

MESHLESS NON-PARAMETRIC SHAPE DESIGN OF PIECEWISE APPROXIMATELY DEVELOPABLE SURFACES USING DISCRETIZED LOCAL GAUSS MAP

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ABSTRACT

A meshless and non-parametric approach is proposed for shape design of curved surfaces consisting of approximately developable surface patches. A local Gauss map is defined at the grid point of a surface, and its developability is evaluated based on the area of the local Gauss map. An optimization problem is formulated for minimizing the sum of square areas of the local Gauss map at the grid points of the surface with specified boundary shape. Internal boundaries between the approximately developable surface patches are generated by relaxing the developability at the selected grid points. The effectiveness of the proposed method for generating various shapes of piecewise approximately developable surface is demonstrated in the numerical examples.

Keywords: shape design, developable surface, discrete differential geometry, Gauss map

1. INTRODUCTION

For the design of curved surfaces for architectural shell roofs, constructional cost can be drastically reduced if the surface is designed as a developable surface [1]. However, the types of developable surfaces are limited to conical, cylindrical, and tangent developable surfaces. Therefore, extensive research has been carried out for designing piecewise developable surfaces consisting of several developable surface patches [2]. Conversely, smooth surfaces may be generated using the parametric representation of a surface [3]. Cui and Ohsaki [4, 5] proposed a method of generating smooth piecewise developable patches using the $(1,n)$ -Bézier surface [6]. However, a discretized model is needed for structural analysis and construction process. Although the formulations of isogeometric analysis [7] can be used to employ the same model consistently in design and analysis, simple triangular or quadrilateral elements are still preferred

especially for nonlinear structural analysis. Therefore, it is preferable that a discretized form is used in the design process.

Recently, discrete differential geometry [8] is extensively utilized for design of curved surfaces in the fields of computer graphics and industrial design [9, 10]. Some studies can also be found in architectural engineering [11]. Various shapes can be generated solving optimization problems formulated with respect to the discrete measures of surface geometry [12]. However, one of the most difficult points for characterizing the geometrical properties of a surface using the definitions of discrete differential geometry is that the values of those properties depend on the mesh division. For the mean curvature and Gaussian curvature, mesh dependency can be reduced by using their definitions utilizing the inner angles of faces and the areas of Voronoi regions around vertices of a triangular mesh [13, 14]. However, when using triangular meshes,

there are still various patterns of mesh division even when the sampling points of the vertices are specified on the surface.

The geometrical invariants can be utilized for generating surfaces with various properties [15]. The geometrical measures of derived surfaces such as offset surfaces and Gauss map can also be utilized for surface generation. Roulier and Rando [16] proposed a method for generating surface shapes by minimizing the areas of derived surfaces, and formulated rounding and flattening measures for generating developable surfaces. As the simplest application of geometrical measures, the developable surface is characterized by vanishing Gaussian curvature everywhere on the surface. However, one of the drawbacks of using the Gaussian curvature as the measure of developability of a discretized surface is that a corrugated surface like a crushed paper also satisfies the developability condition [17]. On the other hand, the properties of Gauss map, which is generated from the unit normal vectors on the surface, can be used for generating developable surfaces [18, 19] so that the Gauss map degenerates into a line or a point, and the aforementioned drawback of generating corrugated surface can be successfully alleviated.

In this paper, the authors' approach of utilizing the area of Gauss map [18] is extended to propose a meshless formulation for computing the area of local Gauss map at the specified points on the surface. The term "meshless" has been taken from the meshless method (also called meshfree method) for structural analysis [20]. In the meshless method, only a scattered set of vertices representing the domain of interest is required; no connectivity information in terms of meshes or elements among these vertices is necessary as in the conventional finite element methods. To compute the area of the local Gauss map, a similar approach to the meshless methods has been adopted in our method; influence of functions at each neighboring vertices is evaluated, and there is no need to explicitly define the connectivity of a grid or randomly assigned vertices on the curved surface. The proposed method is applied for generating piecewise approximately developable surfaces for specified boundary shape.

2. DEVELOPABILITY CONDITION USING AREA OF LOCAL GAUSS MAP

Gauss map is the surface defined by the unit normal of a surface. For example, the Gauss map of a convex surface forms a region on the unit sphere. By

contrast, the Gauss map of a plane and a cylinder degenerate to a point and a line, respectively. Therefore, developability of a surface is characterized by the vanishing area of the Gauss map [18, 19].

Suppose the surface $\mathbf{r}(s,t)$ to be designed, which is called *design surface*, is modeled using the parameters (s,t) . The Gauss map is defined by the unit normal vector $\mathbf{n}(s,t)$ of the design surface as

$$\mathbf{n}(s,t) = \frac{\hat{\mathbf{n}}(s,t)}{\|\hat{\mathbf{n}}(s,t)\|}, \quad \hat{\mathbf{n}}(s,t) = \frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t} \quad (1)$$

An example of a convex surface and its Gauss map are illustrated in Fig. 1. Red arrows represent the unit normal vectors of the design surface.

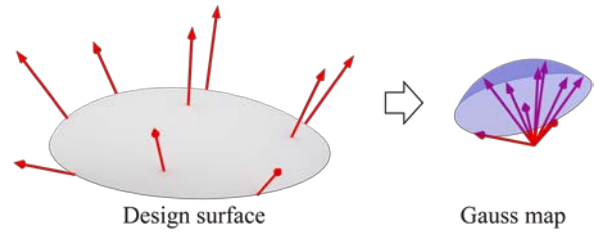


Figure 1: An example of Gauss map

When the design surface is discretized into a polyhedron, there are some definitions of the unit normal vector and the Gauss map. Stein et al. [19] proposed a method for generating piecewise developable discrete surfaces with triangular mesh. However, they defined the Gauss map using the unit normal vectors of triangular faces, and accordingly, it is difficult to compute the area of the Gauss map, because the Gauss map of a surface does not have the same topology as the triangulated design surface. Therefore, it is preferable that the unit normal vector is computed at the vertex so that the Gauss map and the design surface has the same topology of triangularization. There are some definitions of the unit normal vector at a vertex. For example, Hayashi et al. [12] define the unit normal vector as the area-weighted average of the unit normal vectors of the adjacent triangular faces. Sullivan [13] defines the direction of the unit normal vector according to the direction of the mean curvature vector. Ohsaki and Hayakawa [18] proposed a method for generating piecewise developable surfaces based on minimization of the total area of Gauss map defined using the unit normal vectors at vertices computed according to Hayashi et al. [12].

However, the total area need not be computed because developability of the surface can be

determined by its local property. Furthermore, it is desirable to propose a method that does not depend on the type of mesh discretization. Here, the normal vector of a vertex is defined as the average of the unit normal vectors of the adjacent faces, which is not weighted by the area of triangle because we do not consider triangular meshes explicitly. Instead of computing the total area of Gauss map, we define the local Gauss map and compute its area at each vertex or a point on a surface, without resort to a triangular mesh for the whole surface. Let $\mathbf{q}_1, \dots, \mathbf{q}_{m_i}$ denote the location vectors of m_i points, which are simply denoted by points $1, \dots, m_i$, in the neighborhood of point i , for which the location vector is denoted by $\mathbf{p}_i$, as shown in Fig. 2. Auxiliary edges are drawn between $\mathbf{p}_i$ and $\mathbf{q}_1, \dots, \mathbf{q}_{m_i}$ which are denoted by edges $1, \dots, m_i$.

The unit normal vector of the region between edges j and $j+1$, denoted by $\mathbf{n}_j^f$, is computed as

$$\mathbf{n}_j^f = \frac{\hat{\mathbf{n}}_j^f}{\|\hat{\mathbf{n}}_j^f\|}, \quad \hat{\mathbf{n}}_j^f = (\mathbf{q}_j - \mathbf{p}_i) \times (\mathbf{q}_{j+1} - \mathbf{p}_i) \quad (j=1, \dots, m_i) \quad (2)$$

where $j+1$ is replaced with 1 if $j = m_i$. Then the unit normal vector $\mathbf{n}_i$ at point i is computed as the average of the unit normal vectors of the surrounding regions as

$$\mathbf{n}_i = \frac{\hat{\mathbf{n}}_i}{\|\hat{\mathbf{n}}_i\|}, \quad \hat{\mathbf{n}}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \mathbf{n}_j^f \quad (3)$$

In our previous method in Ref. [18], the area of each triangular face is multiplied to the unit normal vector before taking the average. However, a simpler formulation is preferred, because the area of Gauss map in the neighborhood of a point, which is called *area of local Gauss map*, is to be reduced to zero to obtain a developable surface. Furthermore, we propose a method that utilizes only the locations of points and does not depend on a particular triangular mesh discretization.

The local Gauss map is defined by the unit normal vectors at the point and its surrounding regions. The unit normal vector at a point of the design surface, indicated with blue in the left figure of Fig. 2, is computed from those of the regions indicated with red defined by the locations of the neighborhood

points. The normal vectors are translated to have the same origin to generate the local Gauss map as shown in the right figure of Fig. 2.

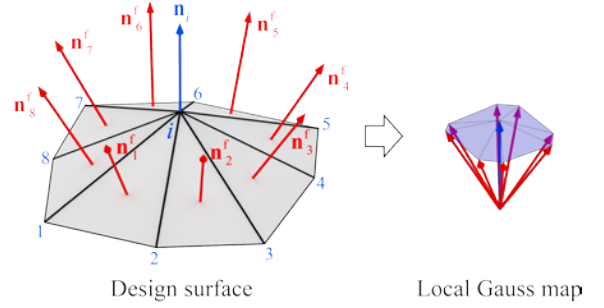


Figure 2: Local Gauss map at a point on a surface

Obviously, the local Gauss map degenerates to a point if the design surface is locally planar at the point. For a cylindrical surface as illustrated in Fig. 3, the area of local Gauss map at the center point of the surface vanishes because the normal vectors 'a', 'b', and 'c' are in the same plane. It is notable that the area of local Gauss map vanishes also for a conical surface, if the points are appropriately located.

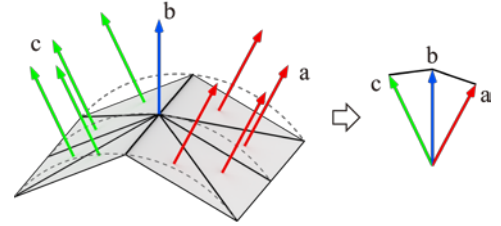


Figure 3: Illustration of local Gauss map at a point on a cylindrical surface; vectors in the same color are parallel to each other

The area of local Gauss map in the neighborhood of point i , denoted by A_i , is computed as

$$A_i = \sum_{j=1}^{m_i} a_{ij} \quad (4)$$

where

$$a_{ij} = \frac{1}{2} \|\mathbf{v}_{ij}\|, \quad \mathbf{v}_{ij} = (\mathbf{n}_j^f - \mathbf{n}_i) \times (\mathbf{n}_{j+1}^f - \mathbf{n}_i) \quad (5)$$

Note that the summation of A_i over the whole surface does not lead to the total area of the Gauss map of the surface. We are interested only in the local property of the Gauss map.

Consider a surface with grid points as shown in Fig. 4, where the dashed grid lines are drawn for convenience of visualizing the surface. The normal

vector is computed at each grid point marked with blue. For this purpose, the auxiliary edges are generated. For example, for point A in Fig. 5, the point is connected to eight neighborhood points with the edges marked with red, and the normal vector at point A is computed as the average of the unit normal vectors of eight regions between the auxiliary edges connected to point A. The auxiliary edges are generated also for points B and C marked with blue and green, respectively. As a result, the auxiliary edges are generated as shown in Fig. 6 from the grid points in Fig. 4. Note that there is no grid point at the intersection of the two diagonal auxiliary edges. The auxiliary edges are added tentatively only when the area of local Gauss map is computed at each grid point.

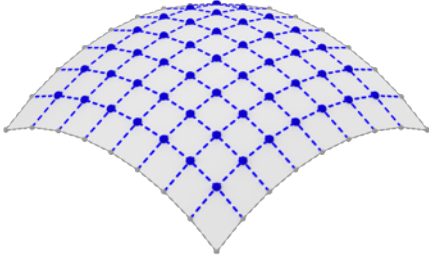


Figure 4: Surface with grid points

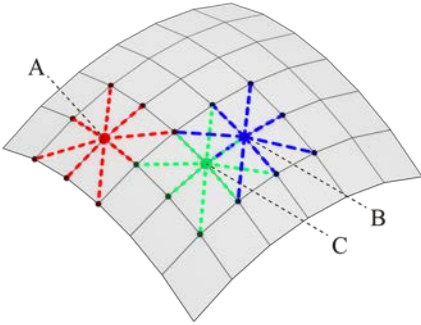


Figure 5: Grid points A, B, C and auxiliary edges connecting to their neighborhood points

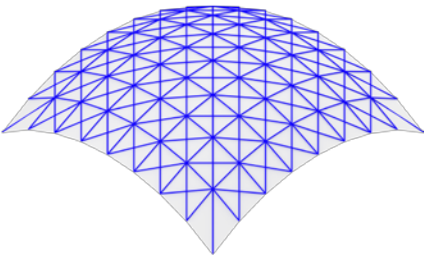


Figure 6: Grid points and auxiliary edges on a curved surface

In the numerical examples, a gradient based nonlinear programming method is used, and the derivatives of the objective function are computed analytically to accelerate the optimization process. Let a prime denote differentiation with respect to a design variable. The derivative of a_{ij} is derived as

$$a_{ij}' = \frac{1}{2} \frac{\mathbf{v}_{ij} \cdot \mathbf{v}_{ij}'}{\|\mathbf{v}_{ij}\|} \quad (6)$$

which diverges if $\|\mathbf{v}_{ij}\| = 0$ although $\|\mathbf{v}_{ij}'\| = 0$ is achieved at the optimal solution. Therefore, the sum of $(a_{ij})^2$ instead of a_{ij} in Eq. (4), which is called *square area of the local Gauss map* for brevity, is computed and their sum over all interior grid points is minimized to obtain a developable surface that is locally planar, cylindrical, or conical. Note that boundary points are excluded for the summation. Furthermore, some selected points are also excluded to obtain piecewise approximately developable surfaces. Let I denote the set of indices of the interior grid points at which the area of local Gauss map is to be computed. The vector of design variables $\mathbf{x}$ consists of the coordinates of grid points that are to be varied. The optimization problem is simply formulated as

$$\text{Minimize } F(\mathbf{x}) = \sum_{i \in I} \sum_{j=1}^{m_i} (a_{ij})^2 \quad (7)$$

3. NUMERICAL EXAMPLES

The proposed method is applied to shape design of piecewise developable surfaces supported at four corners. Optimization is carried out using sequential quadratic programming library SNOPT Ver. 7 [21] with the python interface of pyOpt [22]. Only z -coordinates of the grid points are chosen as design variables. The derivatives of the objective function with respect to the design variables are derived analytically. Computation is carried out on a PC with 12th Gen Intel(R) Core(TM) i9-12900KF CPU 3.19 GHz, 32.0 GB RAM, and 16 cores.

Various piecewise approximately developable surfaces are generated from the initial surface with 25×13 grid points as shown in Fig. 7. The (x, y) -coordinates are on the plane as shown in Fig. 7(b), and the z -coordinate is perpendicular to the plane. The coordinates of four corners and the center point are specified as $(-12, -6, 0)$, $(12, -6, 0)$, $(12, 6, 0)$, $(-12, 6, 0)$, $(0, 0, 6.48)$, respectively. The grid points

are numbered first in y -direction moving sequentially to x -direction; i.e., the surface in Fig. 7 has $23 \times 11 = 253$ interior points, and the points near the four corners are numbered as 1, 11, 243, and 253, as indicated in Fig. 7(b) in green.

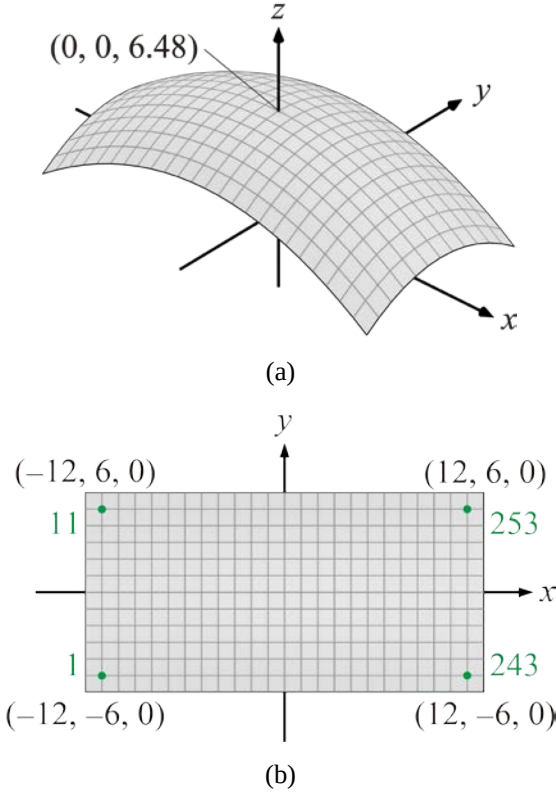


Figure 7: Initial surface with 25×13 grid points; (a) diagonal view, (b) plan view

The optimal shapes are shown in Figs. 8–10, which have fixed points marked with red and the points marked with blue whose Gauss maps are not computed.

If we fix the locations of four corners and the center point indicated with red in Fig. 8(a) and assign developability condition at all the interior grid points, the cylindrical shape as shown in Fig. 8(a) is obtained by solving the optimization problem. As seen from Fig. 8(c), where the vertical axis is in logarithmic scale, the local area of Gauss map at interior grid point is sufficiently smaller than that of the initial surface. The computation time for optimization is 3.607 sec.

The piecewise approximately planar surface as shown in Fig. 9 is obtained, if we fix the locations of five points, as shown in Fig. 9(a), in addition to the four corner points, and do not assign developability condition at the three interior points indicated with blue in Fig. 9(b). It is confirmed from Fig. 9(c) that

the area of local Gauss map is sufficiently small at all the interior grid points as a result of drastic increase of the area at the point at which the developability is not considered. Eight planar patches are connected by the salient ridge lines.

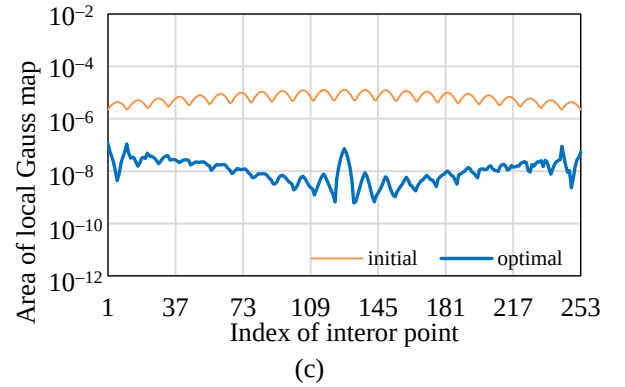
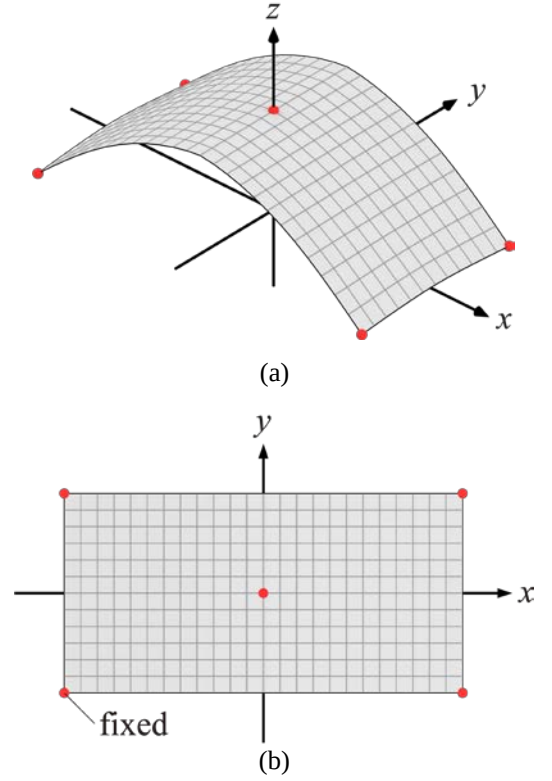


Figure 8: Developable surface with five fixed points in red and no non-developable point; (a) diagonal view, (b) plan view, (c) area of Gauss map at interior point

The computational time for optimization is 2.236 sec.

Next, we obtain a pair of curved surfaces, as shown in Fig. 10, by relaxing the developability condition at the points on xz -plane as shown in Fig. 10(b). The area of local Gauss map at each interior point sufficiently decreases also for this example, as observed in Fig. 10(c), as a result of increase of the

area at the points marked with dark blue. The computational time is 0.950 sec.

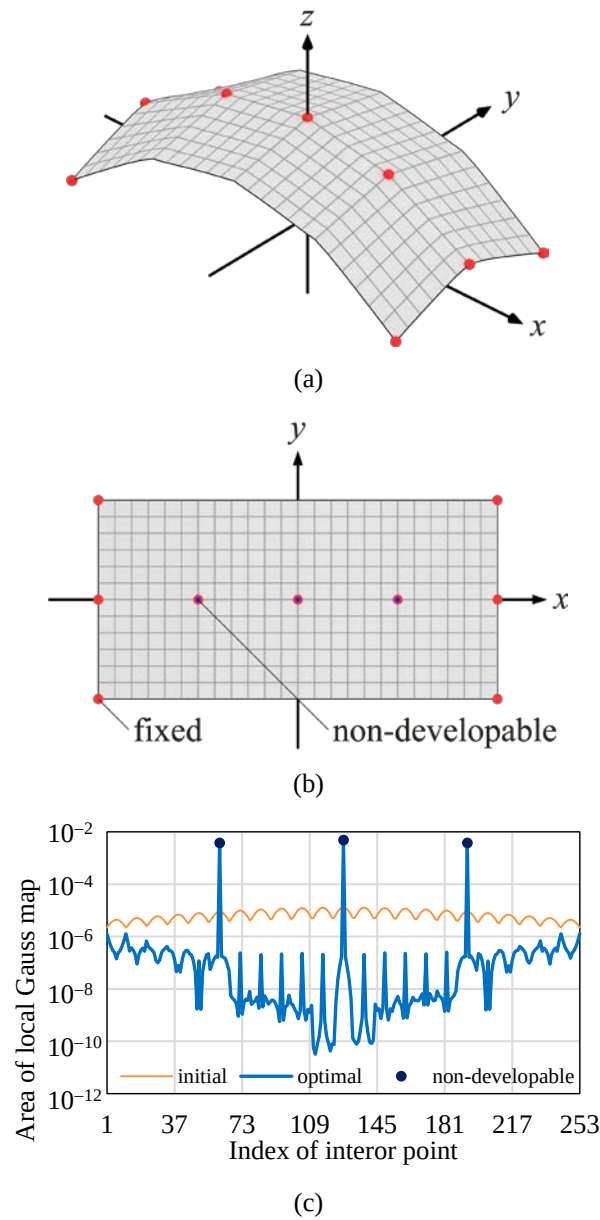


Figure 9: Piecewise developable surface with nine fixed points in red and three non-developable points in blue; (a) diagonal view, (b) plan view, (c) area of Gauss map at interior point

The initial surfaces with 37×19 grid points and 49×25 grid points are also considered for the same case as in Fig. 9. The optimal shapes are shown in Figs. 11 and 12. The computational times are 22.6 sec. and 32.1 sec., respectively. As shown in Figs. 11(c) and 12(c), the area of local gauss map tends to be larger than that in Fig. 9(c) as the number of grid points becomes larger. This may indicate that the optimization is terminated at a local optimal

solution. It is also seen from Figs. 9, 11, and 12 that the ridge lines between the surface patches tend to disappear as the number of grid points increases.

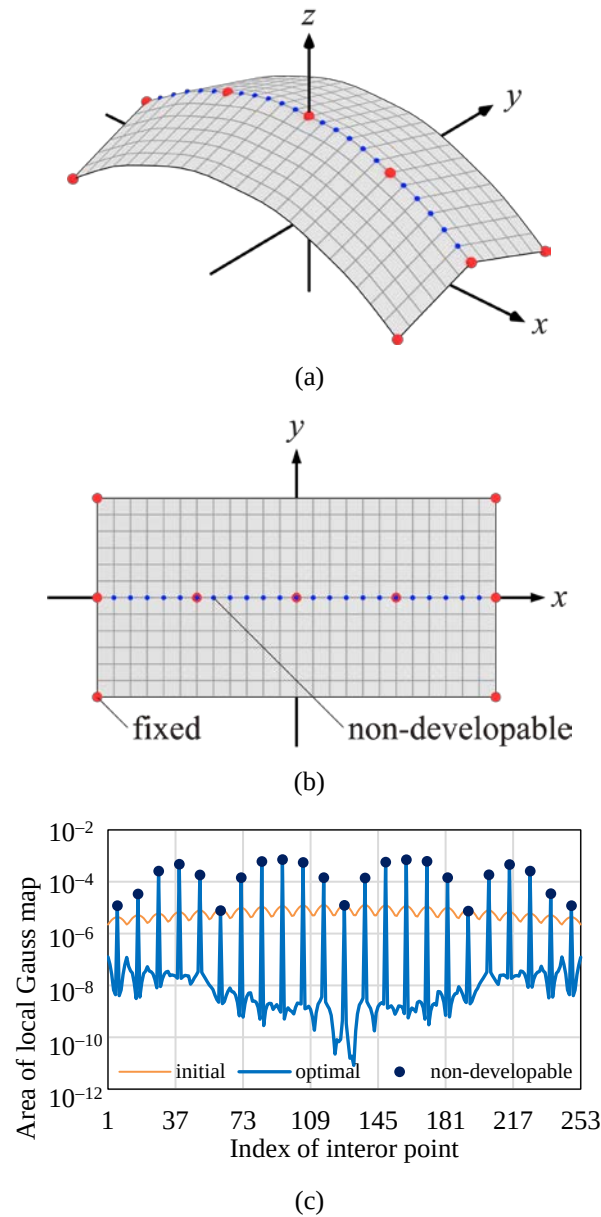


Figure 10: Piecewise developable surface with nine fixed points in red and twenty-three non-developable points in blue; (a) diagonal view, (b) plan view, (c) area of Gauss map at interior point

The reason for this is that the developability condition is formulated locally about each interior point, and more varieties of surfaces consisting of surface patches satisfying local developability conditions can be generated as the number of grid points is increased. In practical design process if architectural shell roofs, it is not necessary to use a very large number of grid points, and the grid points

shown in Fig. 9 are appropriate to avoid a smooth surface as shown in Fig. 12 without explicit ridge lines.

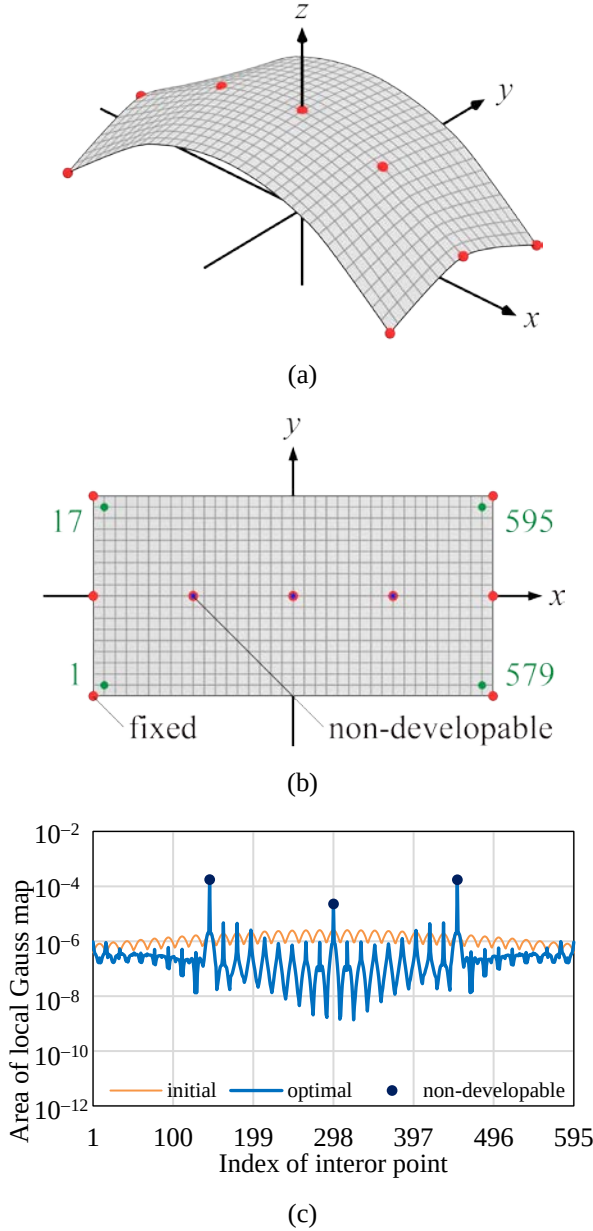


Figure 11: Piecewise developable surface with 37×19 grid points and nine fixed points in red and three non-developable points in blue; (a) diagonal view, (b) plan view, (c) area of Gauss map at interior point

Consider next an initial surface with 25×25 grid points as shown in Fig. 13. The coordinates of four corners and the center point are specified as $(-10, -10, 0)$, $(10, -10, 0)$, $(10, 10, 0)$, $(-10, 10, 0)$, $(0, 0, 4.92)$, respectively, and the locations of points on the perimeter and at the center are fixed. The grid

points are numbered in the same way as in Fig. 7; i.e., the surface in Fig. 11 has $23 \times 23 = 529$ interior points, and the points near the four corners are numbered as 1, 23, 507, and 529, as indicated in Fig. 13(b) in green.

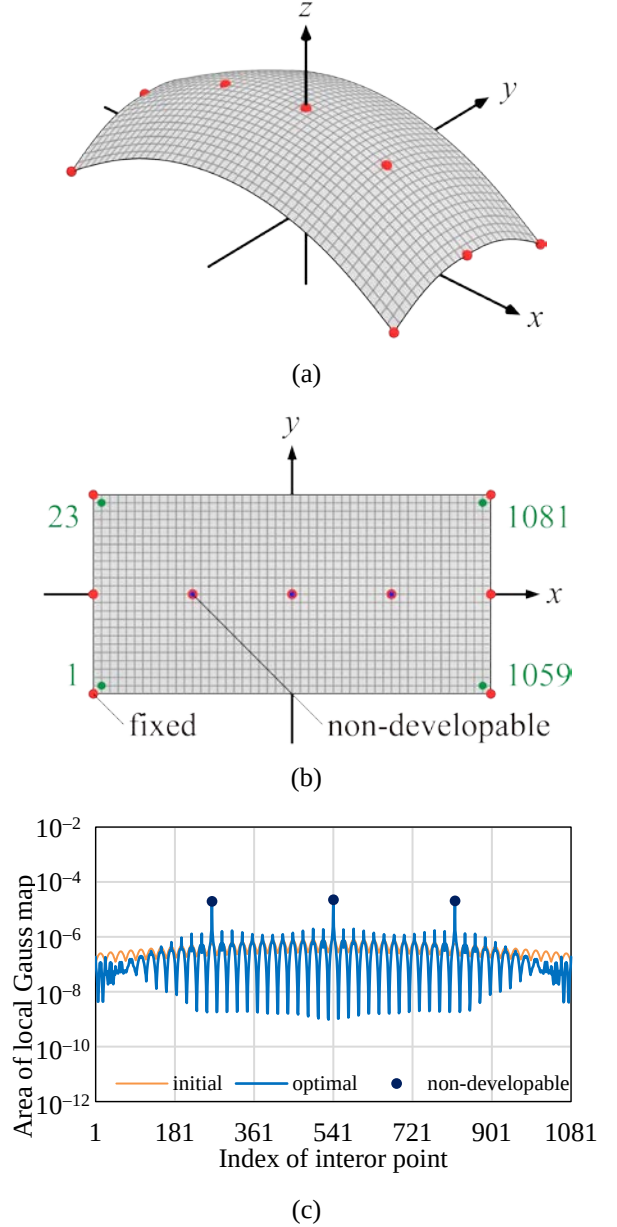


Figure 12: Piecewise developable surface with 49×25 grid points and nine fixed points in red and three non-developable points in blue; (a) diagonal view, (b) plan view, (c) area of Gauss map at interior point

Optimal shape is shown in Fig. 14, whose fixed points are marked with red. The Gauss maps of the points marked with blue are not computed. The surface as shown in Fig. 14 is obtained by relaxing the developability at the grid points on xz - and yz -planes. We can see from Fig. 14(c) that a piecewise

approximately developable surface consisting of four cylindrical surfaces connected on xz - and yz -planes is generated by minimizing the area of local Gauss map.

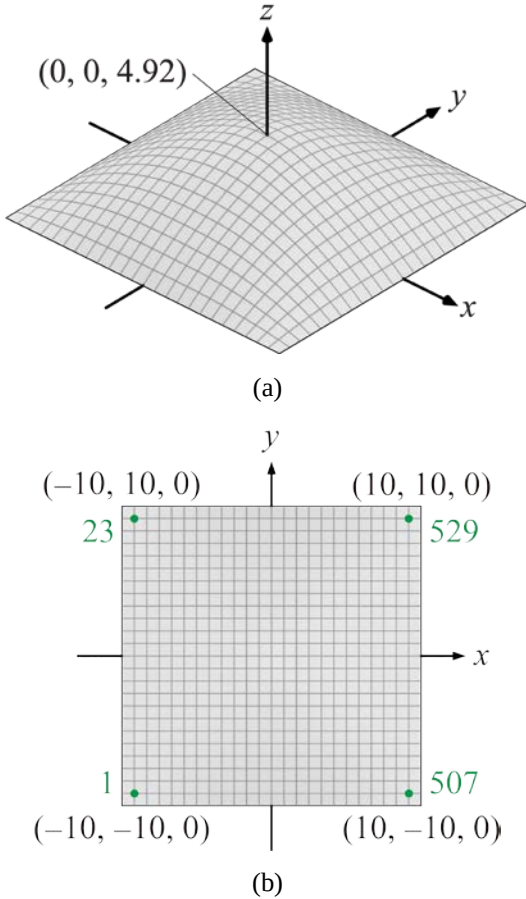


Figure 13: Initial surface with 25×25 grid points; (a) diagonal view, (b) plan view

4. CONCLUSIONS

A meshless and non-parametric approach has been proposed for shape design of curved surfaces consisting of piecewise approximately developable patches. Grid points are specified on the surface without particular mesh discretization. The unit normal vector at a point is defined as the average of the unit normal vectors of the surrounding regions, and the local Gauss map is defined by the unit normal vectors at the grid point and its surrounding regions. Developability is evaluated based on the area of local Gauss map at each grid point on the surface.

Optimization problem is formulated for minimizing the sum of square areas of the local Gauss map at the grid points of the surface with specified boundary shapes. It has been shown in the numerical examples

that various piecewise approximately developable surfaces can be obtained by relaxing the developability at the selected interior grid points and minimizing the objective function using a gradient based nonlinear programming method.

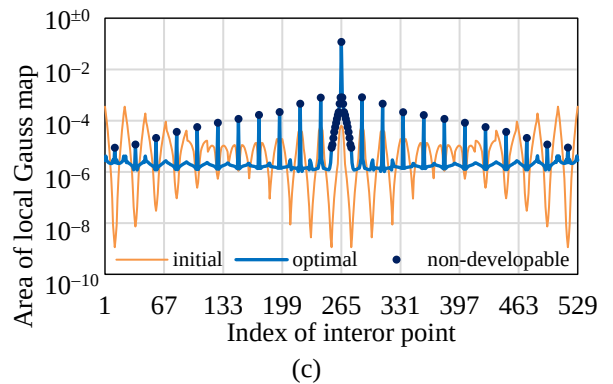
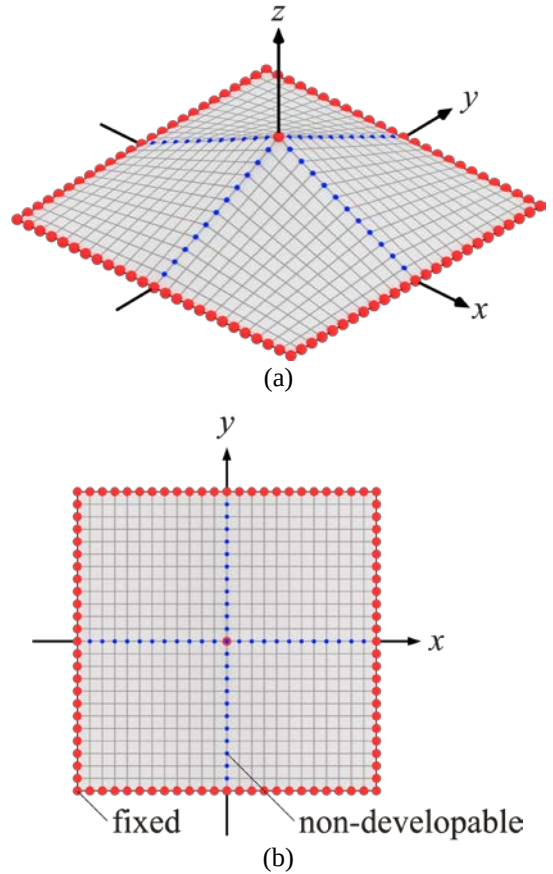


Figure 14: Piecewise developable surface with fixed points in red and non-developable points in blue; (a) diagonal view, (b) plan view, (c) area of Gauss map at interior point

Using the proposed method, mesh dependence of the measure of developability is alleviated with respect to the connectivity of the points on a surface, and a surface as an assembly of piecewise cylindrical or

planar patches can be generated by solving an optimization problem.

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