

STARLIKENESS OF CERTAIN INTEGRAL

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1. Introduction.

Let A be the class of functions $f(z)$ which are analytic in $E = \{z : |z| < 1\}$, with $f(0) = f'(0) - 1 = 0$. A function $f(z) \in A$ is said to be starlike iff

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E.$$

We denote by S^* the subclass of A consisting of functions which are univalently starlike in E .

R. Singh and S. Singh [3] have proved that if $f(z) \in A$ and $\operatorname{Re} f'(z) > 0$ in E , then $F(z) \in S^*$, where

$$F(z) = \int_0^z \frac{f(t)}{t} dt.$$

In this paper, we will improve the above result.

2. Preliminaries.

In this paper, we need the following lemmata.

LEMMA 1. Let $p(z)$ be analytic in E , $p(0) = 1$ and suppose that

$$\operatorname{Re}(p(z) + zp'(z)) > -\frac{\log(4/e)}{(2\log(e/2))} \quad \text{in } E,$$

where $-(\log(4/e)/(2\log(e/2))) = -0.6294\dots$.

Then we have

$$\operatorname{Re} p(z) > 0 \quad \text{in } E.$$

We owe this lemma to [1].

LEMMA 2. Let $p(z)$ be analytic in E , $p(0) = 1$ and suppose that

$$\operatorname{Re}(p(z) + zp'(z)) > 0 \quad \text{in } E.$$

Then we have

$$|\arg p(z)| < \alpha^* \frac{\pi}{2} \quad \text{in } E$$

where

$$1 = \alpha^* + \frac{2}{\pi} \tan^{-1} \alpha^*$$

and

$$0.6383 < \alpha^* < 0.6384.$$

We owe this lemma to [2, Lemma 3].

LEMMA 3. Let $p(z)$ be analytic, $p(0)=1$ and suppose that

$$\operatorname{Re}(p(z) + zp'(z)) > 0 \quad \text{in } E.$$

If $g(z)$ is analytic in E , $g(0)=1$ and if

$$\operatorname{Re} p(z) [zg'(z) + g^2(z) + g(z)] > \frac{\log(4/e)}{6} \left(\tan^2 \alpha^* \frac{\pi}{2} - 3 \right) \quad \text{in } E,$$

then we have

$$\operatorname{Re} g(z) > 0 \quad \text{in } E.$$

We owe this lemma to [2, Lemma 4].

3. Main theorem.

MAIN THEOREM. Let $f(z) \in A$ and suppose that

$$(1) \quad \operatorname{Re} f'(z) > \frac{\log(4/e)}{6} \left(\tan^2 \alpha^* \frac{\pi}{2} - 3 \right) \quad \text{in } E,$$

where

$$-0.03518 < \frac{1}{6} (\log(4/e)) \left(\tan^2 \alpha^* \frac{\pi}{2} - 3 \right) < -0.03502.$$

Then $F(z) \in S^*$, where

$$(2) \quad F(z) = \int_0^z \frac{f(t)}{t} dt.$$

Proof. From (2), we have

$$(3) \quad F'(0)=1, F'(z)=f(z)/z \text{ and } F''(z)=(zf'(z)-f(z))/z^2.$$

Then we have

$$(4) \quad \begin{aligned} \operatorname{Re}(zF''(z) + F'(z)) &= \operatorname{Re} f'(z) \\ &> \frac{\log(4/e)}{6} \left(\tan^2 \alpha^* \frac{\pi}{2} - 3 \right) \quad \text{in } E. \end{aligned}$$

From the assumption (1) and from LEMMA 1, we have

$$(5) \quad \operatorname{Re} F'(z) > 0 \quad \text{in } E.$$

Let us put

$$p(z) = \frac{F(z)}{z}$$

and

$$g(z) = \frac{zF'(z)}{F(z)}.$$

Since $p(0)=1$ and

$$\operatorname{Re}(zp'(z)+p(z)) = \operatorname{Re}F'(z) > 0 \quad \text{in } E,$$

by LEMMA 2, we have

$$|\arg p(z)| < \alpha^* \frac{\pi}{2} \quad \text{in } E.$$

On the other hand, by an easy calculation, and from (3) and (5), we have

$$\begin{aligned} & \operatorname{Re} p(z) [zg'(z) + g^2(z) + g(z)] \\ &= \operatorname{Re} [zF''(z) + 2F'(z)] = \operatorname{Re} \left[f'(z) + \frac{f(z)}{z} \right] \\ &> \operatorname{Re} f'(z) > \frac{1}{6} \left(\tan^2 \alpha^* \frac{\pi}{2} - 3 \right) (\log(4/e)) \quad \text{in } E. \end{aligned}$$

Therefore, from LEMMA 3, we have

$$\operatorname{Re} g(z) > 0 \quad \text{in } E.$$

This shows that

$$\operatorname{Re} \frac{zF'(z)}{F(z)} > 0 \quad \text{in } E.$$

This completes our proof.

References

- [1] M. Nunokawa, Differential inequalities and Caratheodory functions,
Proc. Japan Acad., 65, Ser. A, No. 9(1989). (to appear)
- [2] ———, On starlikeness and convexity of certain integral. submitting.
- [3] R. Singh and S. Singh, Starlikeness and convexity of certain integral,
Ann. Univ. Mariae Curie-Sklodowska. Lublin, XXXV, 16, Ser. A (1981),
145-148.