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Soulé characters in the work of Ihara

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Dedicated to Professor Yasutaka Ihara
on the occasion of his 80th birthday

1 Introduction

My first postdoctoral fellowship was held at the Mathematical Sciences Research Institute, where I worked under the guidance of Yasutaka Ihara during the fall of 1999. Professor Ihara introduced me to his study of the Galois action on the étale fundamental group of the projective line minus three points and a $\mathbb{Z}_p$-Lie algebra constructed out of it. I found, and still find, the structure of this Lie algebra both fascinating and mysterious.

Let $X = \mathbb{P}^1 - \{0, 1, \infty\}$ over $\mathbb{Q}$ and $\bar{X}$ be its base change to $\bar{\mathbb{Q}}$. There is an exact sequence

$$1 \to \pi_1(\bar{X}) \to \pi_1(X) \to G_{\mathbb{Q}} \to 1$$

of étale fundemantal groups, as $\pi_1(\operatorname{Spec}\mathbb{Q}) = G_{\mathbb{Q}}$. This provides us with a homomorphism

$$G_{\mathbb{Q}} \to \operatorname{Out}(\pi_1(\bar{X}, x)),$$

where Out denotes the outer continuous automorphism group. By a theorem of Belyi, this map is injective. Ihara initiated a study of this Galois action in [6].

For a prime p, consider the maximal pro-p quotient $\Pi = \pi_1^{(p)}(\bar{X})$ of $\pi_1(\bar{X})$. Ihara showed that the action of $G_{\mathbb{Q}}$ on Π factors through the Galois group $G_{\mathbb{Q},S}$ of the maximal extension of $\mathbb{Q}$ unramified outside $S = \{p, \infty\}$. In fact, the fixed field of the kernel of

$$\rho\colon G_{\mathbb{Q},S} \to \operatorname{Out}(\Pi)$$

is contained in the maximal pro-p extension M of $\mathbb{Q}(\mu_p)$ unramified outside of S. The fixed field of the kernel of the induced action on the abelianization of Π is $K = \mathbb{Q}(\mu_{p^\infty})$. We set $G = \operatorname{Gal}(M/K)$.

Ihara considers the lower central series of Π defined by $\Pi_1 = \Pi$ and $\Pi_{j+1} = [\Pi, \Pi_j]$ for $j \geq 1$. For $r \geq 1$, the map ρ induces a homomorphism

$$\rho_r\colon G \to \operatorname{Out}(\Pi/\Pi_{r+1}),$$

and we let $F^r(G) = \ker\rho_r$. We set $\operatorname{gr}^r\mathfrak{g} = F^r(G)/F^{r+1}(G)$ and

$$\mathfrak{g} = \bigoplus_{r=1}^{\infty} \operatorname{gr}^r\mathfrak{g}.$$

Ihara's Lie algebra is the graded $\mathbb{Z}_p$-Lie algebra $\mathfrak{g}$ under the commutator induced by the commutator on G, since $[F^r(G), F^s(G)] \subseteq F^{r+s}(G)$ for all $r, s \geq 1$. Each graded piece $\mathrm{gr}^r\mathfrak{g}$ is free of finite rank over $\mathbb{Z}_p$ and via the conjugation action of $G_{\mathbb{Q}}$ on G, is endowed with a $G_{\mathbb{Q}}$-action by the rth power of the p-adic cyclotomic character.

For odd $r \geq 3$, there are elements $\sigma_r \in \mathrm{gr}^r\mathfrak{g}$, which are not canonical but are in some sense dual to classes in $H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r))$ of Kummer cocycles $\kappa_r \colon G_{\mathbb{Q},S} \to \mathbb{Z}_p(r)$ attached to p-adic limits of cyclotomic units known as Soulé characters. Specifically, for any positive integer n and odd positive integer r, we set

$$a_{n,r} = \prod_{\substack{i=1 \\ p \nmid i}}^{p^n-1} (1 - \zeta_{p^n}^i)^{i^{r-1}},$$

and $\kappa_r(\sigma) \in \mathbb{Z}_p^\times$ is defined as the unique element such that

$$\frac{\sigma(a_{n,r}^{1/p^n})}{a_{n,r}^{1/p^n}} = \zeta_{p^n}^{\kappa_r(\sigma)}$$

for all n. For odd primes p, Soulé showed that this κ_r generates $H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r)) \cong \mathbb{Q}_p$. The element $\sigma_r \in \mathrm{gr}^r\mathfrak{g}$ is the image of a choice of $\tilde{\sigma}_r \in F^rG$ such that $\kappa_r(\tilde{\sigma}_r)$ generates $\kappa_r(F^rG)$. Ihara showed in [7] that the latter image is nontrivial.

Deligne made a conjecture equivalent to the statement that $\mathfrak{g} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is freely generated as a $\mathbb{Q}_p$-Lie algebra by the elements σ_r, and Ihara [7] formulated the conjecture as it is stated here. This conjecture, known as the Deligne-Ihara conjecture, is in fact now a theorem: the generation was directly proven in a paper of Hain and Matsumoto [5], and the freeness follows from the work of Brown [3].

Ihara also studied a depth filtration on $\mathfrak{g}$ and, with Takao, observed a relationship between the second depth-graded pieces and cusp forms for GL_2 over $\mathbb{Q}$. The higher depth-graded quotients relate to the structure of certain spaces of automorphic forms for GL_d over $\mathbb{Q}$, as evidenced in work of Goncharov. Additional relationships are seen in work of Brown and of Hain and Matsumoto. It is clear that much fascinating and important mathematics remains to be discovered in these directions, though to say more would take us too far astray.

Ihara [8] asked the finer and more arithmetically interesting question of whether $\mathfrak{g}$ is itself generated by the σ_r, suggesting that the answer is false for irregular primes p. Specifically, Ihara conjectured that a particular relation would exist in $\mathrm{gr}^{12}\mathfrak{g}$ for $p = 691$, and this was verified through work of McCallum and myself [10] and subsequent work [14]. This relation has to do with a mod 691 congruence between the discriminant cusp form and the weight 12 Eisenstein series that exists by the irregularity of 691.

For regular primes p, I showed shortly after my stay at MSRI that $\mathfrak{g}$ is free on the σ_r if p is regular [13], supposing the now-proven Deligne-Ihara conjecture. The crucial point is that p is regular if and only if the maximal pro-p quotient of $G_{\mathbb{Q},S}$ is free pro-p. For regular p, the map $\rho \colon G \to \mathrm{Out}(\Pi)$ is similarly now known to be injective, but whether this is the case for p irregular is a very interesting open question of Anderson and Ihara [1].

In learning about Ihara's amazing work, I first sought to gain an understanding one of the most fundamental theorems underlying it, found in the work of Soulé [12]. It states, in particular, that the κ_r generate $H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r))$ for odd $r \geq 1$.

Theorem 1 (Soulé). *Let p be an odd prime number and r a positive integer. Then*

$$H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r)) \cong \begin{cases} \mathbb{Q}_p & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even.} \end{cases} \tag{1}$$

The cruxes of the matter are Chern class maps of Soulé [11] that give isomorphisms

$$K_{2r-1}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \xrightarrow{\sim} H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r))$$

and a regulator computation of Borel [2] that shows that each $K_{2r-1}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ with r odd is a one-dimensional $\mathbb{Q}$-vector space.

Soulé also showed that $H^2(G_{\mathbb{Q},S}, \mathbb{Q}_p(r))$ vanishes. This is equivalent to showing that the group $H^2(G_{\mathbb{Q},S}, \mathbb{Z}_p(r))$ is finite. For even r, Mazur and Wiles [9] proved that the exact order of the latter group is highest power of p dividing the numerator of the rth Bernoulli number over r. That $H^2(G_{\mathbb{Q},S}, \mathbb{Z}_p(r))$ vanishes for odd r is equivalent to Vandiver's conjecture that p does not divide the the class number of $\mathbb{Q}(\mu_p)^+$.

I hoped to find an elementary proof of Soulé's result. However, this result says that the zeros of certain Kubota-Leopoldt p-adic L-functions cannot be negative even integers. The relevant nonzero p-adic L-value at $1-r$ corresponding to an odd $r \geq 3$ is $(1-p^{r-1})B_r/r$. It is not even known in general that the zeros cannot be negative integers. In the case that p is regular, including $p = 2$, the p-adic L-functions have no zeros, so any obstruction is lifted. Thus, we can prove Soulé's theorem from relatively basic principles. That is the goal of this write-up, and what follows is an edited combination of some notes I wrote during and refined after my stay at MSRI that accomplish this goal.

I thank Yasutaka Ihara for his guidance as I conducted this work and for introducing me to his amazing results. I am deeply grateful to him for the tremendous support he gave me as I began my career.

2 The result

We aim to prove the following theorem for a regular prime p. Most of the subtlety in proving it lies in the case $p = 2$. We prove it for all p at the same time, noting where the arguments simplify for odd p. Even for $p = 2$, the result is certainly not new: it is, for instance, easily subsumed by the work in [15].

Let $G_{\mathbb{Q},S}$ denote the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside $S = \{p, \infty\}$.

Theorem 2. *Let p be a regular prime number and r a nonzero integer. Then*

$$H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r)) \cong \begin{cases} \mathbb{Q}_p & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases} \tag{2}$$

and $H^2(G_{\mathbb{Q},S}, \mathbb{Q}_p(r)) = 0$.

Let n be a positive integer. If $p = 2$, we suppose that $n \geq 3$. Set $F = \mathbb{Q}(\mu_{p^n})$, and let F^+ denote its maximal totally real subfield. Let $G_{F,S}$ denote the Galois group of the maximal extension of F unramified outside of the unique prime $1 - \zeta_{p^n}$ over p.

Set $N = \mathrm{Gal}(F/\mathbb{Q})$. Let $\sigma \in N$ denote the image of complex conjugation. Write $N = \Delta \oplus \Gamma$ as follows. If p is odd, then Γ is the cyclic Sylow p-subgroup of order p^{n-1}, and Δ is the cyclic subgroup of order $p-1$. If $p = 2$, then Γ is the cyclic group of order 2^{n-2} generated by an element τ such that $\tau(\zeta_{2^n}) = \zeta_{2^n}^{-3}$, and $\Delta = \langle \sigma \rangle$ has order 2.

Let $U_S = \mathbb{Z}[\frac{1}{p}, \mu_{p^n}]^\times$ denote the group of p-units in F, and set $U = U_S/U_S^{p^n}$. As p is regular, all abelian unramified outside p extensions of F of exponent dividing p^n are generated by the p^nth roots of p-units in F. Kummer theory then provides a canonical isomorphism

$$U \cong H^1(G_{F,S}, \mu_{p^n})$$

taking an element to the class of its Kummer cocycle. Moreover, the group of cyclotomic p-units that is generated as an N-module by $\lambda_n = 1 - \zeta_{p^n}$ has prime-to-p index in the group U_S of all p-units, in that this index is exactly the class number of F^+. In other words, $H^1(G_{F,S}, \mu_{p^n})$ is the cyclic $(\mathbb{Z}/p^n\mathbb{Z})[N]$-module generated by the Kummer class of λ_n, and this is the entirety of our use of the regularity assumption on p.

For an N-module A, let A^+ denote the maximal submodule fixed under the complex conjugation σ. If $p = 2$, this coincides with the invariant group A^Δ.

Proposition 3. *There is an exact sequence of N-modules*

$$0 \to \mu_{p^n} \to U \to (\mathbb{Z}/p^n\mathbb{Z})[N]^+ \to 0$$

that is canonically split if p is odd.

Proof. Note that $\sigma(\lambda_n)/\lambda_n = -\zeta_{p^n}$. Hence the submodule $(\sigma - 1)U$ of U is isomorphic to μ_{p^n}. The quotient $U/(\sigma-1)U$ is necessarily isomorphic to a quotient A of $\mathbb{Z}/p^n\mathbb{Z}[N]^+$, and we remark that

$$\log_{p^n} |\mathbb{Z}/p^n\mathbb{Z}[\Gamma]| = d^+,$$

where $d^+ = [F^+ : \mathbb{Q}]$. On the other hand, Dirichlet's Unit Theorem says that $\log_{p^n} |U| = d^+ + 1$. Hence $A \cong \mathbb{Z}/p^n\mathbb{Z}[N]^+$. Finally, if p is odd, then $(\sigma - 1)U = \mu_{p^n}$ is canonically a direct summand of U via the projection map given by $\frac{\sigma-1}{2}$. □

Proposition 4. *Let r be an integer. If p is odd, then we have*

$$H^1(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(r))^N \cong \begin{cases} \mathbb{Z}/p^n\mathbb{Z} & \text{if } r \text{ is odd} \\ (\mathbb{Z}/p^n\mathbb{Z}(r))^N & \text{if } r \text{ is even.} \end{cases}$$

and if $p = 2$, we have

$$H^1(G_{F,S}, \mathbb{Z}/2^n\mathbb{Z}(r))^N \cong \begin{cases} \mathbb{Z}/2^{n-1}\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } r \text{ is odd} \\ (\mathbb{Z}/2^n\mathbb{Z}(r))^N \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } r \text{ is even.} \end{cases}$$

Proof. Note that $H^1(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(r))^N = U(r-1)^N$. We have

$$(\mathbb{Z}/p^n\mathbb{Z}(r))^N \cong \begin{cases} (\mathbb{Z}/p^n\mathbb{Z}(r))^\Gamma & \text{if } r \text{ is even} \\ \mu_p(\mathbb{Q}) & \text{if } r \text{ is odd,} \end{cases} \tag{3}$$

and

$$(\mathbb{Z}/p^n\mathbb{Z}[N]^+(r-1))^N \cong \begin{cases} \mu_p(\mathbb{Q}) & \text{if } r \text{ is even} \\ \mathbb{Z}/p^n\mathbb{Z} & \text{if } r \text{ is odd.} \end{cases} \tag{4}$$

For odd p, the fact that the exact sequence in Proposition 3 is a direct sum then yields the result.

Suppose that $p = 2$, and consider the exact sequence

$$0 \to (\mathbb{Z}/2^n\mathbb{Z}(r))^N \to U(r-1)^N \xrightarrow{j} (\mathbb{Z}/2^n\mathbb{Z}[\Gamma](r-1))^N \xrightarrow{d} H^1(N, \mathbb{Z}/2^n\mathbb{Z}(r)),$$

which we have from Proposition 3. We claim that j is either surjective or has cokernel of order 2, which is obvious from (4) if r is even. Proposition 3 tells us that

$$U \cong \mathbb{Z}/2^n\mathbb{Z}[N]/((\sigma-1)(\tau+3)).$$

Recalling that $\mathbb{Z}/2^n\mathbb{Z}(1)$ sits inside U as $(\sigma-1)U$, we view $\mathbb{Z}/2^n\mathbb{Z}[\Gamma](r-1)$ as a Γ-submodule (but not an N-submodule) of $U(r-1)$ via this isomorphism.

Under the above identification, $x \in (\mathbb{Z}/2^n\mathbb{Z}[\Gamma](r-1))^N$ implies $(\sigma+1)x \in U(r-1)^N$ and hence

$$j((\sigma+1)x) = 2x.$$

If $x \in (\mathbb{Z}/2^n\mathbb{Z}[\Gamma](r-1))^N$ then $dx(\tau) = 0$ by definition, and $dx(\sigma) = ((-1)^{r-1}\sigma - 1)x$ inside $U(r-1)$.

If r is odd, then we must consider $x = N_r$, where

$$N_r = \sum_{i=0}^{2^{n-2}-1} (-3)^{i(r-1)}\tau^i,$$

and we see that

$$dN_r(\sigma) = (\sigma-1)N_r = \sum_{i=0}^{2^{n-2}-1} (-3)^{ir} = -2^{n-2}(\sigma-1)$$

considered as an element of $U(r-1)$, or $dN_r(\sigma) = -2^{n-2}$ considered as an element of $\mathbb{Z}/2^n\mathbb{Z}(r-1)$. Furthermore, we must view the cochains in the image of d modulo coboundaries. For $a \in \mathbb{Z}/2^n\mathbb{Z}(r)$, we have $\tau(a) = a$ if and only if $a \equiv 0 \bmod 2^{n-2}$. In this case, we have

$$\sigma(a) - a = -2a \equiv 0 \bmod 2^{n-1}.$$

Hence we see that when r is odd, the image of d has order 2, and we therefore conclude the same about the cokernel of j.

If r is even, then we must consider $x = 2^{n-1}N_r$, and it is easy enough to see that $dx(\sigma) = 0$, so the cokernel of j is trivial.

Let J denote the image of j. To finish the proof of the proposition, it remains to show that the sequence

$$0 \to (\mathbb{Z}/2^n\mathbb{Z}(r))^N \to U(r-1)^N \to J \to 0$$

splits. To see this, we lift any element x of J to an element $x \in \mathbb{Z}/2^n\mathbb{Z}[\Gamma](r-1)^\Gamma \subset U(r-1)^\Gamma$, and then $a + x \in U(r-1)^N$ for some $a \in (\mathbb{Z}/2^n\mathbb{Z}(r))^\Gamma$. Noting equation (3), this immediately yields the splitting when r is even. When r is odd, we must have $a \equiv 0 \bmod 2^{n-2}$ in order that a be fixed under Γ, in which case $2^{n-1}(a+x) = 0$ for $n \geq 3$. Hence, we have the splitting. □

We now prove our main result.

Proof of Theorem 2. We have the following sequence of low degree terms in a Hochschild-Serre spectral sequence

$$0 \to H^1(N, \mathbb{Z}/p^n\mathbb{Z}(r)) \to H^1(G_{\mathbb{Q},S}, \mathbb{Z}/p^n\mathbb{Z}(r)) \to H^1(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(r))^N \to H^2(N, \mathbb{Z}/p^n\mathbb{Z}(r)).$$

The orders of the first and last of these groups are bounded with respect to n. If p is odd, this follows as they are (cyclic) Tate cohomology groups with the same order as $\hat{H}^0(N, \mathbb{Z}/p^n\mathbb{Z}(r))$. In general, this follows by use of the spectral sequence

$$H^s(\Delta, H^t(\Gamma, \mathbb{Z}/p^n\mathbb{Z}(r))) \Rightarrow H^{s+t}(N, \mathbb{Z}/p^n\mathbb{Z}(r)).$$

The orders of the groups $H^i(N, \mathbb{Z}/p^n\mathbb{Z}(r))$ are bounded by the product of the orders of a finite number of terms in this sequence. All of these terms are cyclic of bounded order.

Let $h_i(n) = \log_p |H^i(G_{\mathbb{Q},S}, \mathbb{Z}/p^n\mathbb{Z}(r))|$, and let

$$H^i = H^i(G_{\mathbb{Q},S}, \mathbb{Q}_p/\mathbb{Z}_p(r))$$

for $0 \leq i \leq 2$. Proposition 4 tells us that as n varies, $H^1(G_{\mathbb{Q},S}, \mathbb{Z}/p^n\mathbb{Z}(r))$ is the direct sum of a cyclic group of increasingly large order with a group of bounded order when r is odd and is a group of bounded order when r is even. From this, we have immediately that

$$\lim_{n\to\infty} \frac{h_1(n)}{n} = \begin{cases} 1 & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

We also remark that $H^1(G_{\mathbb{Q},S}, \mathbb{Z}/p^n\mathbb{Z}(r))$ surjects onto the p^n-torsion in H^1, and the kernel of this surjection is isomorphic to the finite cyclic group H^0 for sufficiently large n since r is nonzero. This follows from the exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z}(r)^N \to H^0 \xrightarrow{p^n} H^0 \to H^1(G_{\mathbb{Q},S}, \mathbb{Z}/p^n\mathbb{Z}(r)) \to H^1 \xrightarrow{p^n} H^1,$$

since $H^0 = \mathbb{Z}/p^n\mathbb{Z}(r)^N$ for sufficiently large n. Hence the divisible part of H^1 is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ if r is odd and is trivial if r is nonzero even. But the $\mathbb{Z}_p$-corank of the divisible part of H^1 is exactly the dimension of $H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r))$ as a $\mathbb{Q}_p$-vector space, and therefore $H^1(G_{\mathbb{Q},S}, \mathbb{Q}_p(r))$ is exactly as stated in the theorem.

Now consider the partial Euler-Poincaré characteristic

$$\chi(n) = h_0(n) - h_1(n) + h_2(n).$$

By Poitou-Tate duality, we have

$$\chi(n) = \log_p(|\mathbb{Z}/p^n\mathbb{Z}(r)|^{-1}|\mathbb{Z}/p^n\mathbb{Z}(r)^+|) = \begin{cases} \delta - n & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even,} \end{cases}$$

where $\delta = 0$ if p is odd and $\delta = 1$ if $p = 2$. Now let

$$a = \lim_{n\to\infty} \frac{\chi(n)}{n} = \begin{cases} -1 & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

As

$$\lim_{n\to\infty}\frac{h_0(n)}{n}=0 \quad \text{and} \quad \lim_{n\to\infty}\frac{h_1(n)}{n}=-a,$$

we see that

$$\lim_{n\to\infty}\frac{h_2(n)}{n}=0.$$

Since $H^2(G_{\mathbb{Q},S},\mathbb{Z}/p^n\mathbb{Z}(r))$ surjects onto the p^n-torsion of H^2, we conclude that the divisible part of H^2 is zero. Hence $H^2(G_{\mathbb{Q},S},\mathbb{Q}_2(r))=0$. □

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