

SOP₂ AND ANTICHAINS

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ABSTRACT. In this paper, we pay our attention into 2-strong order property, namely SOP₂, and its relatives. We first observe strongly indiscernible trees and its modeling property, then find the equivalent conditions for each SOP₂ and weak-TP₁. As a main theorem, we prove that the definitions of SOP₂ and TP₁ can be more generalized, by using the notion of strong similarity and antichains.

1. INTRODUCTION

A formula $\phi(x, y)$ has 2-strong order property(SOP₂) if there is a tree of parameters $(a_\eta : \eta \in {}^{\omega>}2)$ such that the types in which parameters consists of a path in tree is consistent, but any two formulas having parameters of incomparable pairs are contradictory.

SOP₂ has similar definition of the tree property of the first kind(TP₁), except TP₁ uses parameters indexed on ${}^{\omega>\omega}$. It is easily proved that SOP₂ is equivalent to the tree property of the first kind(TP₁), but as it is still too strong to deal with, there have been many attempts to weakening the conditions.

In [3], suggesting the notions k -TP₁ and weak k -TP₁, Kim and Kim proved that k -TP₁ is equivalent to TP₁ using the tree indiscernibility called 1-fully-tree-indiscernible. The equivalence of TP₁ and weak k -TP₁ was later proved in [1] by using strongly indiscernible trees. We aim to generalize the notions of tree properties by introducing $\bar{\nu}^{str}$ -SOP₂(Definition 3.6), and then prove that SOP₂ is equivalent to $\bar{\nu}^{str}$ -SOP₂ for any antichain tuple $\bar{\nu}$.

2. PRELIMINARIES ON STRONGLY INDISCRIBIBLE TREES AND SOP₂

Consider a tree ${}^{\lambda>}\kappa$ of height λ which has κ many branches. Each element in tree can be considered as a string. We denote $\langle \rangle$ as an empty string, 0^α as a string of α many zeros, and α as a string $\langle \alpha \rangle$ of length one.

Definition 2.1. Let $\eta, \nu, \xi \in {}^{\lambda>}\kappa$.

- (1) (Ordering) $\eta \triangleleft \nu$ if $\nu \upharpoonright \alpha = \eta$ for some ordinal $\alpha \in \text{dom}(\nu)$.
- (2) (Meet) $\xi = \eta \wedge \nu$ if ξ is the meet of η and ν , i.e., $\xi = \eta \upharpoonright \beta$, when $\beta = \bigcup \{ \alpha \leq \text{dom}(\eta) \cap \text{dom}(\nu) \mid \eta \upharpoonright \alpha = \nu \upharpoonright \alpha \}$. For $\bar{\eta} \in {}^{\lambda>}\kappa$, $\bar{\nu}$ is the meet closure of $\bar{\eta}$ if $\bar{\nu} = \{ \eta_1 \wedge \eta_2 \mid \eta_1, \eta_2 \in \bar{\eta} \}$
- (3) (Incomparability) $\eta \perp \nu$ if they are $\trianglelefteq$ -incomparable, i.e., $\neg(\eta \trianglelefteq \nu)$ and $\neg(\nu \trianglelefteq \eta)$.
- (4) (Lexicographic order) $\eta <_{lex} \nu$ if
 - (a) $\eta \triangleleft \nu$, or
 - (b) $\eta \perp \nu$ and for ordinal $\alpha = \text{dom}(\eta \wedge \nu)$, $\eta(\alpha) < \nu(\alpha)$

Definition 2.2. A *strong language* L_0 is defined by the collection $\{ \triangleleft, \wedge, <_{lex} \}$

We may view the tree ${}^{\lambda>}\kappa$ as an L_0 -structure.

The author was supported by Samsung Science Technology Foundation under Project Number SSTF-BA1301-03, and has been supported by an NRF of Korea grant 2018R1D1A1A02085584.

Fix a complete first order theory T (with language L). Let $\mathfrak{C} \models T$ be a monster model. From now on, we will work in this $\mathfrak{C}$. Note that we distinguish an index structure from $\mathfrak{C}$. We visit [4] to introduce generalized indiscernibility and modeling property.

Definition 2.3. Let L_0 -structure ${}^{\lambda>\kappa}$ be an index structure. For a tree $(b_\eta | \eta \in {}^{\lambda>\kappa})$ in $\mathfrak{C}$, we say it is *strongly indiscernible* if for any finite tuple $\bar{\eta}$ and $\bar{\nu}$ in ${}^{\lambda>\kappa}$,

$$\text{qftp}_{L_0}(\bar{\eta}) = \text{qftp}_{L_0}(\bar{\nu}) \Rightarrow (b_\eta)_{\eta \in \bar{\eta}} \equiv (b_\nu)_{\nu \in \bar{\nu}}.$$

We say $\bar{\eta}$ and $\bar{\nu}$ are strongly similar, $\bar{\eta} \sim_{str} \bar{\nu}$, to denote $\text{qftp}_{L_0}(\bar{\eta}) = \text{qftp}_{L_0}(\bar{\nu})$.

Definition 2.4. Let $\mathcal{I}$ be an index structure. A set $B = \{b_\eta | \eta \in \mathcal{I}\}$ is *based on* a set $A = \{a_\nu | \nu \in \mathcal{I}\}$ if for all $\varphi(x_{i_1}, \dots, x_{i_n})$ in L and for all $\eta_1, \dots, \eta_n \in \mathcal{I}$, there exists some $\nu_1, \dots, \nu_n \in \mathcal{I}$ such that

- (a) $\nu_1 \dots \nu_n \equiv_{\mathcal{I}}^{qf} \eta_1 \dots \eta_n$, and
- (b) $b_{\eta_1} \dots b_{\eta_n} \equiv_{\varphi} a_{\nu_1} \dots a_{\nu_n}$

In particular, when $\mathcal{I}$ is L_0 -structure $<^{\lambda}\kappa$, we say B is *strongly based on* A whenever B is based on A .

Definition 2.5. For an index structure $\mathcal{I}$, we say $\mathcal{I}$ -indexed indiscernibles have the *modeling property* if given any $A = \{a_\nu | \nu \in \mathcal{I}\}$, there is and $\mathcal{I}$ -indexed indiscernible $B = \{b_\eta | \eta \in \mathcal{I}\}$ such that B is based on A .

Fact 2.6. [5] Let $<^\omega\omega$ be the universe of the index structure. The strong indiscernibles have the modeling property.

Note that we cannot use the above fact when the index structure is a binary tree.

Now we introduce SOP_2 and its relatives.

Definition 2.7. [2, 4, 5] Fix $k \geq 2$.

- (1) $\varphi(x; y)$ has SOP_2 if there is a $(a_\eta | \eta \in {}^{\omega>2})$ such that
 - (a) For all $\eta \in {}^{\omega>2}$, $\{\varphi(x; a_{\eta \restriction \alpha}) | \alpha < \omega\}$ is consistent,
 - (b) For all $\xi, \nu \in {}^{\omega>2}$, if $\xi \perp \nu$, then $\{\varphi(x; a_\xi), \varphi(x; a_\nu)\}$ is inconsistent.
- (2) $\varphi(x; y)$ has the tree property of the first kind (TP_1) if there is $(a_\eta | \eta \in {}^{\omega>\omega})$ such that
 - (a) For all $\eta \in {}^{\omega\omega}$, $\{\varphi(x; a_{\eta \restriction \alpha}) | \alpha < \omega\}$ is consistent,
 - (b) For all $\eta \perp \nu \in {}^{\omega>\omega}$, $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.
- (3) $\varphi(x; y)$ has weak k - TP_1 if there is $(a_\eta | \eta \in {}^{\omega>\omega})$ such that
 - (a) For all $\eta \in {}^{\omega\omega}$, $\{\varphi(x; a_{\eta \restriction \alpha}) | \alpha < \omega\}$ is consistent,
 - (b) For any $\eta, \eta_0, \dots, \eta_{k-1} \in {}^{\omega>\omega}$ and $i_0 < \dots < i_{k-1} < \omega$, if $\eta \restriction \langle i_l \rangle \trianglelefteq \eta_i$ for each $l < k$, then $\{\varphi(x; a_{\eta_i}) | i < k\}$ is inconsistent.
- (4) We say T has SOP_2 (resp. TP_1) if there is a formula having SOP_2 (resp. TP_1). If not, we say T is NSOP_2 (resp. NTP_1). We say T has weak- TP_1 if there is a formula having k - TP_1 for some k . If not, we say T is weak- NTP_1 .

In [1], we see that all the notions in Definition 2.7 for theories are equivalent.

By modeling property, any formula having TP_1 or weak k - TP_1 , we have a strongly indiscernible tree $(a_\eta)_{\eta \in <^{\omega>2}}$ that has the same conditions. On the other hand, we cannot use the modeling property on SOP_2 , though there is some trick to obtain strongly indiscernible binary tree. See [1, lemma 4.3 (3)] for more details.

3. MAIN RESULT

We first modify [2, lemma 2.20] to an argument about ω -branched trees.

Lemma 3.1. *Suppose κ is a regular cardinal and we color ${}^{\kappa}>\omega$ by $\theta < \kappa$ colors. Let c be the given coloring.*

(1) *There is ν^* in ${}^{\kappa}>\omega$ and $j < \theta$ such that for any $\nu \supseteq \nu^*$ we can find $\rho \supseteq \nu$ the color of which is j .*

(2) *There is an embedding $h : {}^{\omega}>\omega \rightarrow {}^{\kappa}>\omega$ such that*

- $h(\eta) \frown \langle i \rangle \sqsubseteq h(\eta \frown \langle i \rangle)$ for each $i < \omega$
- $\text{Ran}(h)$ is monochromatic.

Proof. (1) Suppose not. Then for $i < \theta$, we inductively choose $\eta_i \in {}^{\kappa}>\omega$ such that

- if $i < j$, then $\eta_i \sqsubseteq \eta_j$, and
- for $\rho \in {}^{\kappa}>\omega$, if $\eta_{i+1} \sqsubseteq \rho$, then $c(\rho) \neq i$.

Let $\nu = \bigcup_{i < \theta} \eta_i$. Since $\theta < \text{cf}(\kappa)$, $\nu \in {}^{\kappa}>\omega$. But this contradicts that $c(\nu)$ has no color.

(2) Use (1). □

Before the propositions, we give a useful notation.

For each $\eta \in {}^{\omega_1}>2$, $m < \omega$, $\alpha \leq \omega_1$, we say $K_{\eta, m, \alpha}$ to denote the set $\{\eta \frown \nu \frown 0^\beta : \nu \in {}^m 2, \beta < \alpha\}$, and O_η to denote the set $\{\eta \frown 0^\beta : \beta < \omega_1\}$.

Proposition 3.2. *The following are equivalent.*

(1) *T is NSOP₂.*

(2) *For all $\phi(x, y)$ and strongly indiscernible tree $(a_\eta : \eta \in {}^{\omega_1}>2)$, if $\{\phi(x, a_\nu) : \nu \in O_\eta\}$ is consistent, then for each $\eta \in {}^{\omega_1}>2$ and $m < \omega$, $\{\phi(x, a_\nu) : \nu \in K_{\eta, m, \omega_1}\}$ is consistent.*

Proof. (Sketch)

(1 $\Leftarrow$ 2) Suppose ϕ has SOP₂. By [1, lemma 4.3 (3)], we may assume there is a strongly indiscernible tree $(a_\eta : \eta \in {}^{\omega}>2)$ witnessing SOP₂. Use compactness to obtain a tree where (2) does not hold.

(1 $\Rightarrow$ 2) Suppose not. Fix ϕ and $(a_\eta : \eta \in {}^{\omega_1}>2)$. We inductively choose a finite subset $w_\eta \subseteq {}^{\omega_1}>2$ and $\nu_\eta \in {}^{\omega_1}>2$ for each $\eta \in {}^{\omega_1}>2$ so that the following conditions holds after the construction;

- (a) for each $i = 0, 1$, the union of $\{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta \frown \alpha} : \alpha \leq \text{len}(\eta)\}\}$ and $\{\phi(x, a_\nu) : \nu \in O_{\nu_\eta}\}$ is consistent,
- (b) the union of $\{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta \frown \alpha} : \alpha \leq \text{len}(\eta)\}\}$ and $\{\phi(x, a_\nu) : \nu \in w_{\eta \frown 0} \cup w_{\eta \frown 1}\}$ is inconsistent.

Set $w_\emptyset = \emptyset$ and $\nu_\emptyset = \langle \rangle$. At limit case, $w_\eta = \emptyset$ and $\nu_\eta = \bigcup_{\xi \leq \eta} \nu_\xi$.

Assume $w_{\eta \frown \alpha}$, $\nu_{\eta \frown \alpha}$ is chosen for all $\alpha \leq \text{len}(\eta)$. Let $p_\eta = \{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta \frown \alpha} : \alpha \leq \text{len}(\eta)\}\}$. We take the least $m_\eta < \omega$ where $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta, m_\eta, \omega_1}\}$ is inconsistent.

By minimality of m_η and strong indiscernibility, $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta \frown i, m_\eta - 1, \omega_1}\}$ is consistent for $i = 0, 1$.

By compactness and strong indiscernibility, we have $l_\eta < \omega$ such that $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta, m_\eta, l_\eta}\}$ is inconsistent.

Take $w_{\eta \frown i} = K_{\nu_\eta \frown i, m_\eta - 1, l_\eta}$ and $\nu_{\eta \frown i} = \nu_\eta \frown i \frown 0^{m_\eta - 1} \frown 0^{l_\eta + 1}$ for $i = 0, 1$.

Having done the construction, we choose a finite subset $q_\eta \subseteq p_\eta$ for each η such that $q_\eta \cup \{\phi(x, a_\nu) : \nu \in w_{\eta \frown 0} \cup w_{\eta \frown 1}\}$ is inconsistent.

Let $\tau_\eta = \{a_\nu : \phi(x, a_\nu) \in q_\eta\}$. We may assume τ_η is a finite collection of $K_{\nu, m, \alpha}$ s. Considering τ_η as a tuple, the number of $\sim_{str}$ -equivalent classes in $\{\bar{\tau}_\eta : \eta \in {}^{\kappa}>2\}$ and $\{\bar{w}_{\eta \frown i} : \eta \in {}^{\kappa}>2, i = 0, 1\}$ are both countable.

By [2, lemma 2.20], we have an embedding $h : {}^{\omega}>2 \rightarrow {}^{\kappa}>2$ whose range is monochromatic.

Define a formula $\psi(x, y)$ and a tree $(b_\eta : \eta \in {}^{\omega}>2)$ such that $\psi(x, b_\eta) = \bigwedge q_{h(\langle \rangle)} \wedge \bigwedge \{\phi(x, a_\nu) : \nu \in w_{h(0 \frown \eta)}\}$. Then $\psi(x, y)$ and $(b_\eta : \eta \in {}^{\omega}>2)$ witness SOP₂. □

We analogously give another proposition about weak-TP₁.

For each $\eta \in {}^{\omega_1}>\omega$, $k \leq \omega$, $m < \omega$, $\alpha \leq \omega_1$, we say $K_{\eta,k,m,\alpha}$ to denote the set $\{\eta \smallfrown \nu \smallfrown 0^\beta : \nu \in {}^m k, \beta < \alpha\}$, and O_η to denote the set $\{\eta \smallfrown 0^\beta : \beta < \omega_1\}$.

Proposition 3.3. *The following are equivalent.*

- (1) *T does not have weak-TP₁.*
- (2) *For all $\phi(x, y)$ and strongly indiscernible tree $(a_\eta : \eta \in {}^{\omega_1}>\omega)$, if $\{\phi(x, a_\nu) : \nu \in O_\eta\}$ is consistent, then for each $\eta \in {}^{\omega_1}>\omega$ and $m < \omega$, $\{\phi(x, a_\nu) : \nu \in K_{\eta,m,\omega_1}\}$ is consistent.*

Proof. (Sketch)

(1 $\Leftarrow$ 2) Use modeling property and compactness.

(1 $\Rightarrow$ 2) Suppose not. Fix $\phi(x, y)$ and $(a_\eta : \eta \in {}^{\omega_1}>\omega)$. We inductively choose a finite subset $w_\eta \subseteq {}^{\omega_1}>\omega$ and $\nu_\eta \in {}^{\omega_1}>\omega$ for each $\eta \in {}^{\omega_1}>\omega$ so that the following conditions holds after the construction;

- (a) the union of $\{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta \smallfrown \alpha} : \alpha \leq \text{len}(\eta)\}\}$ and $\{\phi(x, a_\nu) : \nu \in O_{\nu_\eta}\}$ is consistent,
- (b) the union of $\{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta \smallfrown \alpha} : \alpha \leq \text{len}(\eta)\}\}$ and $\{\phi(x, a_\nu) : \nu \in \bigcup_{i < k} w_{\eta \smallfrown i}\}$ is inconsistent.

Set $w_\emptyset = \emptyset$ and $\nu_\emptyset = \langle \rangle$. At limit case, $w_\eta = \emptyset$ and $\nu_\eta = \bigcup_{\xi \leq \eta} \nu_\xi$.

Assume $w_{\eta \smallfrown \alpha}$, $\nu_{\eta \smallfrown \alpha}$ is chosen for all $\alpha \leq \text{len}(\eta)$. Let $p_\eta = \{\phi(x, a_\nu) : \nu \in \bigcup\{w_{\eta \smallfrown \alpha} : \alpha \leq \text{len}(\eta)\}\}$. We take the least $m_\eta < \omega$ where $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta, \omega, m_\eta, \omega_1}\}$ is inconsistent.

By minimality of m_η and strong indiscernibility, $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\nu_\eta, i, \omega, m_\eta - 1, \omega_1}\}$ is consistent for any $i < \omega$.

To argue inconsistency, we need an observation on strongly indiscernible trees.

Observation 3.4. *Let $(a_\eta : \eta \in {}^{\omega_1}>\omega)$ be strongly indiscernible. If $\{\phi(x, a_\nu) : \nu \in K_{\eta, \omega, m, \omega_1}\}$ is inconsistent for some $m < \omega$ and $\eta \in {}^{\omega_1}>\omega$, then there is some $k, l < \omega$ such that $\{\phi(x, a_\nu) : \nu \in K_{\eta, k, m, l}\}$ is inconsistent.*

By the above observation, we have $k_\eta, l_\eta < \omega$ such that $p_\eta \cup \{\phi(x, a_\nu) : \nu \in K_{\eta, k_\eta, m_\eta, l_\eta}\}$ is inconsistent. Take $w_{\eta \smallfrown i} = K_{\eta \smallfrown i, k_\eta, m_\eta - 1, l_\eta}$ and $\nu_{\eta \smallfrown i} = \nu_\eta \smallfrown i \smallfrown 0^{m_\eta - 1} \smallfrown 0^{l_\eta + 1}$ for $i < \omega$. Note that for any $i_0 < \dots < i_{k_\eta - 1} < \omega$, $p_\eta \cup \bigcup_{j < k_\eta} \{\phi(x, a_\nu) : \nu \in w_{\eta \smallfrown i_j}\}$ is inconsistent by strong indiscernibility.

Having done the construction, we choose a finite subset $q_\eta \subseteq p_\eta$ for each η such that $q_\eta \cup \bigcup_{j < k_\eta} \{\phi(x, a_\nu) : \nu \in w_{\eta \smallfrown j}\}$ is inconsistent.

Let $\tau_\eta = \{a_\nu : \phi(x, a_\nu) \in q_\eta\}$. By observation again, we may assume τ_η is a finite collection of $K_{\nu, k, m, l}$ s. Considering τ_η as a tuple, the number of $\sim_{str}$ -equivalent classes of $\{\tau_\eta : \eta \in {}^{\kappa}>\omega\}$ and $\{\bar{w}_{\eta \smallfrown i} : \eta \in {}^{\kappa}>\omega, i < \omega\}$ are both countable. By lemma 3.1, we have an embedding $h : {}^{\omega}>\omega \rightarrow {}^{\omega_1}>\omega$ whose range is monochromatic.

Define a formula $\psi(x, y)$ and a tree $(b_\eta : \eta \in {}^{\omega}>\omega)$ such that $\psi(x, b_\eta) = \bigwedge q_{h(\eta)} \wedge \bigwedge \{\phi(x, a_\nu) : \nu \in w_{h(0 \smallfrown \eta)}\}$. Then $\psi(x, y)$ and $(b_\eta : \eta \in {}^{\omega}>\omega)$ witness weak-TP₁. $\square$

Now we turn our intention to antichains.

Definition 3.5. A subset $A \subseteq {}^{\kappa}>\lambda$ is called an antichain if for all $\eta, \nu \in A$, $\eta \perp \nu$.

- Definition 3.6.**
- (1) Let A be a set of tuples in ${}^{\omega}>2$. We say ϕ has A^{str} -SOP₂ if there is $(a_\eta : \eta \in {}^{\omega}>2)$ such that
 - (a) for all $\eta \in {}^{\omega}>2$, $\{\phi(x, a_{\eta \smallfrown m}) : m < \omega\}$ is consistent, and
 - (b) for all $\bar{\nu} \in {}^{\omega}>2$, if $\bar{\nu} \sim_{str} \bar{\xi}$ for some $\bar{\xi} \in A$, then $\{\phi(x, a_\nu) : \nu \in \bar{\nu}\}$ is inconsistent.
 - (2) Let A be a set of tuples in ${}^{\omega}>\omega$. We say ϕ has A^{str} -TP₁ if there is $(a_\eta : \eta \in {}^{\omega}>\omega)$ such that

- (a) for all $\eta \in {}^\omega\omega$, $\{\phi(x, a_{\eta \upharpoonright m}) : m < \omega\}$ is consistent, and
- (b) for all $\bar{\nu} \in {}^\omega > \omega$, if $\bar{\nu} \sim_{str} \bar{\xi}$ for some $\bar{\xi} \in A$, then $\{\phi(x, a_\nu) : \nu \in \bar{\nu}\}$ is inconsistent.
- (3) We say T has A^{str} -SOP₂ (resp. A^{str} -TP₁) if it has a A^{str} -SOP₂ (resp. A^{str} -TP₁) formula. If $A = \{\bar{\nu}\}$, then we say ϕ (or T) has $\bar{\nu}^{str}$ -SOP₂ (resp. $\bar{\nu}^{str}$ -TP₁).

Remark 3.7. (1) ϕ has SOP₂ if and only if ϕ has $\langle\langle 0 \rangle, \langle 1 \rangle\rangle^{str}$ -SOP₂.
 (2) ϕ has TP₁ if and only if ϕ has $\langle\langle 0 \rangle, \langle 1 \rangle\rangle^{str}$ -TP₁.
 (3) ϕ has weak k -TP₁ if and only if ϕ has $\langle\langle 0 \rangle, \dots, \langle k-1 \rangle\rangle^{str}$ -TP₁.

Theorem 3.8. *Let $1 < k < \omega$ be given.*

- (1) *Let $\bar{\nu} \in {}^\omega > \omega$ be any antichain of size k . Then T has TP₁ if and only if T has $\bar{\nu}^{str}$ -TP₁.*
- (2) *Let $\bar{\nu} \in {}^\omega > 2$ be any antichain of size k . Then T has SOP₂ if and only if T has $\bar{\nu}^{str}$ -SOP₂.*

Proof. (1) Suppose ϕ has TP₁. Since any antichain tuple contains $\leq$ -incomparable pairs, ϕ has $\bar{\nu}^{str}$ -TP₁ for any antichain $\bar{\nu}$.

The converse is clear by proposition 3.3, and that T has TP₁ if and only if T has weak-TP₁.

(2) Note that T has SOP₂ if and only if T has TP₁. Then by (1), T has $\bar{\nu}^{str}$ -TP₁. Let $(a_\eta : \eta \in {}^\omega > \omega)$ be the witness of $\bar{\nu}^{str}$ -TP₁-ness. Then ϕ with the subtree $(a_\eta : \eta \in {}^\omega > 2)$ satisfies $\bar{\nu}^{str}$ -SOP₂.

The converse is clear by proposition 3.2. □

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