

Geometrical View of the Furuta Inequality

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1. Introduction. Throughout this note, we use a capital letter as an operator on a Hilbert space H . An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible.

The original form of the Furuta inequality [5] given by Furuta himself is the following(cf.[6],[17]).

Furuta inequality: If $A \geq B \geq 0$,

then for each $r \geq 0$,

$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for p and q such that $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p+r$$

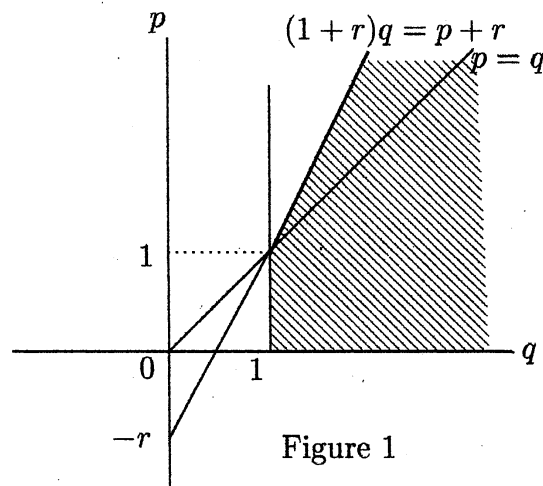


Figure 1

The case of $r = 0$ in this inequality is the Löwner-Heinz inequality:

$$(LH) \quad A^\alpha \geq B^\alpha \quad \text{for } A \geq B \geq 0 \text{ and } 0 \leq \alpha \leq 1.$$

From the viewpoint of operator mean ([2],[3],[10],[11] etc.), the Furuta inequality is rewritten as follows;

$$A^u \sharp_{\frac{1-u}{p-u}} B^p \leq A \quad \text{and} \quad B \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

for $p \geq 1$ and $u \leq 0$. The notations $\sharp_\alpha$ and $\natural_\alpha$ are defined for positive operators A and B by

$$A \natural_\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, \quad \text{for } \alpha \in \mathbb{R}$$

and $\sharp_\alpha = \natural_\alpha$ when $\alpha \in [0, 1]$. Note that $\sharp_\alpha$ is an operator mean in the sense of Kubo-Ando [16] which corresponds to the operator monotone function x^α in the Löwner theory.

As shown in [11], we had arranged these inequalities in one line by using the operator mean $\sharp_\alpha$ as follows:

Satellite theorem of the Furuta inequality: *If $A \geq B \geq 0$, then*

$$A^u \sharp_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

for all $p \geq 1$ and $u \leq 0$.

We can generalize this inequality as follows, in which the case of $\delta = 1$ is the satellite theorem([13], [14]).

Theorem A. *If $A \geq B > 0$, then for $0 \leq \delta \leq 1$, $\delta \leq p$ and $u \leq 0$*

$$A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \leq A^\delta \leq B^u \sharp_{\frac{\delta-u}{p-u}} A^p,$$

and for $-1 \leq \gamma \leq 0$, $u \leq \gamma$ and $p \geq 0$

$$A^u \sharp_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma \leq B^\gamma \leq B^u \sharp_{\frac{\gamma-u}{p-u}} A^p.$$

More generally we have the following and called it a parametrization of the Furuta inequality ([13], [14]).

Theorem A'. *If $A \geq B > 0$, then for $0 \leq \delta \leq p$ and $u \leq 0$*

$$A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \text{ and } B^u \sharp_{\frac{\delta-u}{p-u}} A^p \geq A^\delta,$$

and for $u \leq \gamma \leq 0$ and $p \geq 0$

$$A^u \sharp_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma \text{ and } B^u \sharp_{\frac{\gamma-u}{p-u}} A^p \geq B^\gamma.$$

We can explain these relations by the following Figure 2.

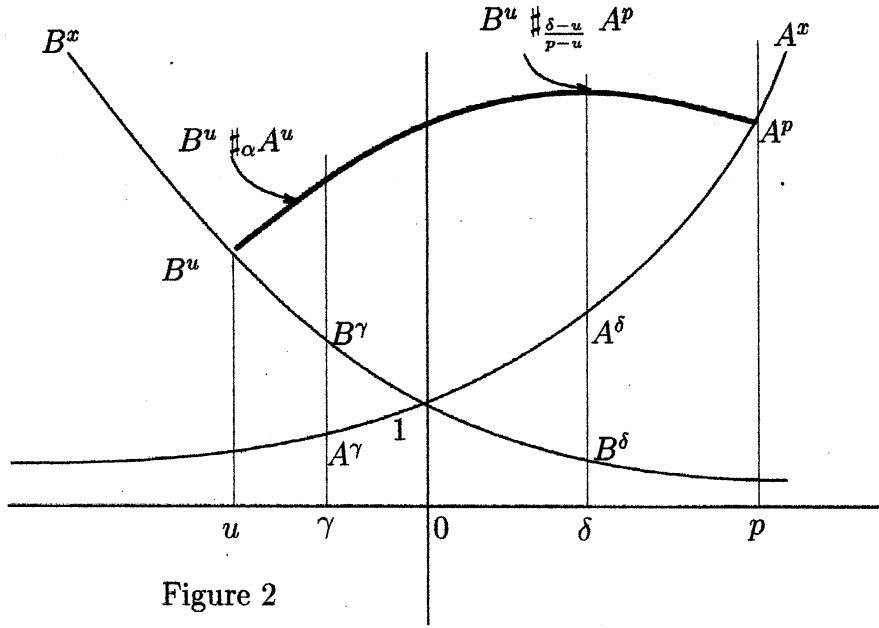


Figure 2

As a generalization of the Furuta inequality, Furuta [7] had given an inequality which we called the grand Furuta inequality. It interpolates the Furuta inequality and the Ando-Hiai inequality [1] equivalent to the main result of log majorization. We cite here it in terms of operator mean ([3]):

The grand Furuta inequality: If $A \geq B \geq 0$ and A is invertible, then for each $p \geq 1$ and $0 \leq t \leq 1$,

$$A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A$$

holds for $r \geq t$ and $s \geq 1$.

The best possibility of $\frac{1-t+r}{(p-t)s+r}$ is shown in [18]. We can state this theorem also by the satellite form as follows [14];

Theorem B. If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$, $0 \leq \delta \leq 1$ and $\delta \leq \beta$

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^\delta \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}} \leq B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

More generally we have shown the next theorem as a parametrized form of the grand Furuta inequality [15].

Theorem B'. If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$ and $0 \leq \delta \leq \beta$

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}}$$

and

$$B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}}.$$

On the complementary domain of the Furuta inequality, that is, $0 \leq t < p \leq 1$, the following inequality holds ([12],[15]).

Theorem C. *If $A \geq B > 0$, then for $0 \leq t < p \leq 1$, $p \leq \delta \leq \min\{1, 2p\}$ and $\beta \geq \delta$*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^\delta \leq A^\delta \leq B^t \natural_{\frac{\delta-t}{p-t}} A^p \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}}.$$

If $A \geq B > 0$, then for $0 \leq t \leq 1 \leq p$, $p \neq t$ and $\beta \geq p$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}}.$$

2. Results and Proofs. At the begining, we give a generalization of Theorem C.

Theorem 1. *If $A \geq B > 0$, then for $0 \leq t \leq 1$ and $0 \leq t < p \leq \beta$*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p \quad \text{and} \quad (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{p}{\beta}} \geq A^p.$$

We prepare the next lemma to prove this theorem.

Lemma. *If $A \geq B > 0$, then*

$$A^t \natural_{\frac{q-t}{p-t}} B^p \leq B^q \quad \text{and} \quad B^t \natural_{\frac{q-t}{p-t}} A^p \geq A^q$$

for $0 \leq t \leq 1$, $t < p$ and q such that $1 \leq \frac{q-t}{p-t} \leq 2$.

Proof. We have

$$\begin{aligned} A^t \natural_{\frac{q-t}{p-t}} B^p &= B^p \natural_{1-\frac{q-t}{p-t}} A^t = B^p \natural_{\frac{p-q}{p-t}} A^t \\ &= B^p (B^{-p} \sharp_{\frac{q-p}{p-t}} A^{-t}) B^p \leq B^p (B^{-p} \sharp_{\frac{q-p}{p-t}} B^{-t}) B^p = B^q. \end{aligned}$$

Proof of Theorem 1. In the case of $p \leq 1$, it is already shown in Theorem C. So we have only to see the case of $p \geq 1$. If $1 \leq \frac{\beta-t}{p-t} \leq 2$, then by the use of we Lemma and (LH) we have $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p$.

Secondly, if we choose β_1 such as $1 \leq \frac{\beta_1-t}{\beta-t} \leq 2$ for the above β , then we can use

Lemma for A and $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$ since $B_1 \leq A$ by Theorem C. So we have $A^t \natural_{\frac{\beta_1-t}{\beta-t}} B_1^{\beta} \leq B_1^{\beta_1}$. That is,

$$A^t \natural_{\frac{\beta_1-t}{p-t}} B^p = A^t \natural_{\frac{\beta_1-t}{\beta-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}}.$$

By (LH) we have

$$(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}} \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p.$$

Thirdly, we choose β_2 such that $1 \leq \frac{\beta_2-t}{\beta_1-t} \leq 2$ for the above β_1 , then Lemma is usable for A and $B_2 = (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{1}{\beta_1}}$ by Theorem C. So $A^t \natural_{\frac{\beta_2-t}{\beta_1-t}} B_2^{\beta_1} \leq B_2^{\beta_2}$ holds. That is,

$$A^t \natural_{\frac{\beta_2-t}{p-t}} B^p = A^t \natural_{\frac{\beta_2-t}{\beta_1-t}} (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{\beta_2}{\beta_1}}.$$

Hence by using (LH), we have

$$(A^t \natural_{\frac{\beta_2-t}{p-t}} B^p)^{\frac{p}{\beta_2}} \leq (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}}.$$

Combining this with the above cases, we have

$$(A^t \natural_{\frac{\beta_2-t}{p-t}} B^p)^{\frac{p}{\beta_2}} \leq (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}} \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p.$$

Repeating this method, we can obtain the desired inequality.

In parallel with Theorem B, we have the following:

Theorem 2. *If $A \geq B > 0$ and $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$, then*

(1) $0 \leq \delta \leq 1$ and $\delta \leq p$,

$$A^u \#_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \#_{\frac{\delta-u}{p-u}} B^p \leq B^{\delta} \leq A^{\delta} \leq B^u \#_{\frac{\delta-u}{p-u}} A^p \leq B^u \#_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

(2) $-1 \leq \gamma \leq 0$ and $u \leq \gamma$,

$$A^u \#_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \#_{\frac{\gamma-u}{p-u}} B^p \leq A^{\gamma} \leq B^{\gamma} \leq B^u \#_{\frac{\gamma-u}{p-u}} A^p \leq B^u \#_{\frac{\gamma-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The case of $\delta = 1$ in (1) shows the order between the Furuta inequality and grand Furuta inequality. We can more loosen the condition on δ and γ as follows:

Theorem 3 *If $A \geq B > 0$ and $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$, then*

(1) $0 \leq \delta \leq p$,

$$A^u \#_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \#_{\frac{\delta-u}{p-u}} B^p \leq B^{\delta}$$

and

$$B^u \#_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^u \#_{\frac{\delta-u}{p-u}} A^p \geq A^\delta.$$

(2) $u \leq \gamma \leq 0$

$$A^u \#_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \#_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma$$

and

$$B^u \#_{\frac{\gamma-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^u \#_{\frac{\gamma-u}{p-u}} A^p \geq B^\gamma.$$

Proof. (1) It follows from Theorem B and Theorem 1 that

$$\begin{aligned} & A^u \#_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \\ &= A^u \#_{\frac{\delta-u}{p-u}} (A^u \#_{\frac{p-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)) \\ &\leq A^u \#_{\frac{\delta-u}{p-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq A^u \#_{\frac{\delta-u}{p-u}} B^p. \end{aligned}$$

The case of (2) is also obtained as follows:

$$\begin{aligned} & A^u \#_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \\ &= A^u \#_{\frac{\gamma-u}{p-u}} (A^u \#_{\frac{p-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)) \\ &\leq A^u \#_{\frac{\gamma-u}{p-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq A^u \#_{\frac{\gamma-u}{p-u}} B^p. \end{aligned}$$

In Theorem B', if we restrict the condition on δ to $0 \leq \delta \leq p$ the same as Theorem 3, we have the following parallel formulas to Theorem 3 (1).

$$A^u \#_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta$$

and

$$B^u \#_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}} \geq A^\delta.$$

We can explain the relations in Theorem 2 and Theorem 3 in the following Figure 3.

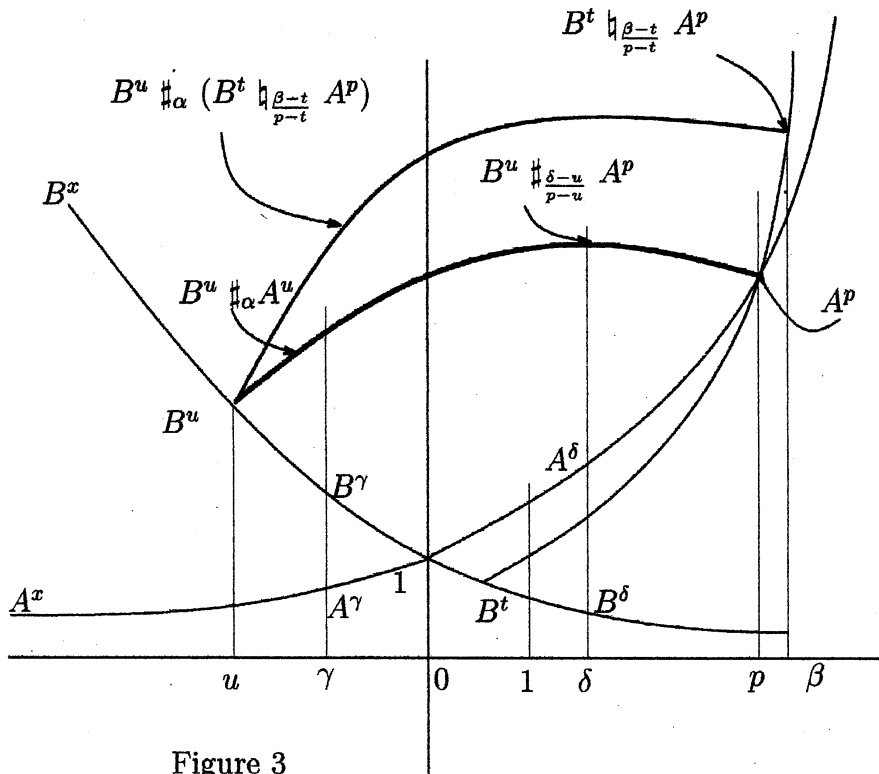


Figure 3

3. Remark. The results of Theorem 2 and Theorem 4 can be led by using some monotone property of operator function on the grand Furuta inequality. This viewpoint is an indication of Professor Furuta. In the papers [8] and [9], Furuta et al., the grand Furuta inequality is treated as an operator function with monotone properties with respect to two variables. We also translated their assertions into our terms in [15] and showed this property being the structural necessity of the Furuta inequality in [4]. The representation of this function by us and our result [4] are the following.

Theorem D. If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$ and $0 \leq \delta \leq \beta$

$$H_{p,\delta,t}(A, B, u, \beta) = A^u \#_{\frac{\delta-u}{\beta-u}} (A^t \sharp_{\frac{\beta-t}{p-t}} B^p)$$

is increasing for $u \leq 0$ and decreasing for $\beta \geq p$.

In this theorem, the case of $\beta = p$ is the form of the Furuta inequality and the monotone property of this function for β leads Theorem 2.

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