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Applications of Phase Plane Analysis of a Liénard System to Positive Solutions of Schrödinger Equations

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1. INTRODUCTION

We consider the semilinear elliptic equation

$$\Delta u + f(x,u) = 0, \qquad x \in \Omega, \tag{1}$$

where Ω is an exterior domain of $\mathbb{R}^N$ with $N \geq 3$, that is, $G_a = \{x \in \mathbb{R}^N\colon |x| > a\} \subset \Omega$ for some $a > 0$. Throughout this paper, we assume that $f(x,u)$ is nonnegative and locally Hölder continuous with exponent $\alpha \in (0,1)$ in $\overline{M} \times \overline{J}$ for every bounded domain $M \subset \Omega$ and for every bounded interval $J \subset \mathbb{R}$.

It is very famous that de Broglie's wave function

$$\psi(x,t) = \exp\left(-\frac{iEt}{\hbar}\right) v(x)$$

is a solution of the Schrödinger equation for a free particle of mass m, momentum p and kinetic energy E:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi,$$

where $\hbar = h/2\pi$ (h is Planck's constant) and

$$v(x) = A\exp\left(\frac{i(p\cdot x)}{\hbar}\right).$$

This equation is generalized into the Schrödinger equation with the potential V and the nonlinearity

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - g(x,|\psi|)\psi.$$

If it has standing waves solutions of the form

$$\psi(x,t) = \exp\left(-\frac{iEt}{\hbar}\right) u(x),$$

then the function $u(x)$ must satisfy the elliptic equation

$$\Delta u + \frac{2m}{\hbar^2}(E - V(x))u + g(x,|u|)u = 0,$$

which is of the form (1). In quantum mechanics, such are called stationary Schrödinger equations.

The aim of this paper is to give sufficient conditions under which equation (1) has a positive solution in an exterior domain of $\mathbb{R}^N$.

For a bounded domain $M \subset \Omega$, let $C^{2+\alpha}(\overline{M})$ denote the usual Hölder space. For simplicity, $C^{2+\alpha}_{\mathrm{loc}}(\Omega)$ is defined as the set of all functions $u \colon \Omega \to \mathbb{R}$ such that $u \in C^{2+\alpha}_{\mathrm{loc}}(\overline{M})$ for every bounded domain $M \subset \Omega$. A function $u \in C^{2+\alpha}_{\mathrm{loc}}(\Omega)$ is called a *solution* of (1) in Ω if it satisfies equation (1) at every point $x \in \Omega$. Similarly, a function $u \in C^{2+\alpha}_{\mathrm{loc}}(\Omega)$ is called a *supersolution* (resp., *subsolution*) of (1) in Ω if it satisfies the inequality $\Delta u + f(x,u) \leq 0$ (resp., ≥ 0) at every point $x \in \Omega$.

A typical example of (1) is the Emden-Fowler equation

$$\Delta u + p(x)u^{\gamma} = 0, \qquad x \in \Omega,$$

where $p(x)$ is nonnegative and locally Hölder continuous in Ω and γ is a positive number. From this fact, equation (1) is often discussed under a sublinear or a superlinear hypothesis. For instance, equation (1) is said to be *sublinear* (resp., *superlinear*) if there exists a γ with $0 < \gamma < 1$ (resp., $\gamma > 1$) such that $f(x,u)/u^{\gamma}$ is nonincreasing (resp., nondecreasing) in u for each fixed $r = |x| > 0$.

Many studies have been made on the existence of a positive solution of (1) in the linear case, the sublinear case and the superlinear case (see [2, 4, 5, 6, 7]). In this paper, we intend to examine another case in addition to these cases. For example, consider the case that

$$f(x,u) = p(x)\left(u + \frac{u}{4(\log u)^2}\right) \tag{2}$$

for all sufficiently small u. Then equation (1) is neither sublinear nor superlinear (of course, equation (1) is not linear). In fact, differentiating $f(x,u)/u^{\gamma}$, we have

$$\frac{d}{du}\left(\frac{f(x,u)}{u^{\gamma}}\right) = \frac{p(x)}{u^{\gamma}}\left\{(1-\beta) + \frac{1-\beta-2/\log u}{4(\log u)^2}\right\}.$$

Hence, if $0 < \gamma < 1$ (resp., $\gamma > 1$), then $f(x,u)/u^{\gamma}$ is increasing (resp., decreasing) for $u > 0$ sufficiently small. In the case (2), for any $k > 1$, there exists a positive interval I such that

$$p(x)u < f(x,u) < kp(x)u$$

for all $x \in \Omega$ and $u \in I$. Hence, from this point of view, we may say that equation (1) is *almost linear* in such cases as (2).

For sublinear Schrödinger equations, Swanson [7, Theorem 2.4] gave the following sufficient condition for the existence of a positive solution under the assumption that

$$0 \leq f(x,u) \leq u\varphi(|x|,u) \tag{3}$$

for all $x \in \Omega$ and $u > 0$, where $\varphi(r,u)$ is locally Hölder continuous with exponent $\alpha \in (0,1)$ and nonincreasing in u for each fixed $r > 0$.

Theorem A. *Under the assumption* (3), *equation* (1) *has a positive solution in an exterior domain if*

$$\int^{\infty} r\varphi(r,c)\,dr < \infty \tag{4}$$

for some $c > 0$.

Consider the case that $f(x,u) = u/4|x|^\beta$ with $\beta \geq 2$. Then assumption (3) is satisfied with $\varphi(r,u) = 1/4r^\beta$. Since

$$\int^\infty r\varphi(r,c)dr = \int^\infty \frac{1}{4r^{\beta-1}}dr,$$

for any $c > 0$, condition (4) is satisfied if $\beta > 2$, but it does not hold if $\beta = 2$. Hence, Theorem A is inapplicable to the case $\beta = 2$. However, the equation

$$\Delta u + \frac{u}{4|x|^2} = 0$$

has a positive solution, because its radial solutions are represented as the form of

$$u(x) = \begin{cases} (K_1 + K_2 \log|x|)|x|^{-1/2} & \text{if } N = 3, \\ K_3|x|^z + K_4|x|^{2-N-z} & \text{if } N \geq 4, \end{cases}$$

where K_i $(i = 1,2,3,4)$ are arbitrary constants and z is the root of $z^2+(N-2)z+1/4 = 0$.

Assumption (3) is not compatible with the superlinear case and the almost linear case. Hence, instead of (3), we assume that

$$0 \leq f(x,u) \leq \frac{h(u)}{|x|^2} \tag{5}$$

for all $x \in \Omega$ and $u \geq 0$, where $h(u)$ is locally Lipschitz continuous and positive for $u > 0$, and $h(0) = 0$. We also prepare the following notation to present a theorem which can be applied to these cases. Write

$$L_1(u) = 1, \qquad L_{n+1}(u) = L_n(u)\, l_n(u), \qquad n = 1,2,\cdots,$$

where

$$l_1(u) = 2|\log u|, \qquad l_{n+1}(u) = \log\{l_n(u)\},$$

and set

$$S_n(u) = \sum_{k=1}^{n} \frac{1}{\{L_k(u)\}^2}.$$

Define $e_0 = 1$ and $e_n = \exp(e_{n-1})$. Then we have

$$l_{n+1}(u) = \log\{l_n(u)\} > 0 \quad \text{for } 0 < u < 1/\sqrt{e_n},$$

and therefore, the function sequences $\{L_n(u)\}$, $\{l_n(u)\}$ and $\{S_n(u)\}$ are well-defined for $u > 0$ sufficiently small. To take some concrete forms of $S_n(u)$, for $u > 0$ sufficiently small,

$$S_1(u) = 1, \quad S_2(u) = 1 + \frac{1}{4(\log u)^2}, \quad S_3(u) = 1 + \frac{1}{4(\log u)^2} + \frac{1}{4(\log u)^2\,(\log(2|\log u|))^2},$$

and so on.

Our main result is stated in the following:

Theorem 1. *Assume* (5) *and suppose that there exists a positive integer n such that*

$$\frac{h(u)}{u} \leq \frac{(N-2)^2}{4} S_n(u) \tag{6}$$

for all $u > 0$ sufficiently small. Then equation (1) *has a positive solution $u(x)$ in an exterior domain with $\lim_{|x|\to\infty} u(x) = 0$.*

2. A SUPERSOLUTION AND A SUBSOLUTION

We will prove the main result by use of the so-called "supersolution-subsolution" method. The lemma below yields from a result of Noussair and Swanson [5, Theorem 3.3].

Lemma 2. *If there exists a positive supersolution $\overline{u}$ of* (1) *and a positive subsolution $\underline{u}$ of* (1) *in G_b such that $\underline{u}(x) \leq \overline{u}(x)$ for all $x \in G_b \cup C_b$, where $b \geq a$* and *$C_b = \{x \in \mathbb{R}^N$: $|x| = b\}$, then equation* (1) *has at least one solution u satisfying $u(x) = \overline{u}(x)$ on C_b and $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ through G_b.*

To apply Lemma 2, we have to find a suitable positive supersolution of (1) and a positive subsolution of (1) which is not greater than the supersolution. For this purpose, we consider the nonlinear differential equation

$$\frac{d^2}{dr^2}w + \frac{N-1}{r}\frac{d}{dr}w + \frac{1}{r^2}g(w) = 0, \qquad r > a, \tag{7}$$

where $g(w)$ satisfies a suitable smoothness condition for the uniqueness of solutions of the initial value problem and the signum condition

$$wg(w) > 0 \quad \text{if } w \neq 0. \tag{8}$$

Then we have the following nonoscillation theorem for equation (7).

Lemma 3. *Assume* (8). *If there exists a positive integer n such that*

$$\frac{g(w)}{w} \leq \frac{(N-2)^2}{4} S_n(|w|) \tag{9}$$

for $w > 0$ or $w < 0$, $|w|$ sufficiently small, then all nontrivial solutions of (7) *are nonoscillatory.*

Proof. Using phase plane analysis of a Liénard system, Sugie *et al.* [10, Lemma 3.2] proved that under the assumption (8), all nontrivial solutions of the equation

$$\frac{d^2}{dr^2}w + \frac{2}{r}\frac{d}{dr}w + \frac{1}{r^2}g(w) = 0 \tag{10}$$

are nonoscillatory if

$$\frac{g(w)}{w} \leq \frac{1}{4} S_n(|w|) \tag{11}$$

for $w > 0$ or $w < 0$, $|w|$ sufficiently small. Hence, the lemma is true for $N = 3$.

Suppose that $N \geq 4$. Let

$$\tau = (N-2)r^{N-2} \qquad \text{and} \qquad v(\tau) = w(r).$$

Then equation (7) becomes

$$\frac{d^2}{d\tau^2}v + \frac{2}{\tau}\frac{d}{d\tau}v + \frac{1}{\tau^2}g^*(v) = 0,$$

where $g^*(v) = g(v)/(N-2)^2$. This equation has the form of (10). It follows from (9) that

$$\frac{g^*(w)}{w} = \frac{g(w)}{(N-2)^2 w} \leq \frac{1}{4}S_n(|w|)$$

for $w > 0$ or $w < 0$, $|w|$ sufficiently small, that is, (11) is satisfied with $g(w) = g^*(w)$. Hence, by Lemma 3.2 in [10] again, we see that all nontrivial solutions of (7) are nonoscillatory in the case $N \geq 4$. □

By virtue of Lemma 3, we can choose a solution of (7) which is eventually positive. In the next section, we will show that the positive solution is a supersolution of (1). To get a positive subsolution of (1), we need to estimate the asymptotic behavior of positive solutions of (7) as follows.

Lemma 4. *Assume* (8) *and* (9). *Then there exist a positive number* $b \geq a$ *and a positive solution* $w(r)$ *of* (7) *such that* $\lim_{r\to\infty} w(r) = 0$

$$b^{N-2}w(b) \leq r^{N-2}w(r) \quad \text{for } r \geq b.$$

Proof. From Lemma 3 we see that equation (7) has a positive solution. Let $w(r)$ be the positive solution. Then there exists a $b \geq a$ such that

$$w(r) > 0 \quad \text{for } r \geq b.$$

The change of variables $r = e^s$ and $w(r) = \xi(s)$ transforms equation (7) into the Liénard system

$$\begin{aligned} \frac{d}{ds}\xi &= \eta - (N-2)\xi, \\ \frac{d}{ds}\eta &= -g(\xi). \end{aligned} \tag{12}$$

Let $(\xi(s), \eta(s))$ be the solution of (12) corresponding to $w(r)$. Then we have

$$\xi(s) > 0 \quad \text{for } s \geq \log b. \tag{13}$$

By (8) we obtain

$$\frac{d}{ds}\eta(s) < 0 \quad \text{for } s \geq \log b. \tag{14}$$

It is well known that the zero solution of (12) is globally asymptotically stable (for example, see [1, 3, 8]). Hence, we conclude that the solution $(\xi(s), \eta(s))$ tends to the origin as $s \to \infty$. This means that $w(r)$ approaches the zero as $r \to \infty$.

We will show that $\eta(s) \geq 0$ for $s \geq \log b$. Suppose that $\eta(s_0) < 0$ for some $s_0 \geq \log b$. Then, by (12)–(14) we have

$$\frac{d}{ds}\xi(s) < \frac{d}{ds}\xi(s) + (N-2)\xi(s) = \eta(s) \leq \eta(s_0)$$

for $s \geq s_0$. Integrate this inequality from s_0 to s to obtain

$$\xi(s) < \xi(s_0) + \eta(s_0)(s - s_0) \to -\infty \quad \text{as } s \to \infty.$$

This is a contradiction to (13).

Since $\eta(s) \geq 0$ for $s \geq \log b$, we see that

$$\frac{d}{ds}\xi(s) \geq -(N-2)\xi(s) \quad \text{for } s \geq \log b.$$

Hence, integrating the both sides, we have

$$b^{N-2}\xi(\log b) \leq e^{(N-2)s}\xi(s) \quad \text{for } s \geq \log b,$$

namely, $b^{N-2}w(b) \leq r^{N-2}w(r)$ for $r \geq b$. Thus, the lemma is proved. □

We are now ready to prove the main theorem.

3. PROOF OF THE MAIN THEOREM

Consider the nonlinear differential equation

$$\frac{d^2}{dr^2}w + \frac{N-1}{r}\frac{d}{dr}w + \frac{1}{r^2}h^*(w) = 0, \qquad r \geq a, \tag{15}$$

where a is the number given in (1) and

$$h^*(w) = \begin{cases} h(w) & \text{for } w \geq 0, \\ -h(-w) & \text{for } w < 0. \end{cases}$$

Then, from assumption (5) we see that $h^*(w)$ satisfies the signum condition (8), and therefore, equation (15) is in the type of (7). Also, by condition (6) we have

$$\frac{h^*(w)}{w} \leq \frac{1}{4}S_n(|w|)$$

for $w > 0$ and $w < 0$, $|w|$ sufficiently small. Hence, from Lemma 3 we conclude that all nontrivial solutions of (15) are nonoscillatory. For this reason, we can choose a solution $w(r)$ which is positive for all $r \geq b$ with some $b \geq a$ (we may regard b as the positive number in Lemma 4). As in the proof of Lemma 4, we can show that $w(r)$ approaches the zero as r tends to ∞. Note that $w(r)$ is also a positive solution of the equation

$$\frac{d^2}{dr^2}w + \frac{N-1}{r}\frac{d}{dr}w + \frac{1}{r^2}h(w) = 0.$$

Let $\overline{u}$ be the function defined in G_b by $\overline{u}(x) = w(r)$, $r = |x| \geq b$. Then, by assumption (5) we obtain

$$\begin{aligned}
\Delta\overline{u}(x) + f(x, \overline{u}(x)) &= \frac{d^2}{dr^2}w(r) + \frac{N-1}{r}\frac{d}{dr}w(r) + f(x, w(r)) \\
&\leq \frac{d^2}{dr^2}w(r) + \frac{N-1}{r}\frac{d}{dr}w(r) + \frac{1}{|x|^2}h(w(r)) \\
&= \frac{d^2}{dr^2}w(r) + \frac{N-1}{r}\frac{d}{dr}w(r) + \frac{1}{r^2}h(w(r)) = 0
\end{aligned}$$

Hence, $\overline{u}$ is a supersolution of (1) in G_b. We next denote $\underline{u}(x) = b^{N-2}w(b)/|x|^{N-2}$ for $|x| \geq b$. Then, since $f(x, u)$ is nonnegative, we get

$$\begin{aligned}
\Delta\underline{u}(x) + f(x, \underline{u}(x)) &\geq \frac{d^2}{dr^2}\left(\frac{b^{N-2}w(b)}{r^{N-2}}\right) + \frac{N-1}{r}\frac{d}{dr}\left(\frac{b^{N-2}w(b)}{r^{N-2}}\right) \\
&= \frac{(N-2)(N-1)b^{N-2}w(b)}{r^N} - \frac{N-1}{r}\frac{(N-2)b^{N-2}w(b)}{r^{N-1}} = 0.
\end{aligned}$$

This means that $\underline{u}(x)$ is a subsolution of (1) in G_b.

From Lemma 4 we see that

$$\underline{u}(x) = \frac{b^{N-2}w(b)}{|x|^{N-2}} = \frac{b^{N-2}w(b)}{r^{N-2}} \leq w(r) = \overline{u}(x)$$

for $|x| \geq b$. Hence, by means of Lemma 2, we conclude that there exists a positive solution $u(x)$ of (1) satisfying $\underline{u}(x) = u(x) = \overline{u}(x)$ on C_b and $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ through G_b. Since $w(r)$ tends to the zero as $r \to \infty$, the positive solution $u(x)$ also tends to the zero as $|x| \to \infty$. This completes the proof. □

4. DISCUSSION

To illustrate the main theorem, we will give some examples which are the almost linear case. One cannot apply previous results on the existence of a positive solution to those examples. For brevity, we define the function $\phi(u; \lambda)$ by $\phi(0; \lambda) = 0$ for any $\lambda \geq 0$ and

$$\phi(u; \lambda) = \begin{cases} u + \dfrac{\lambda u}{(\log|u|)^2} & \text{for } 0 < u \leq \dfrac{1}{e}, \\ (3\lambda + 1)u - \dfrac{2\lambda}{e} & \text{for } u > \dfrac{1}{e}. \end{cases}$$

Then it is easy to check that $\phi(u; \lambda)$ is continuous for $u \geq 0$ and is continuously differentiable for $u > 0$.

We first consider the elliptic equation

$$\Delta u + p(x)\phi(u; 1/4) = 0 \tag{16}$$

in an exterior domain Ω of $\mathbb{R}^N$ with $N \geq 3$. Let

$$f(x,u) = p(x)\phi(u; 1/4) \qquad \text{and} \qquad h(u) = \frac{(N-2)^2}{4}\phi(u; 1/4).$$

Then condition (5) holds and condition (6) is satisfied with $n = 2$. Hence, as a direct consequence of Theorem 1, we have the following result.

Example 5. If

$$0 \leq p(x) \leq \frac{(N-2)^2}{4|x|^2}$$

for $x \in \Omega$, then equation (16) has a decaying positive solution.

Let us take another example to show how sharp Theorem 1 is. For this purpose, we restrict $p(x)/|x|^2$ to any constant.

Example 6. Consider the equation with two parameters

$$\Delta u + \frac{\mu}{|x|^2}\phi(u; \lambda) = 0 \tag{17}$$

instead of (16). Then, from Theorem 1 we have the following conclusions:

(i) if $0 \leq \mu < (N-2)^2/4$, then equation (17) has a decaying positive solution for all $\lambda \geq 0$;

(ii) if $\mu = (N-2)^2/4$, then equation (17) has a decaying positive solution for $0 \leq \lambda \leq 1/4$.

Proof. Let $f(x,u) = \mu\phi(u;\lambda)/|x|^2$ and $h(u) = \mu\phi(u;\lambda)$. Since λ and μ are nonnegative, condition (5) is satisfied. Hence, it is enough to check that condition (6) holds for $u > 0$ sufficiently small. If $\lambda = 0$, then $h(u)/u = \mu \leq (N-2)^2/4$ for all $u > 0$, that is, condition (6) is satisfied with $n = 1$. We assume that λ is positive.

(i) We can choose an $\varepsilon_0 > 0$ so small that $\mu(1+\varepsilon_0) < (N-2)^2/4$. For any $\lambda > 0$, we see that

$$\frac{h(u)}{u} = \mu\left(1 + \frac{\lambda}{(\log u)^2}\right) < \mu(1+\varepsilon_0) < \frac{(N-2)^2}{4}$$

for $0 < u < \exp(-\sqrt{\lambda/\varepsilon_0})$. Hence, condition (6) is satisfied with $n = 1$.

(ii) In this case, we have

$$\frac{h(u)}{u} = \mu\left(1 + \frac{\lambda}{(\log u)^2}\right) \leq \frac{(N-2)^2}{4}\left(1 + \frac{1}{4(\log u)^2}\right)$$

for u sufficiently small, namely, condition (6) with $n = 2$. □

Recently, by use of phase plane analysis of a Liénard system, Sugie *et al.* [9, Lemma 4.4] gave an oscillation theorem for equation (10) under the assumption (8) as follows.

Theorem B. *Assume* (8) *and suppose that there exists a* λ *with* $\lambda > 1/4$ *satisfying*

$$\frac{g(w)}{w} \geq \frac{1}{4} + \frac{\lambda}{(2\log|w|)^2} \tag{18}$$

for $|w|$ *sufficiently small. Then all nontrivial solutions of* (10) *are oscillatory.*

To compare with the conclusion (ii) of Example 6, we consider the equation

$$\Delta u + \frac{(N-2)^2}{4|x|^2}\phi^*(u;\lambda) = 0, \tag{19}$$

where

$$\phi^*(u;\lambda) = \begin{cases} \phi(u;\lambda) & \text{for } u \geq 0, \\ -\phi(-u;\lambda) & \text{for } u < 0. \end{cases}$$

It is clear that $\phi^*(u;\lambda)$ is odd, and therefore, it satisfies the signum condition (8). As shown in Sections 2 and 3, the change of variables

$$v(\tau) = w(r) = u(x), \qquad r = |x| \qquad \text{and} \qquad \tau = (N-2)r^{N-2}$$

reduces equation (19) to

$$\frac{d^2}{d\tau^2}v + \frac{2}{\tau}\frac{d}{d\tau}v + \frac{1}{4\tau^2}\phi^*(v;\lambda) = 0.$$

This is of the form (10). Since

$$\frac{\phi^*(v;\lambda)}{4v} = \frac{1}{4} + \frac{\lambda}{(2\log|v|)^2}$$

for $|v|$ sufficiently small, from Theorem B it turns out that if $\lambda > 1/4$, then equation (19) fails to have positive radial solutions. Hence, together with the second conclusion in Example 6, we see that equation (19) has a positive radial solution if and only if $\lambda \leq 1/4$.

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