

On the solutions of an extended Chebyshev's Equations

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Abstract

In this article, we treat the extended homogeneous Chebyshev's equations

$$\varphi_2 \cdot (z^2 - 2bz - 1) + \varphi_1 \cdot (z - b) - \varphi \cdot \nu^2 = 0,$$

where

$$\varphi_0 = \varphi = \varphi(z), \quad f = f(z), \quad \varphi_\alpha = \frac{d^\alpha \varphi}{dz^\alpha} \text{ (for } \alpha > 0 \text{)}$$

and $z^2 - 2bz - 1 \neq 0$ in the view of N-fractional calculus and discuss the solutions by means of N- fractional calculus operator. We present the familiar form of the solution like as

$$\varphi(z) = -e^{i\pi\nu} \frac{\sqrt{\pi}}{2^\nu \nu \Gamma(\nu + \frac{1}{2})} (z - b)^{-\nu} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu}{2} + \frac{1}{2}; \nu + 1; \frac{b^2 + 1}{(z - b)^2}\right)$$

where $|(b^2 + 1)/(z - b)^2| < 1$ and ${}_2F_1(\cdots)$ is the Gauss hypergeometric function.

1 Definition of fractional calculus and some properties

We define the N-fractional calculus and N-fractional operator N^α as follows.

For a regular function $f = f(z)$ and a arbitrary number α , N-fractional differintegration of order α is defined as follows,

$$\begin{aligned} N^\alpha f &= f_\alpha = (f)_\alpha = {}_C(f)_\alpha \\ &= \frac{\Gamma(\alpha + 1)}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{\alpha+1}} \quad (\alpha \notin \mathbb{Z}^-), \end{aligned}$$

$$(f)_{-m} = \lim_{\alpha \rightarrow -m} (f)_\alpha \quad (m \in \mathbb{Z}^+),$$

where $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

$$-\pi \leq \arg(\zeta - z) \leq \pi \text{ for } C_-, \quad 0 \leq \arg(\zeta - z) \leq 2\pi \text{ for } C_+,$$

$$\zeta \neq z, \quad z \in C, \quad \nu \in R, \quad \Gamma; \text{ Gamma function,}$$

and C_- is a curve along the cut joining two points z and $-\infty + iIm(z)$, D_- is a domain surrounded by C_- , C_+ is a curve along the cut joining two points z and $\infty + iIm(z)$, D_+ is a domain surrounded by C_+ .

When $\alpha > 0$, $(f)_\alpha$ is the fractional derivative of arbitrary order α , and when $\alpha < 0$, it is the integral of order $-\alpha$, if $|(f)_\alpha| < \infty$.

We denote

$$N^\alpha \varphi = \frac{d^\alpha \varphi}{dz^\alpha} = (\varphi)_\alpha.$$

and the binary operation $\circ$ is

$$(N^\beta \circ N^\alpha)f = (N^\beta N^\alpha)f = N^\beta(N^\alpha f) = N^\alpha(N^\beta f) \quad (\alpha, \beta \in R), \quad (1)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \quad (2)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| \leq \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

As for the properties of the operator, see [2], [3], [5]. We introduce here two necessary lemmas.

Lemma I. We have ([1])

(i)

$$((z - c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} (z - c)^{\beta - \alpha} \quad (|\frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(\log(z - c))_\alpha = -e^{i\pi\alpha} \Gamma(\alpha) (z - c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z - c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z - c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z - c \neq 0$ in (i), and $z - c \neq 0, 1$ in (ii) and (iii),

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} u_{\alpha - k} v_k \quad (u = u(z), v = v(z))$$

Lemma II We have ([4])

(i)

$$(((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta}\right)^k, \quad \left(\left|\frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)}\right| < \infty\right) \quad (3)$$

and

(ii) for $n \in Z_0^+$

$$(((z-b)^\beta - c)^\alpha)_n = (-1)^n (z-b)^{\alpha\beta-n} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta}\right)^k, \quad \left|\frac{c}{(z-b)^\beta}\right| < 1, \quad (4)$$

where

$$[\lambda]_k = \lambda(\lambda+1) \cdots (\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \quad [\lambda]_0 = 1.$$

2 Solutions to an extended homogeneous Chebyshev's equations by means of N-fractional calculus

We discuss the following type of an extended Chebyshev's equation

$$\varphi_2 \cdot (z^2 - 2bz - 1) + \varphi_1 \cdot (z - b) - \varphi \cdot \nu^2 = 0$$

by means of N-fractional calculus.

The above equation is solved by means of N-fractional calculus as follows.

Theorem 1 Let $\varphi \in F = \{\varphi : 0 \neq |\varphi_\nu| < \infty, \nu \in R\}$, then the homogeneous extended Chebyshev's equation

$$\varphi_2 \cdot (z^2 - 2bz - 1) + \varphi_1 \cdot (z - b) - \varphi \cdot \nu^2 = 0 \quad (1)$$

has particular solutions of the forms,

(i)

$$\varphi = (((z-b)^2 - (b^2 + 1))^{-(\frac{1}{2}+\nu)})_{-(1+\nu)} \equiv \varphi_{[1]}^* \quad (2)$$

and

(ii)

$$\varphi = (((z-b)^2 - (b^2 + 1))^{-(\frac{1}{2}-\nu)})_{-(1-\nu)} \equiv \varphi_{[2]}^*. \quad (3)$$

Proof.

We set $g = z - b$, $h = z^2 - 2bz - 1$ and operating N^α to the both sides of equation (1), we have then

$$(\varphi_2 \cdot h)_\alpha + (\varphi_1 \cdot g)_\alpha - (\varphi \cdot \nu^2)_\alpha = 0, \quad (4)$$

hence

$$\varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot (2\alpha + 1) \cdot g + \varphi_{\alpha} \cdot (\alpha^2 - \nu^2) = 0 \quad (5)$$

since

$$N^{\alpha} \varphi_m = (\varphi_m)_{\alpha} = \varphi_{m+\alpha} \quad (m = 2, 1) \quad (6)$$

by our index law, and from Lemma (iv) we have

$$N^{\alpha}(\varphi_1 \cdot g) = (\varphi_1 \cdot g)_{\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + 1)}{k! \Gamma(\alpha + 1 - k)} (\varphi_1)_{\alpha-k} \cdot g_k \quad (7)$$

$$= \varphi_{1+\alpha} \cdot g + \varphi_{\alpha} \cdot \alpha \quad (8)$$

and

$$N^{\alpha}(\varphi_2 \cdot h) = (\varphi_2 \cdot h)_{\alpha} \quad (9)$$

$$= \varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot 2\alpha g + \varphi_{\alpha} \cdot \alpha(\alpha - 1). \quad (10)$$

Choose α such that $\alpha^2 - \nu^2 = 0$, that is $\alpha = \nu$ or $-\nu$. We have then

$$\varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot (2\alpha + 1)g = 0. \quad (11)$$

When $\alpha = \nu$, we set

$$\psi = \psi(z) = \varphi_{1+\nu}, \quad (\varphi = \psi_{-(1+\nu)}) \quad (12)$$

and we obtain

$$\psi_1 \cdot h + \psi \cdot (2\nu + 1)g = 0. \quad (13)$$

Then a particular solution to this linear first order equation is given by

$$\psi = h^{-(\frac{1}{2}+\nu)} \quad (14)$$

Therefore we have

$$\begin{aligned} \varphi &= \psi_{-(1+\nu)} = ((h^{-(\frac{1}{2}+\nu)})_{-(1+\nu)}) \\ &= (((z-b)^2 - (b^2 + 1))^{-(\frac{1}{2}+\nu)})_{-(1+\nu)} = \varphi_{[1]}^*. \end{aligned} \quad (15)$$

When $\alpha = -\nu$, in the similar way we obtain the second solution

$$\begin{aligned} \varphi = \psi_{-(1-\nu)} &= ((h^{-(\frac{1}{2}-\nu)})_{-(1-\nu)}) \\ &= (((z-b)^2 - (b^2 + 1))^{-(\frac{1}{2}-\nu)})_{-(1-\nu)} = \varphi_{[2]}^*. \end{aligned} \quad (16)$$

Inversely these functions shown by (15) and (16) satisfy the equation (1) clearly.

Indeed we have

$$\begin{aligned}
LHS \text{ of (1)} &= \varphi_2 \cdot h + \varphi_1 \cdot g - \varphi \cdot \nu^2 \\
&= \left((\varphi_2 \cdot h)_\alpha + (\varphi_1 \cdot g)_\alpha - \varphi_\alpha \cdot \nu^2 \right)_{-\alpha} \\
&= \left((\varphi_{2+\alpha} \cdot h + \varphi_{1+\alpha} \cdot (2\alpha + 1)g + \varphi_\alpha \cdot (\alpha^2 - \nu^2)) \right)_{-\alpha} \\
&= (\psi_1 \cdot h + \psi \cdot (2\alpha + 1)g)_{-\alpha} \\
&= 0
\end{aligned} \tag{17}$$

with applying (12) and (15) or (16).

3 Familiar forms of Solutions of the extended homogeneous Chebyshev's equation

Applying Lemma II, in case of $\alpha = \nu$ we have

$$\begin{aligned}
\varphi &= \left(((z-b)^2 - (b^2 + 1))^{-\nu + \frac{1}{2}} \right)_{-(1+\nu)} = e^{-i\pi(-1-\nu)}(z-b)^{-(\nu + \frac{1}{2}) \cdot 2 + (1+\nu)} \\
&\times \sum_{k=0}^{\infty} \frac{[\nu + \frac{1}{2}]_k \Gamma(2k + (\nu + \frac{1}{2}) \cdot 2 - (1 + \nu))}{k! \Gamma(2k + (\nu + \frac{1}{2}) \cdot 2)} \left(\frac{(b^2 + 1)}{(z-b)^2} \right)^k \\
&= e^{i\pi(1+\nu)}(z-b)^{-\nu} \sum_{k=0}^{\infty} \frac{[\nu + \frac{1}{2}]_k \Gamma(2k + \nu)}{k! \Gamma(2k + 2\nu + 1)} \left(\frac{(b^2 + 1)}{(z-b)^2} \right)^k \\
&= -e^{i\pi\nu}(z-b)^{-\nu} \sum_{k=0}^{\infty} \frac{[\frac{\nu}{2}]_k [\frac{\nu}{2} + \frac{1}{2}]_k \Gamma(\nu)}{k! [\nu + 1]_k \Gamma(2\nu + 1)} \left(\frac{(b^2 + 1)}{(z-b)^2} \right)^k
\end{aligned}$$

Now we notice following relations.

$$\Gamma(2k + \nu) = [\nu]_{2k} \Gamma(\nu)$$

$$\Gamma(2k + 2\nu + 1) = [2\nu + 1]_{2k} \Gamma(2\nu + 1)$$

$$[\nu]_{2k} = 2^{2k} \left[\frac{\nu}{2} \right]_k \left[\frac{\nu}{2} + \frac{1}{2} \right]_k$$

$$[2\nu + 1]_{2k} = 2^{2k} \left[\nu + \frac{1}{2} \right]_k [\nu + 1]_k$$

So we can get

$$\begin{aligned}
\varphi &= -e^{i\pi\nu} \frac{\Gamma(\nu)}{\Gamma(2\nu + 1)} (z-b)^{-\nu} \sum_{k=0}^{\infty} \frac{[\frac{\nu}{2}]_k [\frac{\nu}{2} + \frac{1}{2}]_k}{k! [\nu + 1]_k} \left(\frac{(b^2 + 1)}{(z-b)^2} \right)^k \\
&= -e^{i\pi\nu} \frac{\Gamma(\nu)}{\Gamma(2\nu + 1)} (z-b)^{-\nu} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu}{2} + \frac{1}{2}; \nu + 1; \frac{b^2 + 1}{(z-b)^2}\right). \tag{1}
\end{aligned}$$

Here ${}_2F_1(\dots)$ denote the Gauss's hypergeometric function.

In the case of $\alpha = -\nu$, we have the following form in according to the same way of the case $\alpha = \nu$,

$$\begin{aligned}\varphi &= \left(((z-b)^2 - (b^2 + 1))^{\nu - \frac{1}{2}} \right)_{\nu-1} \\ &= -e^{-i\pi\nu} \frac{\Gamma(-\nu)}{\Gamma(1-2\nu)} (z-b)^\nu \sum_{k=0}^{\infty} \frac{[-\frac{\nu}{2}]_k [-\frac{\nu}{2} + \frac{1}{2}]_k}{k! [-\nu + 1]_k} \left(\frac{(b^2 + 1)}{(z-b)^2} \right)^k \\ &= -e^{-i\pi\nu} \frac{\Gamma(-\nu)}{\Gamma(1-2\nu)} (z-b)^\nu {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2} + \frac{1}{2}; -\nu + 1; \frac{b^2 + 1}{(z-b)^2}\right). \quad (2)\end{aligned}$$

4 Illustrative Example

We show some examples.

(i) We consider the case of $\nu = \frac{1}{2}$ and $b = \frac{1}{2}$. The equation is

$$\varphi_2 \cdot (z^2 - z - 1) + \varphi_1 \cdot (z - \frac{1}{2}) - \varphi \cdot (\frac{1}{2})^2 = 0.$$

Operating N^α to the equation, we have

$$\varphi_{2+\alpha} \cdot (z^2 - z - 1) + \varphi_{1+\alpha} \cdot (2\alpha + 1) \cdot (z - \frac{1}{2}) + \varphi_\alpha \cdot (\alpha^2 - (\frac{1}{2})^2) = 0$$

We adopt $\alpha = 1/2$, then

$$\varphi_{2+\frac{1}{2}} \cdot (z^2 - z - 1) + \varphi_{1+\frac{1}{2}} \cdot 2 \cdot (z - \frac{1}{2}) = 0.$$

Setting $\psi = \psi(z) = \varphi_{1+\frac{1}{2}}$, we have the following equation

$$\psi_1 \cdot (z^2 - z - 1) + \psi \cdot (2)(z - \frac{1}{2}) = 0.$$

and the solution

$$\psi = (z^2 - z - 1)^{-1}.$$

Therefore we have the solution as follows,

$$\begin{aligned}\varphi &= (((z^2 - z - 1)^{-1})_{-(1+\frac{1}{2})}) \\ &= (((z - \frac{1}{2})^2 - ((\frac{1}{2})^2 + 1))^{-1})_{-(1+\frac{1}{2})}\end{aligned}$$

With applying Lemma II, we have

$$\begin{aligned}\varphi &= \left(((z - \frac{1}{2})^2 - (\frac{5}{4}))^{-1} \right)_{-(\frac{3}{2})} \\ &= e^{i\pi(\frac{3}{2})} (z - \frac{1}{2})^{-\frac{1}{2}} \times \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(2k + \frac{1}{2})}{k! \Gamma(2k + 2)} \left(\frac{\frac{5}{4}}{(z - \frac{1}{2})^2} \right)^k\end{aligned}$$

Here we notice the following relations.

$$\Gamma(2k + \frac{1}{2}) = [\frac{1}{2}]_{2k} \Gamma(\frac{1}{2}) = [\frac{1}{2}]_{2k} \sqrt{\pi}$$

$$\Gamma(2k + 2) = [2]_{2k} \Gamma(2) = [2]_{2k}$$

$$[1]_k = 1 \cdot 2 \cdots (1 + k - 1) = k!$$

Furthermore,

$$[\frac{1}{2}]_{2k} = 2^{2k} [\frac{1}{4}]_k [\frac{1}{4} + \frac{1}{2}]_k$$

$$[2]_{2k} = 2^{2k} [1]_k [\frac{3}{2}]_k = 2^{2k} k! [\frac{3}{2}]_k$$

At last we obtain

$$\begin{aligned} \varphi &= e^{i\pi(\frac{3}{2})} \sqrt{\pi} (z - \frac{1}{2})^{-\frac{1}{2}} \times \sum_{k=0}^{\infty} \frac{[\frac{1}{4}]_k [\frac{3}{4}]_k}{k! [\frac{3}{2}]_k} \left(\frac{\frac{5}{4}}{(z - \frac{1}{2})^2} \right)^k \\ &= e^{i\pi(\frac{3}{2})} \sqrt{\pi} (z - \frac{1}{2})^{-\frac{1}{2}} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}; \frac{\frac{5}{4}}{(z - \frac{1}{2})^2}\right) \end{aligned}$$

(ii) We consider the case of $\nu = \frac{1}{3}$ and $b = \frac{1}{2}$. Then the equation is

$$\varphi_2 \cdot (z^2 - z - 1) + \varphi_1 \cdot (z - \frac{1}{2}) - \varphi \cdot (\frac{1}{3})^2 = 0.$$

Operating N^α to this equation, we have

$$\varphi_{2+\alpha} \cdot (z^2 - z - 1) + \varphi_{1+\alpha} \cdot (2\alpha + 1) \cdot (z - \frac{1}{2}) + \varphi_\alpha \cdot (\alpha^2 - (\frac{1}{3})^2) = 0$$

We adopt $\alpha = 1/3$, then

$$\varphi_{2+\frac{1}{3}} \cdot (z^2 - z - 1) + \varphi_{1+\frac{1}{3}} \cdot 2 \cdot (z - \frac{1}{2}) = 0.$$

Setting $\psi = \psi(z) = \varphi_{1+\frac{1}{3}}$, we have the following equation

$$\psi_1 \cdot (z^2 - z - 1) + \psi \cdot (\frac{5}{3})(z - \frac{1}{2}) = 0.$$

and the solution

$$\psi = (z^2 - z - 1)^{-\frac{5}{6}}.$$

Therefore we have the solution as follows,

$$\begin{aligned} \varphi &= (((z^2 - z - 1)^{-\frac{5}{6}})_{-(1+\frac{1}{3})}) \\ &= (((z - \frac{1}{2})^2 - ((\frac{1}{2})^2 + 1))^{-\frac{5}{6}})_{-(1+\frac{1}{3})} \end{aligned}$$

Applying Lemma II,

$$\begin{aligned}\varphi &= \left(\left(\left(z - \frac{1}{2} \right)^2 - \left(\frac{5}{4} \right) \right)^{-\frac{5}{6}} \right)_{-(\frac{4}{3})} \\ &= e^{i\pi(\frac{4}{3})} \left(z - \frac{1}{2} \right)^{-\frac{1}{3}} \times \sum_{k=0}^{\infty} \frac{[\frac{5}{6}]_k \Gamma(2k + \frac{1}{3})}{k! \Gamma(2k + \frac{5}{3})} \left(\frac{\frac{5}{4}}{(z - \frac{1}{2})^2} \right)^k.\end{aligned}$$

We notice the following relations.

$$\begin{aligned}\Gamma(2k + \frac{1}{3}) &= [\frac{1}{3}]_{2k} \Gamma(\frac{1}{3}) = 2^{2k} [\frac{1}{6}]_k [\frac{2}{3}]_k \Gamma(\frac{1}{3}) \\ \Gamma(2k + \frac{5}{3}) &= [\frac{5}{3}]_{2k} \Gamma(\frac{5}{3}) = 2^{2k} [\frac{5}{6}]_k [\frac{4}{3}]_k \Gamma(\frac{5}{3})\end{aligned}$$

So we have

$$\begin{aligned}\varphi &= -e^{i\pi(\frac{1}{3})} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})} \times \sum_{k=0}^{\infty} \frac{[\frac{1}{6}]_k [\frac{2}{3}]_k}{k! [\frac{4}{3}]_k} \left(\frac{\frac{5}{4}}{(z - \frac{1}{2})^2} \right)^k \\ \times &= -e^{i\pi(\frac{1}{3})} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})} \left(z - \frac{1}{2} \right)^{-\frac{1}{3}} {}_2F_1 \left(\frac{1}{6}, \frac{2}{3}; \frac{4}{3}; \frac{\frac{5}{4}}{(z - \frac{1}{2})^2} \right).\end{aligned}$$

We can conclude that the familiar form of the solution can write as

$$\varphi(z) = -e^{i\pi\nu} \frac{\Gamma(\nu)}{\Gamma(2\nu + 1)} (z - b)^{-\nu} {}_2F_1 \left(\frac{\nu}{2}, \frac{\nu}{2} + \frac{1}{2}; \nu + 1; \frac{b^2 + 1}{(z - b)^2} \right)$$

where $|(b^2 + 1)/(z - b)^2| < 1$, including the case of ν be integer formally.

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