

ON MAXIMUM MODULUS OF ANALYTIC FUNCTIONS

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ABSTRACT. Let A be the class of functions $f(z)$ analytic in $|z| < R$ with $f(0) = 1$. The object of the present paper is to investigate analyticity conditions of $M(r)$ which is the maximum modulus of $f(z)$ in A . Our results proved here provide a generalization of some results due to W. K. Hayman (J. Analyse Math. 1(1951), 135 - 154).

1. INTRODUCTION. Let A be the class of functions $f(z)$ analytic in the disk $|z| < R$ with $f(0) = 1$. The maximum modulus $M(r)$ given by

$$(1.1) \quad M(r) = \max\{f(z) \in A: z = re^{i\theta}, 0 < r < R, 0 \leq \theta \leq 2\pi\}$$

was investigated by Hadamard [1], Hayman [3], et al. (see [2], [4]).

The curves from the origin to $|z| = R$ consisting of all points $z = re^{i\theta}$ which satisfy

$$(1.2) \quad \frac{\partial |f(re^{i\theta})|}{\partial \theta} = 0$$

are called quasi-extremal curves of $f(z) \in A$. Some of quasi-extremal curves are called extremal curves of $f(z) \in A$ if

$$(1.3) \quad |f(re^{i\theta})| = M(r).$$

In the present paper, we show analyticity conditions of the maximum modulus $M(r)$ of $f(z) \in A$.

2. MAIN RESULTS. In order to discuss our problems, we have to use the following lemmas.

LEMMA 1 (Hayman [3]). Let $f(z)$ be analytic on the circle $|z| = r$. If it satisfies

$$(2.1) \quad \frac{\partial |f(re^{i\theta_0})|}{\partial \theta} = 0, \quad f(re^{i\theta_0}) \neq 0,$$

then $g(z) = zf'(z)/f(z)$ is real at the point $z_0 = re^{i\theta_0}$ and

$$(2.2) \quad g(z_0) = r \frac{d}{dr} \log |f(\alpha(r))|,$$

where $\alpha(t)$ is the parametric representation with $t = |\alpha(t)|$ of any given curve which is analytic at a point z_0 and non-tangential with the circle $|z| = r$.

LEMMA 2. Let Γ be an analytic curve and intersect every circle $|z| = r$ ($0 < r \leq R$) at only one point. Then Γ is of the form

$$(2.3) \quad \Gamma = \{z: z = re^{i\theta(r)}, 0 \leq r \leq R\},$$

where $\theta(r)$ is a real analytic function.

PROOF. Since Γ is a simple analytic curve, there exists a parametric representation of Γ

$$(2.4) \quad \Gamma = \{z: z = \alpha(t), 0 \leq t \leq 1\}$$

such that $\alpha(t)$ is an analytic injection with $\alpha(0) = 0$. Letting

$$(2.5) \quad \log \frac{\alpha(t)}{t} = \phi(t) + i\psi(t),$$

we see that $\phi(t)$ and $\psi(t)$ are real analytic. Further, we know that

$$(2.6) \quad r = te^{\phi(t)}$$

is exactly increasing in $t \in [0,1]$, and its inverse function is $t = t(r)$.

Defining $\theta(r)$ by

$$(2.7) \quad \theta(r) = \psi(t(r)),$$

we obtain the representation (2.4) of Γ .

Now we prove

THEOREM 1. If there exists a quasi-extremal curve

$$\Gamma = \{z: z = \alpha(r), 0 \leq r < R\}$$

of $f(z)$ which is in the class A with modulus r such that $\forall z_0 \in \Gamma$, $g(z_0)$ is the maximum real value of $g(z) = zf'(z)/f(z)$ on the circle $|z| = |z_0|$, then Γ is an extremal curve of $f(z)$ and $M(r)$ is real analytic on $[0, R)$.

PROOF. We may assume that $f(z)$ is analytic on the closed disk $|z| \leq R$, other wise we consider a closed subdisk of $|z| < R$. Divide the disk $|z| < R$ into several annular domain D_k given by

$$D_k = \{z: r_{k-1} < |z| < r_k; r_0 = 0; r_n = R; k = 1, 2, 3, \dots, n\}$$

such that $f(z) \neq 0$ in every D_k . It is known that there are 2^{n_k} quasi-extremal arcs

$$C_{kj} = \{z: z = \alpha_{kj}(r), j = 1, 2, 3, \dots, 2^{n_k}\}$$

of $f(z)$ in D_k , which are analytic in (r_{k-1}, r_k) . Applying Lemma 1, we know that

$$(2.8) \quad g(\alpha(r)) = r \frac{d}{dr} \log |f(\alpha(r))|$$

and

$$(2.9) \quad g(\alpha_{k,j}(r)) = r \frac{d}{dr} \log |f(\alpha_{k,j}(r))|.$$

Then we have

$$(2.10) \quad \log |f(\alpha(r))| = \int_0^r g(\alpha(t)) \frac{dt}{t} \quad (0 \leq r < R)$$

and

$$(2.11) \quad \log |f(\alpha_{1j}(r))| = \int_0^r g(\alpha_{1j}(t)) \frac{dt}{t} \quad (0 \leq r < r_1),$$

which give

$$(2.12) \quad |f(\alpha_{1j}(r))| \leq |f(\alpha(r))| \quad (0 \leq r < r_1).$$

Also the inequality (2.12) is still true for $r = r_1$. Therefore we obtain

$$(2.13) \quad M(r) = |f(\alpha(r))| \quad (0 \leq r \leq r_1).$$

For $r_1 < r' < r'' < r_2$, it follows from (2.9) and (2.9) that

$$(2.14) \quad \frac{|f(\alpha_{2j}(r''))|}{|f(\alpha_{2j}(r'))|} = \exp \left\{ \int_{r'}^{r''} g(\alpha_{2j}(r)) \frac{dt}{t} \right\} \\ \leq \exp \left\{ \int_{r'}^{r''} g(\alpha(t)) \frac{dt}{t} \right\} \\ = \frac{|f(\alpha(r''))|}{|f(\alpha(r'))|},$$

so

$$(2.15) \quad \frac{|f(\alpha_{2j}(r''))|}{|f(\alpha(r''))|} \leq \frac{|f(\alpha_{2j}(r'))|}{|f(\alpha(r'))|}.$$

Letting $r' \rightarrow r_1$ in (2.15), we see that

$$(2.16) \quad |f(\alpha_{2j}(r''))| \leq |f(\alpha(r''))|,$$

which proves that

$$(2.17) \quad M(r) = |f(\alpha(r))| \quad (0 \leq r \leq r_2).$$

To do the above process again and again, it follows that Γ is an extremal curve of $f(z)$ and $M(r)$ is real analytic on $[0, R)$.

Next, we derive

THEOREM 2. If $g(z)$ is analytic and univalent in $|z| < R$, and if there exists a quasi-extremal curve of $f(z) \in \Lambda$ which intersects every circle $|z| = r$ ($0 < r < R$) at only one point, then $M(r)$ is real analytic.

PROOF. Because $g(z)$ maps every quasi-extremal curve of $f(z)$ onto

a half open interval on the real axis with the origin as an end point, the intersection of the real axis and the range of $g(z)$ is an open interval L which is divided by the origin into two intervals L_1 (left hand interval) and L_2 (right hand interval). These inverse images Γ_1 and Γ_2 are two quasi-extremal curves of $f(z)$ which are analytic curves.

Using Lemma 1 and Lemma 2, we have

$$(2.18) \quad \Gamma_j = \{z: z = re^{i\theta_j(r)}, j = 1, 2\},$$

where $\theta_j(r)$ are real analytic and

$$(2.19) \quad g(re^{i\theta_j(r)}) = r \frac{d}{dr} \log |f(re^{i\theta_j(r)})|.$$

Then we have

$$(2.20) \quad |f(re^{i\theta_j(r)})| = \exp \left\{ \int_0^r g(te^{i\theta_j(t)}) \frac{dt}{t} \right\}.$$

Since

$$(2.21) \quad g(re^{i\theta_1(r)}) < 0 < g(re^{i\theta_2(r)}),$$

it follows that

$$(2.22) \quad M(r) = |f(re^{i\theta_2(r)})|$$

which is real analytic.

Further, we prove

THEOREM 3. Under the assumption in Theorem 2, there exists a certain open disk $|z| < R_1$ such that $M(r)$ is the maximum modulus of its complex analytic extension $M(z)$ in $|z| < R_1$.

PROOF. Let D be the domain in which $M(z)$ is analytic. Then the origin is contained in D , and there exists a open disk $|z| < R_1$ in D such that, for any $|z| < R_1$,

$$(2.23) \quad |ze^{i\theta_2(z)}| < R,$$

where $\theta_2(z)$ is the complex analytic extension of $\theta_2(r)$. From (2.19), (2.21), and (2.22), we know that

$$(2.24) \quad \mu(r) = r \frac{M'(r)}{M(r)} = g(re^{i\theta_2(r)})$$

is the maximum real value of $g(z)$ on the circle $|z| = r$, and $\mu(r)$ is exactly increasing and real analytic. It follows from (2.24) that

$$(2.25) \quad \mu(z) = z \frac{M'(z)}{M(z)} = g(ze^{i\theta_2(z)})$$

and

$$(2.26) \quad \mu(\bar{z}) = \overline{\mu(z)}, \quad M(\bar{z}) = \overline{M(z)}, \quad \theta_2(\bar{z}) = \overline{\theta_2(z)}.$$

Suppose that $M(r)$ is not the maximum modulus of $M(z)$ in $|z| < R_1$, that is, there exists r_0 ($0 < r_0 < R_1$) such that $M(r_0)$ is not the maximum modulus of $M(z)$ on the circle $|z| = r_0$. Then there is a point $z_0 = r_0 e^{i\theta_0}$ such that

$$(2.27) \quad \mu(\bar{z}_0) = \mu(z_0) > \mu(r_0)$$

by Theorem 1. This gives us that

$$(2.28) \quad g(\bar{z}_0 e^{i\theta_2(\bar{z}_0)}) = g(z_0 e^{i\theta_2(z_0)}) > \mu(r_0).$$

But, since

$$(2.29) \quad |\bar{z}_0 re^{i\theta_2(\bar{z}_0)}| |z_0 e^{i\theta_2(z_0)}| = r_0^2 |e^{i(\theta_2(\bar{z}_0) + \theta_2(z_0))}| = r_0^2,$$

(2.28) contradicts the fact that $\mu(r)$ is the maximum real value on the circle $|z| = r$ and exactly increasing. This completes the proof of Theorem 3.

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