

Continuous and Discrete Fourier Coefficients of Equi-distant Piecewise Linear Continuous Periodic Functions

- Application to Mathematical Analysis of An FEM-CSM Combined Method for 2D Exterior Laplace Problems -

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Abstract

The author has investigated an FEM-CSM combined method for 2D exterior Laplace problems during these years ([2], [3]). Here the abbreviation of CSM is employed for the charge simulation method (See [1]). In the mathematical analysis for the method, especially in the proof of an a priori error estimate for the approximate solutions obtained by the method, a relation between continuous and discrete Fourier coefficients of equi-distant piecewise linear continuous 2π -periodic function plays a key role. In this paper, the relation is introduced with illustrative examples of application to the mathematical analysis mentioned above.

1. Relation between continuous and discrete Fourier coefficients for equi-distant piecewise linear continuous 2π -periodic functions

Let $f(\theta)$ be a complex valued continuous 2π -periodic function of θ . For $n \in \mathbf{Z}$, a continuous Fourier coefficient f_n of the function $f(\theta)$ is defined through

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta.$$

Fix a positive integer N . Set

$$\theta_1 = \frac{2\pi}{N}, \quad \theta_j = j\theta_1 \quad \text{for } j \in \mathbf{Z}.$$

For $n \in \mathbf{Z}$, a discrete Fourier coefficient $f_n^{(N)}$ of the function $f(\theta)$ is defined through

$$f_n^{(N)} = \frac{1}{N} \sum_{j=0}^{N-1} f(\theta_j) e^{-in\theta_j}.$$

It is to be noted that we have for any continuous 2π -periodic function $f(\theta)$,

$$(1) \quad f_{n+Nr}^{(N)} = f_n^{(N)}, \quad n \in \mathbf{Z}, \quad r \in \mathbf{Z} - \{0\}.$$

Let $\hat{w}(\theta)$ be the reference roof function defined through

$$\hat{w}(\theta) = \begin{cases} 1 - |\theta| & : |\theta| \leq 1, \\ 0 & : |\theta| \geq 1. \end{cases}$$

For any $j \in \mathbf{Z}$, define a piecewise linear basis function $w_j^{(N)}(\theta)$ through the following formula:

$$w_j^{(N)}(\theta) = \hat{w}\left(\frac{\theta - \theta_j}{\theta_1}\right), \quad -\infty < \theta < \infty.$$

A complex valued function $f(\theta)$ is said to be an **equi-distant piecewise linear continuous 2π -periodic function** (with N nodal points) in this paper if $f(\theta)$ is represented as

$$f(\theta) = \sum_{j=0}^N f(\theta_j) w_j^{(N)}(\theta), \quad 0 \leq \theta \leq 2\pi,$$

with

$$f(2\pi) = f(0).$$

Introduce a function $\alpha(\theta)$ through the formula:

$$\alpha(\theta) = \frac{2(1 - \cos \theta)}{\theta^2} \quad \text{for } \theta \neq 0, \quad \text{with } \alpha(0) = 1.$$

Theorem 1 *We have the following relation for any equi-distant piecewise linear continuous 2π -periodic function (with N nodal points) $f(\theta)$,*

$$(2) \quad f_n = \alpha(\theta_n) f_n^{(N)}, \quad n \in \mathbf{Z}.$$

Proof A straightforward calculus leads the relation. $\square$

Corollary *We have the following identity for any equi-distant piecewise linear continuous 2π -periodic function (with N nodal points) $f(\theta)$,*

$$(3) \quad f_{n+Nr} = \left(\frac{n}{n+Nr} \right)^2 f_n, \quad n \in \mathbf{Z}, \quad r \in \mathbf{Z} - \{0\}.$$

Proof Since we have

$$\alpha(\theta_{n+Nr}) = \left(\frac{n}{n+Nr} \right)^2 \alpha(\theta_n), \quad n \in \mathbf{Z}, \quad r \in \mathbf{Z} - \{0\},$$

Theorem 1 together with Equality (1) implies Equality (3). $\square$

2. Boundary bilinear forms of Steklov type for exterior Laplace problems and its CSM-approximation forms

Let D_a be the interior of the disc with radius a being the origin as its center, and let Γ_a be the boundary of D_a . Let $\Omega_e = (D_a \cup \Gamma_a)^C$, which is said to be the exterior domain. We use the notation $\mathbf{r} = \mathbf{r}(\theta)$ for the point in the plane corresponding to the complex number $re^{i\theta}$ with $r = |\mathbf{r}|$ where $|\mathbf{r}|$ is the Euclidean norm of $\mathbf{r} \in \mathbf{R}^2$. Similarly we use $\mathbf{a} = \mathbf{a}(\theta)$, and $\vec{\rho} = \vec{\rho}(\theta)$, corresponding to $ae^{i\theta}$ with $a = |\mathbf{a}|$, and $\rho e^{i\theta}$ with $\rho = |\vec{\rho}|$, respectively.

For functions $u(\mathbf{a}(\theta))$ and $v(\mathbf{a}(\theta))$ of $H^{1/2}(\Gamma_a)$, let us introduce the boundary bilinear form of Steklov type for exterior Laplace problem through the following formula:

$$(4) \quad b(u, v) = 2\pi \sum_{n=-\infty}^{\infty} |n| f_n \overline{g_n},$$

where f_n , and g_n , are continuous Fourier coefficients of $u(\mathbf{a}(\theta))$, and $v(\mathbf{a}(\theta))$, respectively.

It is to be noted that the following fact:

If $u(\mathbf{a}(\theta))$ is the boundary value on Γ_a of the function $u(\mathbf{r})$ satisfying the following boundary value problem (E):

$$(E) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega_e, \\ u = \varphi & \text{on } \Gamma_a, \\ \sup_{\Omega_e} |u| < \infty, \end{cases}$$

with

$$\varphi = u(\mathbf{a}(\theta)),$$

then

$$(5) \quad b(u, v) = - \int_{\Gamma_a} \frac{\partial u}{\partial r} v d\Gamma.$$

The CSM approximate form for $b(u, v)$ of the first type, which is denoted by $b^{(N)}(u, v)$, is represented through the following formula (6):

$$(6) \quad b^{(N)}(u, v) = - \int_{\Gamma_a} \frac{\partial u^{(N)}}{\partial r} v^{(N)} d\Gamma,$$

where $u^{(N)}(\mathbf{r})$ is a CSM-approximate solution for $u(\mathbf{r})$ satisfying (E) with $\varphi = u(\mathbf{a}(\theta))$. Namely $u^{(N)}(\mathbf{r})$ is determined through the following problem $(E^{(N)})$:

$$(E^{(N)}) \quad \begin{cases} u^{(N)}(\mathbf{r}) = \sum_{j=0}^{N-1} q_j G_j(\mathbf{r}) + q_N, \\ u^{(N)}(\mathbf{a}_j) = u(\mathbf{a}_j), \quad 0 \leq j \leq N-1, \\ \sum_{j=0}^{N-1} q_j = 0, \end{cases}$$

where

$$\mathbf{a}_j = \mathbf{a}(\theta_j), \quad \vec{\rho}_j = \vec{\rho}(\theta_j) \quad \text{with} \quad 0 < \rho < a,$$

$$G_j(\mathbf{r}) = E(\mathbf{r} - \vec{\rho}_j) - E(\mathbf{r}), \quad E(\mathbf{r}) = -\frac{1}{2\pi} \log r.$$

Problem $(E^{(N)})$ is to find $N + 1$ unknowns q_j , $0 \leq j \leq N$, and it is uniquely solvable for any fixed $\rho \in (0, a)$.

The CSM approximate form for $b(u, v)$ of the second type, which is denoted by $\bar{b}^{(N)}(u, v)$, is represented through the following formula (7):

$$(7) \quad \bar{b}^{(N)}(u, v) = -\frac{2\pi a}{N} \sum_{j=0}^{N-1} \frac{\partial u^{(N)}(\mathbf{a}_j)}{\partial r} v^{(N)}(\mathbf{a}_j),$$

which is the quadrature formula for $b^{(N)}(u, v)$ with the use of trapezoidal rule.

We use the following notations:

$$b(v) = b(v, v)^{1/2}, \quad b^{(N)}(v) = b(v, v)^{1/2}, \quad \bar{b}^{(N)}(v) = \bar{b}^{(N)}(v, v)^{1/2}.$$

Denote the totality of equi-distant piecewise linear continuous 2π -periodic functions (with N nodal points) $v(\mathbf{a}(\theta))$ by V_N :

$$V_N = \{v(\mathbf{a}(\theta)) = \sum_{j=0}^N v(\mathbf{a}_j) w_j^{(N)}(\theta)\}.$$

Let

$$N(\gamma) = \frac{\log 2}{-\log \gamma} \quad \text{with} \quad \gamma = \frac{\rho}{a}.$$

Theorem 2 *We have the following inequalities for any $v \in V_N$.*

$$\frac{1}{4\sqrt{1+2\zeta(3)}} b(v) \leq b^{(N)}(v) \leq \frac{\pi^2}{2} b(v)$$

provided that $N \geq N(\gamma)$, where

$$\zeta(3) = \sum_{r=1}^{\infty} \frac{1}{r^3}.$$

Theorem 3 *For $u, v \in V_N$, we have*

$$|b^{(N)}(u, v) - \bar{b}^{(N)}(u, v)| \leq 8\gamma^{2N} b^{(N)}(u) b^{(N)}(v)$$

provided that $N \geq N(\gamma)$.

3. Proof of Theorem 2

For a fixed positive integer N , introduce sets of integers $\mathcal{N}_r$ through

$$\mathcal{N}_r = \{n : -\frac{N}{2} \leq n - Nr < \frac{N}{2}, \quad n \neq Nr\}$$

with

$$r = 0, \pm 1, \pm 2, \dots.$$

For any integer $n \in [1, N-1]$, define a function $s_n^{(N)}(\gamma)$ of $\gamma \in (0, 1)$, numbers $\Lambda_n^{(N)}$ and $\overline{\Lambda}_n^{(N)}$ as follows.

$$s_n^{(N)}(\gamma) = \int_0^\gamma \frac{x^{n-1} + x^{N-n-1}}{1-x^N} dx,$$

$$\Lambda_n^{(N)} = \frac{s_n^{(N)}(\gamma^2)}{\{s_n^{(N)}(\gamma)\}^2}, \quad \overline{\Lambda}_n^{(N)} = \frac{\gamma \frac{d}{d\gamma} s_n^{(N)}(\gamma)}{s_n^{(N)}(\gamma)}.$$

We admit the validity of the following Proposition 1 without proof.

Proposition 1 *For $u, v \in V_N$, we have*

$$b^{(N)}(u, v) = 2\pi \sum_{n \in \mathcal{N}_0} \Lambda_{|n|}^{(N)} f_n^{(N)} \overline{g_n^{(N)}}$$

and

$$\overline{b}^{(N)}(u, v) = 2\pi \sum_{n \in \mathcal{N}_0} \overline{\Lambda}_{|n|}^{(N)} f_n^{(N)} \overline{g_n^{(N)}},$$

where $f_n^{(N)}$, and $g_n^{(N)}$, are discrete Fourier coefficients of $u(\mathbf{a}(\theta))$, and $v(\mathbf{a}(\theta))$, respectively.

Using the representation of $\Lambda_n^{(N)}$, we obtain

Proposition 2 *If $N \geq N(\gamma)$, then*

$$\frac{n}{16} \leq \Lambda_n^{(N)} \leq 4n, \quad 1 \leq n \leq \frac{N}{2}.$$

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Proposition 3 *It holds*

$$\frac{4}{\pi^2} \leq \alpha(\theta) \leq 1, \quad -\pi \leq \theta \leq \pi.$$

Proposition 4 *For $v \in V_N$, we have*

$$\frac{1}{16} \left\{ 2\pi \sum_{n \in \mathcal{N}_0} |n| |g_n|^2 \right\} \leq b^{(N)}(v, v) \leq \frac{\pi^4}{4} \left\{ 2\pi \sum_{n \in \mathcal{N}_0} |n| |g_n|^2 \right\}$$

provided that $N \geq N(\gamma)$.

Proof Due to Theorem 1 and Proposition 1, we have

$$b^{(N)}(v, v) = 2\pi \sum_{n \in \mathcal{N}_0} \Lambda_{|n|}^{(N)} \frac{1}{|\alpha(\theta_n)|^2} |g_n|^2.$$

Propositions 2 and 3 imply the conclusion of Proposition 4. $\square$

Proposition 5 For $v \in V_N$, we have

$$\left\{ 2\pi \sum_{n \in \mathcal{N}_0} |n| |g_n|^2 \right\} \leq b(v, v) \leq (1 + 2\zeta(3)) \left\{ 2\pi \sum_{n \in \mathcal{N}_0} |n| |g_n|^2 \right\}.$$

Proof Due to Corollary of Theorem 1, we have

$$\frac{1}{2\pi} b(v, v) = \sum_{r \in \mathbf{Z}} \sum_{n \in \mathcal{N}_0} \left| \frac{n}{n + Nr} \right|^3 |n| |g_n|^2.$$

For $r \in \mathbf{Z} - \{0\}$, we have

$$\left| \frac{n}{n + Nr} \right| \leq \frac{1}{|r|}, \quad n \in \mathcal{N}_0.$$

Therefore

$$b(v, v) \leq \left(1 + 2 \sum_{r=1}^{\infty} \frac{1}{r^3} \right) \left\{ 2\pi \sum_{n \in \mathcal{N}_0} |n| |g_n|^2 \right\}.$$

Hence the second inequality of the conclusion is valid, while the first one is trivial by definition of $b(u, v)$. $\square$

Propositions 4 and 5 complete the proof of Theorem 2.

4. Proof of Theorem 3

Proposition 6 For an integer $n \in [1, N-1]$, define B_n through the following formula:

$$B_n = \sum_{p \in \mathbf{Z}} \sum_{q \in \mathbf{Z}} \left\{ \gamma^{|n+Np|} \frac{\gamma^{|n+Nq|}}{|n + Nq|} \right\}.$$

Then we have

$$s_n^{(N)}(\gamma^2) \leq B_n \leq (1 + 8\gamma^2) s_n^{(N)}(\gamma^2)$$

provided that $N \geq N(\gamma)$.

Proof A lengthy but straightforward calculus leads the conclusion. $\square$

Proposition 7 For $N \geq N(\gamma)$, we have

$$\Lambda_n^{(N)} \leq \bar{\Lambda}_n^{(N)} \leq (1 + 8\gamma^2) \Lambda_n^{(N)}.$$

Proof Let

$$\Gamma_n = s_n^{(N)}(\gamma).$$

Then we have

$$\Lambda_n^{(N)} = \frac{s_n^{(N)}(\gamma^2)}{\Gamma_n^2},$$

and

$$\bar{\Lambda}_n^{(N)} = \frac{B_n}{\Gamma_n^2}.$$

Hence Proposition 6 implies the conclusion. $\square$

The proof of Theorem 3 is now straightforward. In fact, we have

$$b^{(N)}(u, v) - \bar{b}^{(N)}(u, v) = 2\pi \sum_{n \in \mathcal{N}_0} (\Lambda_{|n|}^{(N)} - \bar{\Lambda}_{|n|}^{(N)}) f_n^{(N)} \overline{g_n^{(N)}}.$$

Hence it holds

$$|b^{(N)}(u, v) - \bar{b}^{(N)}(u, v)| \leq 2\pi \left\{ \sum_{n \in \mathcal{N}_0} |\Lambda_{|n|}^{(N)} - \bar{\Lambda}_{|n|}^{(N)}| |f_n^{(N)}|^2 \right\}^{1/2} \times \left\{ \sum_{n \in \mathcal{N}_0} |\Lambda_{|n|}^{(N)} - \bar{\Lambda}_{|n|}^{(N)}| |g_n^{(N)}|^2 \right\}^{1/2}.$$

Let $N \geq N(\gamma)$. Proposition 7 implies

$$0 \leq |\Lambda_{|n|}^{(N)} - \bar{\Lambda}_{|n|}^{(N)}| \leq 8\gamma^2 \Lambda_{|n|}^{(N)}, \quad n \in \mathcal{N}_0.$$

Therefore we get

$$|b^{(N)}(u, v) - \bar{b}^{(N)}(u, v)| \leq 8\gamma^2 \times \left\{ 2\pi \sum_{n \in \mathcal{N}_0} \Lambda_{|n|}^{(N)} |f_n^{(N)}|^2 \right\}^{1/2} \times \left\{ 2\pi \sum_{n \in \mathcal{N}_0} \Lambda_{|n|}^{(N)} |g_n^{(N)}|^2 \right\}^{1/2}$$

provided that $N \geq N(\gamma)$. Due to Proposition 1 we have the conclusion of Theorem 3.

5. Application to mathematical analysis of an FEM-CSM combined method for exterior Laplace problems

Fix a simply connected bounded domain $\mathcal{O}$ in the plane. Assume that the boundary $\mathcal{C}$ of $\mathcal{O}$ is sufficiently smooth. The exterior domain of $\mathcal{C}$ is denoted by Ω .

Fix a function $f \in L^2(\Omega)$. Assume that the support of f , $\text{supp}(f)$, is compact.

Choose a so large that the open disc D_a may contain the union $\mathcal{O} \cup \text{supp}(f)$ in its interior.

As a model problem the following Poisson equation (E) is employed.

$$(E) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathcal{C}, \\ \sup_{|\mathbf{r}| > a} |u| < \infty. \end{cases}$$

The intersection of the domain Ω and the disc D_a is said to be the interior domain, denoted by Ω_i :

$$\Omega_i = \Omega \cap D_a.$$

Consider the Dirichlet inner product $a(u, v)$ for $u, v \in H^1(\Omega_i)$:

$$a(u, v) = \int_{\Omega_i} \text{grad} u \cdot \text{grad} v \, d\Omega.$$

Since the trace $\gamma_a v$ on Γ_a is an element of $H^{1/2}(\Gamma_a)$ for any $v \in H^1(\Omega_i)$, the boundary bilinear form of Steklov type $b(u, v)$ is well defined for $u, v \in H^1(\Omega_i)$. Therefore we can define a continuous symmetric bilinear form :

$$t(u, v) = a(u, v) + b(u, v)$$

for $u, v \in H^1(\Omega_i)$.

Let $F(v)$ be a continuous linear functional on $H^1(\Omega_i)$ defined through the following formula:

$$F(v) = \int_{\Omega_i} f v \, d\Omega.$$

A function space V is defined as follows:

$$V = \{v \in H^1(\Omega_i) : v = 0 \text{ on } \mathcal{C}\}.$$

Using these notations, the following weak formulation problem (II) is defined.

$$(II) \quad \begin{cases} t(u, v) = F(v), & v \in V, \\ u \in V. \end{cases}$$

Admitting the equivalence between the equation (E) and the problem (II), we consider the problem (II) and its approximate ones.

Fix a positive number ρ so as to satisfy $0 < \rho < a$. For a fixed positive integer N , set the points $\vec{\rho}_j, \mathbf{a}_j, 0 \leq j \leq N-1$, as is defined in Section 2.

A family of finite dimensional subspaces of V :

$$\{V_N : N = N_0, N_0 + 1, \dots\}$$

is supposed to have the following properties:

$$(V_N - 1) \quad V_N \subset C(\overline{\Omega_i}).$$

$$(V_N - 2) \quad \begin{cases} \text{For any } v \in V_N, v(\mathbf{a}(\theta)) \text{ is an equi-distant piecewise linear} \\ \text{continuous } 2\pi\text{-periodic function with respect to } \theta. \end{cases}$$

$$(V_N - 3) \quad \min_{v \in V_N} a(v - v_N) \leq \frac{C}{N} \|v\|_{H^2(\Omega_i)}, \quad v \in V \cap H^2(\Omega_i).$$

In the property $(V_N - 3)$, C is a constant independent of N and v , and

$$a(v) = a(v, v)^{1/2}, \quad v \in V.$$

For $u, v \in H^1(\Omega_i) \cap C(\overline{\Omega_i})$, we define bilinear forms $t^{(N)}(u, v)$ and $\bar{t}^{(N)}(u, v)$ as follows.

$$t^{(N)}(u, v) = a(u, v) + b^{(N)}(u, v),$$

and

$$\bar{t}^{(N)}(u, v) = a(u, v) + \bar{b}^{(N)}(u, v).$$

Now two approximate problems $(\Pi^{(N)})$ and $(\bar{\Pi}^{(N)})$ are stated as follows.

$$(\Pi^{(N)}) \quad \begin{cases} t^{(N)}(u_N, v) = F(v), & v \in V_N, \\ u_N \in V_N. \end{cases}$$

$$(\bar{\Pi}^{(N)}) \quad \begin{cases} \bar{t}^{(N)}(\bar{u}_N, v) = F(v), & v \in V_N, \\ \bar{u}_N \in V_N. \end{cases}$$

With the aide of Theorems 2 and 3 and other necessary discussions, we can show the following error estimate.

Theorem 4 *For a constant C , we have the following estimate.*

$$\left. \begin{aligned} \|u - u_N\|_{H^1(\Omega_i)} \\ \|u - \bar{u}_N\|_{H^1(\Omega_i)} \end{aligned} \right\} \leq \frac{C}{N} \|u\|_{H^2(\Omega_i)}.$$

In the above, the constant C is independent of the solution u of (Π) and N .

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