\Gamma}(S^2, E^{(q)})$, we decompose $\tilde{\Gamma}(S^2, E^{(q)})^4$ into the direct sum of the simultaneous eigenspaces $\mathcal{R}_{j,m,k}^{(q)}$ of J^2 , J_3 and $K^\dagger$). The restriction of H to the partial wave subspace $L^2((0, \infty), dr) \hat{\otimes} \mathcal{R}_{j,m,k}^{(q)}$, $h_{j,m,k}$, is represented on $L^2((0, \infty), dr)^2$ by radial terms.

In §4 we show under what condition the total Hamiltonian H is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ (As for $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$, see the lower half part of this page.). Arnold-Kalf-Schneider's theorems [16] are useful for the essential self-adjointness of $h_{j,m,k}$. Then we obtain the three main results, Theorems 4.2, 4.3, 4.4 by setting some reasonable assumptions on the behavior of $V = u(r)I_4 + v(r)\beta + iw(r)\beta(\alpha \cdot e_r)$ in a neighbourhood of the origin.

§2 Formulation of Dirac operator with a monopole

We first construct two line bundles. Let $\{W_N, W_S\}$ be an open covering of a base space S^2 as follows:

$$W_N := \left\{ (\theta, \varphi); 0 < \theta < \frac{\pi}{2} + \delta, 0 < \varphi < 2\pi \right\}, \quad \left(0 < \delta < \frac{\pi}{2} \right) \quad (2.1)$$

$$W_S := \left\{ (\theta, \varphi); \frac{\pi}{2} - \delta < \theta < \pi, 0 < \varphi < 2\pi \right\}. \quad (2.2)$$

A transition function τ_{NS} of $W_N \cap W_S$ into the unitary group $U(1)$ is defined by

$$\tau_{NS}(\theta, \varphi) := e^{2iq\varphi}. \quad (2.3)$$

Using these quantities, we build up a complex line bundle $E^{(q)}$. Subsequently, let $D^{(q)}$ be the pull-back of $E^{(q)}$ by the smooth mapping f of $\mathbb{R}^3 \setminus \{0\}$ onto S^2 defined as $f(\mathbf{x}) := \frac{\mathbf{x}}{\|\mathbf{x}\|}$. The open covering $\{\{r; r > 0\} \times W_N, \{r; r > 0\} \times W_S\}$ of $\mathbb{R}^3 \setminus \{0\}$ is chosen and the transition function $t_{NS}(r, \theta, \varphi)$ of $D^{(q)}$ is essentially the same as that of $E^{(q)}$: $t_{NS}(r, \theta, \varphi) = e^{2iq\varphi}$.

Furthermore, let $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ denote the set of all C^∞ -class global sections of $D^{(q)}$ with compact support and $\Gamma^\infty(S^2, E^{(q)})$ the set of all C^∞ -class global sections of $E^{(q)}$. They are complex linear spaces. We equip $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and $\Gamma^\infty(S^2, E^{(q)})$ with an inner product as follows:

$$\langle \eta, \xi \rangle = \int_{\mathbb{R}^3 \setminus \{0\}} \eta(r, \theta, \varphi)^* \xi(r, \theta, \varphi) r^2 \sin \theta dr d\theta d\varphi, \quad (2.4)$$

$$\langle \Xi, \Psi \rangle = \int_{S^2} \Xi(\theta, \varphi)^* \Psi(\theta, \varphi) \sin \theta d\theta d\varphi. \quad (2.5)$$

[†]) J : total angular momentum operator. See (2.11).

Then we obtain the two Hilbert spaces by completing $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and $\Gamma^\infty(S^2, E^{(q)})$. We denote them by $\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and $\tilde{\Gamma}(S^2, E^{(q)})$, respectively.

Obviously we get

$$\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \cong C_0^\infty(0, \infty) \otimes \Gamma^\infty(S^2, E^{(q)}) \quad (2.6)$$

and

$$\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)}) \cong L^2((0, \infty), dr) \hat{\otimes} \tilde{\Gamma}(S^2, E^{(q)}). \quad (2.7)$$

Since any wave function satisfying Dirac equation has 4 components, the next decomposition provides a starting point

$$\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \cong L^2((0, \infty), dr) \hat{\otimes} \tilde{\Gamma}(S^2, E^{(q)})^4. \quad (2.8)$$

We have now reached the stage of construction of the vector potential in a free Hamiltonian H_0 . It must be described with a connection form of the principal fibre bundle associated to $D^{(q)}$. Since the magnetic field induced by a monopole q is a curvature form of the connection form, we choose Wu-Yang's connection form $\mathcal{A}$ [13] and take the vector potential $\mathbf{A}$ to be the dual of $\mathcal{A}$:

$$\begin{cases} \mathbf{A}_N = \frac{iq(1 - \cos \theta)}{r \sin \theta} \mathbf{e}_\varphi^{\S} & \text{on } \{r; r > 0\} \times W_N, \\ \mathbf{A}_S = \frac{-iq(1 + \cos \theta)}{r \sin \theta} \mathbf{e}_\varphi & \text{on } \{r; r > 0\} \times W_S. \end{cases} \quad (2.9)$$

With the help of $\mathbf{A}$ we can define H_0 as

$$H_0 := c\boldsymbol{\alpha} \cdot (-i\nabla + i\mathbf{A}) + \beta m_0 c^2 \mathbb{1}. \quad (2.10)$$

We shall here assume that the perturbed potential V is spherically symmetric and that $V(r)$ is 4×4 Hermitian matrix composing of continuous functions on $(0, \infty)$. The total Hamiltonian $H := H_0 + V$ operates on $\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. We take the domain $\text{Dom}(H)$ to be $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ for the present.

To decompose H into the direct sum of radial terms on the basis of (2.8), we rewrite H_0 by four new operators L, S, J and K .

$$\begin{aligned} L &:= \mathbf{M} - q\mathbf{e}_r, & S &:= \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \\ J &:= L I_4 + S, & K &:= \beta(2\mathbf{S} \cdot \mathbf{M} + I_4), \end{aligned} \quad (2.11)$$

^{\S} $\mathbf{e}_r = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, $\mathbf{e}_\theta = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)$, $\mathbf{e}_\varphi = (-\sin \varphi, \cos \varphi, 0)$.

^{\P} α_j ($j = 1, 2, 3$), $\beta = \alpha_0$ are 4×4 constant Hermitian matrices satisfying the anti-commutation relations $\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4$.

where M is the auxiliary operator in $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ given by

$$M := \mathbf{x} \wedge (-i\nabla + i\mathbf{A}). \quad (2.12)$$

Then L is a symmetric operator defined on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})$ and S, J, K are symmetric operators defined on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. The operators J and K are called the total angular momentum operator and the generalized spin-orbit coupling one, respectively. These operators enable us to deduce

$$H_0 = -ic(\boldsymbol{\alpha} \cdot \mathbf{e}_r) \left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r}\beta K \right) + \beta m_0 c^2. \quad (2.13)$$

§3 Decomposition of Dirac operator

We first decompose $\tilde{\Gamma}(S^2, E^{(q)})^4$ into the direct sum of simultaneous eigenspaces of J^2, J_3 and K . We here put

$$\Xi_q := \left\{ |q| - \frac{1}{2}, |q| + \frac{1}{2}, |q| + \frac{3}{2}, \dots \right\} \quad \left(q = \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \dots \right), \quad (3.1)$$

$$\kappa_j^{(q)} := \sqrt{\left(j + \frac{1}{2}\right)^2 - q^2} \quad (j \in \Xi_q). \quad (3.2)$$

There exists an orthonormal basis

$$\{\Phi_{j,m,k}^\pm \mid j \in \Xi_q, m = -j, -j+1, \dots, j-1, j, k = \pm \kappa_j^{(q)}\} \quad (3.3)$$

of $\tilde{\Gamma}(S^2, E^{(q)})^4$ whose elements satisfy the following simultaneous eigenequations of J^2, J_3 and K , according to Y. Kazama *et al* [8].

$$\begin{cases} J^2 \Phi_{j,m,k}^\pm = j(j+1) \Phi_{j,m,k}^\pm, \\ J_3 \Phi_{j,m,k}^\pm = m \Phi_{j,m,k}^\pm, \\ K \Phi_{j,m,k}^\pm = -k \Phi_{j,m,k}^\pm, \end{cases} \quad \begin{matrix} m = -j, -j+1, \dots, j-1, j, \\ k = -\kappa_j^{(q)}, \kappa_j^{(q)}. \end{matrix} \quad (3.4)$$

All $\Phi_{j,m,k}^\pm$ are constructed with Wu-Yang's monopole harmonic sections $Y_{l,m}^q$ [7].

The above consideration leads us to the following decomposition theorem.

Theorem 3.1. When setting $\mathfrak{R}_{j,m,k}^{(q)} := \text{span}\{\Phi_{j,m,k}^+, \Phi_{j,m,k}^-\}$ we obtain

$$\tilde{\Gamma}(S^2, E^{(q)})^4 \cong \bigoplus_{j \in \Xi_q} \bigoplus_{m=-j}^j \bigoplus_{k=\pm \kappa_j^{(q)}} \mathfrak{R}_{j,m,k}^{(q)} \quad (3.5)$$

owing to [7] and [8].

Combination of Eqs.(2.8) and (3.5) yields the relation

$$\tilde{\Gamma}(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4 \cong \bigoplus_{j \in \Xi_q} \bigoplus_{m=-j}^j \bigoplus_{k=\pm\kappa_j^{(q)}} \left(L^2((0, \infty), dr) \hat{\otimes} \mathfrak{K}_{j,m,k}^{(q)} \right). \quad (3.6)$$

Each subspace $L^2((0, \infty), dr) \hat{\otimes} \mathfrak{K}_{j,m,k}^{(q)}$ is called a partial wave subspace and isomorphic to $L^2((0, \infty), dr)^2$.

Assume that V has the form of

$$V(r) = u(r)I_4 + v(r)\beta + iw(r)\beta(\boldsymbol{\alpha} \cdot \mathbf{e}_r), \quad (3.7)$$

where u, v and w are real-valued C^1 -class functions on $(0, \infty)$. Since $\beta\Phi_{j,m,k}^\pm = \pm\Phi_{j,m,k}^\pm$ and $-i(\boldsymbol{\alpha} \cdot \mathbf{e}_r)\Phi_{j,m,k}^\pm = \pm\Phi_{j,m,k}^\mp$, we obtain the following fundamental theorem.

Theorem 3.2. *Let $h_{j,m,k}$ denote the restriction of the total Hamiltonian H to the partial wave subspace. Then we have*

$$H \cong \bigoplus_{j \in \Xi_q} \bigoplus_{m=-j}^j \bigoplus_{k=\pm\kappa_j^{(q)}} h_{j,m,k} \quad (3.8)$$

and $h_{j,m,k}$ is represented by

$$h_{j,m,k} = \begin{pmatrix} m_0c^2 + u(r) + v(r) & c \left\{ -\frac{d}{dr} + \frac{k}{r} \right\} + w(r) \\ c \left\{ \frac{d}{dr} + \frac{k}{r} \right\} + w(r) & -m_0c^2 + u(r) - v(r) \end{pmatrix} \quad (k = \pm\kappa_j^{(q)}) \quad (3.9)$$

on $C_0^\infty(0, \infty)^2$.

The operator $h_{j,m,k}$ is called a radial Dirac operator.

§4 Essential self-adjointness of Dirac operator

We are now in a position to state a sufficient condition that Dirac operator be essentially self-adjoint. The following theorem serves well for the purpose.

Theorem 4.1. *Let $u, v \in C^1(0, \infty)$ and $f_\pm := u \pm v$. Suppose $\lim_{r \rightarrow 0} r f_\pm(r)$ exist. Put $l_\pm := \frac{1}{c} \lim_{r \rightarrow 0} r f_\pm(r)$. If $l_+ l_- < (\kappa_j^{(q)})^2 - \frac{1}{4}$, then $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0, \infty)^2$ for all $j \in \Xi_q$.*

The proof is easily given owing to V. Arnold, H. Kalf, and A. Schneider [16].

Theorem 4.2. Let $g \in C^1(0, \infty)$. If $\lim_{r \rightarrow 0} g(r)$ exists and $|g(+0)| > \frac{1}{2}$, then the total Hamiltonian $H = H_0 + \frac{cg(r)}{r}\beta$ is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$ for all $|q| \geq \frac{1}{2}$. $\left(u = w = 0, v = \frac{cg(r)}{r}\right)$

Proof. It is sufficient to prove the essential self-adjointness of the radial Dirac operator $h_{j,m,k}$ for each $j \in \Xi_q$. The constants $\pm m_0 c^2$ in the diagonal part of $h_{j,m,k}$ may be omitted in discussion of essential self-adjointness. Then we have

$$-g(+0)^2 < (\kappa_j^{(q)}) - \frac{1}{4}$$

for all $j \in \Xi_q$. Hence $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0, \infty)^2$ for all $j \in \Xi_q$. This implies that H is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. $\square$

Theorem 4.3. Let $|q| \geq \frac{1}{2}$. If the inequalities $\frac{1}{2} < |b| < \sqrt{2|q|+1} - \frac{1}{2}$ hold, then the total Hamiltonian $H = H_0 + i\frac{cb}{r}\beta(\alpha \cdot e_r)$ is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. $\left(u = v = 0, w = \frac{cb}{r}\right)$

Proof. The constants $\pm m_0 c^2$ in the diagonal part may be omitted in argument on the essential self-adjointness of $h_{j,m,k}$.

Case I. $j \geq |q| + \frac{1}{2}$: Assume the inequalities $\frac{1}{2} < b < \sqrt{2|q|+1} - \frac{1}{2}$ hold. In the case of $k = \kappa_j^{(q)}$, we have

$$(b+k)^2 - \frac{1}{4} \geq b^2 - \frac{1}{4} > 0.$$

In the case of $k = -\kappa_j^{(q)}$, we have

$$(b+k)^2 - \frac{1}{4} = \left(b - \kappa_j^{(q)} + \frac{1}{2}\right) \left(b - \kappa_j^{(q)} - \frac{1}{2}\right). \quad (*)$$

Since $\kappa_j^{(q)} \geq \sqrt{2|q|+1}$, we get $b - \kappa_j^{(q)} \leq -\frac{1}{2}$ and the right-hand side of Eq.(*) is non-negative. Hence it follows from Theorem 4.1 that $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0, \infty)^2$. Likewise in the case of $b < 0$, we can obtain the assertion.

Case II. $j = |q| - \frac{1}{2}$: In this case, we have $0 < b^2 - \frac{1}{4}$ ($\kappa_j^{(q)} = 0$), and so $h_{j,m,k}$ is essentially self-adjoint.

As a consequence, $h_{j,m,k}$ is essentially self-adjoint on $C_0^\infty(0, \infty)^2$ for all $j \in \Xi_q$. This means that H is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. $\square$

Theorem 4.4. Let $|q| \geq \frac{1}{2}$. Assume that u is a C^1 -class function on $(0, \infty)$ and $p_0 := \frac{1}{c} \lim_{r \rightarrow 0} ru'(r)$ exists. If the inequalities $\frac{1}{2} < |p_0\lambda| < \sqrt{2|q|+1} - \frac{1}{2}$ hold, then the total Hamiltonian $H = H_0 + u(r)I_4 + i\lambda u'(r)\beta(\alpha \cdot e_r)$ is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. ($v = 0, w = \lambda u'(r)$)^{||}

Proof. The constants $\pm m_0 c^2$ in the diagonal part may be omitted.

Case I. $j \geq |q| + \frac{1}{2}$: Assume the inequalities $\frac{1}{2} < p_0\lambda < \sqrt{2|q|+1} - \frac{1}{2}$ hold. In a similar way to the proof of Theorem 4.3 we get

$$(k + p_0\lambda)^2 - \frac{1}{4} > 0$$

for $k = \pm \kappa_j^{(q)}$. Likewise in the case of $p_0\lambda < 0$ we obtain the above inequality. Hence it follows from Corollary 2 of Theorem 3 of Ref.[16] that $h_{j,m,k}$ is in the limit-point case at the origin. Consequently, $h_{j,m,k}$ is essentially self-adjoint.

Case II. $j = |q| - \frac{1}{2}$: In this case, we have $0 < (p_0\lambda)^2 - \frac{1}{4}$ ($\kappa_j^{(q)} = 0$).

The both cases imply that H is essentially self-adjoint on $\Gamma_0^\infty(\mathbb{R}^3 \setminus \{0\}, D^{(q)})^4$. $\square$

§5 Discussion

In §4 we have proved the essential self-adjointness of H by the limit-point case at the origin of every radial Dirac operator $h_{j,m,k}$ (Theorem 4.1, [16]) and the decomposition theorem (Theorem 3.1 and 3.2). In our case (a monopole exists), it is an interesting fact that although the unperturbed operator

$$h_{j,m,k}^{(0)} = \begin{pmatrix} m_0 c^2 & -c \frac{d}{dr} \\ c \frac{d}{dr} & -m_0 c^2 \end{pmatrix}$$

for $j = |q| - 1/2$ ($\kappa_j^{(q)} = 0$) is not essentially self-adjoint, $h_{j,m,k}$ becomes essentially self-adjoint if H has a special-type potential.

The investigation of the essential self-adjointness the usual n -dimensional Dirac operator was treated by Kalf and Yamada [19]. Under the assumption that m and V are spherically symmetric, they reduced the problem to that of every radial Dirac operator h with $k \in \pm\{\mathbb{N}_0 + (n-1)/2\}$. Their method^{**} is the same as ours. But since $k = \pm\sqrt{(j+1/2)^2 - q^2}$ and $j \in \Xi_q$ in our case, it is more difficult to study the essential self-adjointness of $h_{j,m,k}$.

^{||}) Behncke and Thaller already discussed this case for the usual Dirac operator (No monopole) [10, 14]. cf. Corollaries 2 and 3 of Theorem 3 in [16].

^{**}) Kalf and Yamada's varying mass term $m(r)$ corresponds to $m_0 c^2 + v(r)$ in our case.

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