

Inverse eigenvalue problems for nonlinear ordinary differential equations

広島大学・大学院工学研究科 柴田徹太郎 (Tetsutaro Shibata)

Graduate School of Engineering

Hiroshima University

1 Introduction

We consider the following problem

$$(1.1) \quad -u''(t) + f(u(t)) = \lambda u(t), \quad t \in I,$$

$$(1.2) \quad u(t) > 0, \quad t \in I,$$

$$(1.3) \quad u(0) = u(1) = 0,$$

where $I := (0, 1)$ and $\lambda > 0$ is a parameter. We assume the following conditions.

(A.1) $f(u)$ is a function of C^1 for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.

(A.2) $g(u) := f(u)/u$ is strictly increasing for $u \geq 0$ ($g(0) := 0$).

(A.3) $g(u) \rightarrow \infty$ as $u \rightarrow \infty$.

The typical examples of $f(u)$ are as follows.

$$f(u) = u^p \quad (p > 1),$$

$$f(u) = u^p \log(u+1) \quad (p > 1),$$

$$f(u) = u^p \left(1 - \frac{1}{1+u^q}\right) \quad (p > 1, q > 1),$$

$$f(u) = u^2 \left(1 - \frac{u-4}{2} e^{-u}\right),$$

$$f(u) = u^p e^u \quad (p > 1).$$

The equation (1.1)–(1.3) has been studied by many authors. We refer to the papers in the references. We know that for any given $\alpha > 0$, there exists a unique solution pair of (1.1)–(1.3) $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ such that $\|u_\alpha\|_2 = \alpha$. Moreover, the set $\{(\lambda(\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions of (1.1)–(1.3), which is an unbounded C^1 -bifurcation curve emanating from $(\pi^2, 0)$ in $\mathbf{R}_+ \times L^2(I)$, and $\lambda(\alpha)$ is C^1 and strictly increasing for $\alpha > 0$. We know that for any given $\lambda > \pi^2$, there exists a unique solution $u_\lambda \in C^2(\bar{I})$. Further, for $\lambda \gg 1$,

$$(1.4) \quad \lambda = g(\|u_\lambda\|_\infty) + O(1).$$

For instance, let $f(u) = u^p$. Then since $g(u) = f(u)/u = u^{p-1}$, for $\lambda \gg 1$,

$$(1.5) \quad \lambda = \|u_\lambda\|_\infty^{p-1} + O(1).$$

More precisely, we know that as $\lambda \rightarrow \infty$

$$\lambda = \|u_\lambda\|_\infty^{p-1} + \lambda e^{-\sqrt{(p-1)\lambda(1+o(1))}/2}.$$

Further, we know that as $\lambda \rightarrow \infty$

$$(1.6) \quad \frac{u_\lambda(t)}{g^{-1}(\lambda)} \rightarrow 1$$

uniformly on any compact set in I . Therefore,

$$\alpha = \|u_\alpha\|_2 = \left(\int_I g^{-1}(\lambda)^2 dt \right)^{1/2} (1 + o(1)) = g^{-1}(\lambda)(1 + o(1)).$$

Then in many cases, we have

$$(1.7) \quad \lambda(\alpha) = g(\alpha) + o(g(\alpha)).$$

For instance, if $f(u) = u^p$, then for $\alpha \gg 1$

$$(1.8) \quad \lambda(\alpha) = \alpha^{p-1} + o(\alpha^{p-1}).$$

We here consider L^2 -inverse spectral problems. More precisely, it is valid that the L^2 -bifurcation curve $\lambda(\alpha)$ is determined by the nonlinear term $f(u)$. Our problem here is, conversely, to investigate whether we determine $f(u)$ by the asymptotic formula for $\lambda(\alpha)$ as $\alpha \rightarrow \infty$ or not.

We know the following fact.

Theorem 1 [16]. Let $f(u) = u^p$ ($p > 1$). Then for any fixed $n \in \mathbb{N}_0$, as $\alpha \rightarrow \infty$:

$$\lambda(\alpha) = \alpha^{p-1} + C_1 \alpha^{(p-1)/2} + \sum_{k=0}^n \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2} \alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}),$$

where

$$C_1 = (p+3) \int_I \sqrt{\frac{p-1}{p+1} - s^2} + \frac{2}{p+1} s^{p+1} ds$$

and $a_k(p)$ ($\deg a_k(p) \leq k+1$) is a polynomial determined by $a_0 = 1, a_1, \dots, a_{k-1}$.

Motivated by Theorem 1, we consider the following Problems.

Problem A. Assume that the following asymptotic formula is valid as $\alpha \rightarrow \infty$.

$$(1.9) \quad \lambda(\alpha) = \alpha^{p-1} + C_0 \alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}).$$

Then does $f(u) = u^p$ hold ?

Problem B. Assume that the following asymptotic formula is valid as $\alpha \rightarrow \infty$.

$$(1.10) \quad \lambda(\alpha) = \alpha^{p-1} + C_1 \alpha^{(p-1)/2} + \frac{1}{p-1} C_1^2 + o(1).$$

Then does $f(u) = u^p$ hold ?

Problem C. Assume that the asymptotic formula in Theorem 1 with some $p > 1$ is valid for any $n \in \mathbb{N}$ as $\alpha \rightarrow \infty$. Then can we conclude $f(u) = u^p$?

Theorem 2. For $p, q > 1$, let

$$f(u) = u^p \left(1 - \frac{1}{1+u^q} \right).$$

(i) Assume that $(p-1)/2 < q < p+1$. Then (1.9) holds as $\alpha \rightarrow \infty$.

(ii) Assume that $p-1 < q < p+1$. Then (1.10) holds as $\alpha \rightarrow \infty$.

Theorem 3. Assume that

$$f(u) = u^2 \left(1 - \frac{u-4}{2} e^{-u} \right).$$

Then the asymptotic formula for $\lambda(\alpha)$ in Theorem 1 with $p = 2$ holds for any $n \in \mathbb{N}$.

Therefore, unfortunately, we find that the assumptions in Problem A–C are not sufficient to obtain the desired results for L^2 -inverse problems. The next problem we have to consider is to find the suitable setting for nonlinear inverse eigenvalue problems.

2 New and direct proof of Theorem 1

The proofs of Theorems 2 and 3 are variant of those used in [16]. We here introduce a new and direct proof of Theorem 1. We consider (λ, u_λ) for $\lambda \gg 1$. We put

$$\begin{aligned} R_\lambda(s) &:= 1 - s^2 - \frac{2}{p+1} \lambda^{-1} \|u_\lambda\|_\infty^{p-1} (1 - s^{p+1}), \\ S(s) &:= 1 - s^2 - \frac{2}{p+1} (1 - s^{p+1}). \end{aligned}$$

Lemma 2.1. For $\lambda \gg 1$

$$\|u_\lambda\|_\infty^2 - \|u_\lambda\|_2^2 = \lambda^{-1/2} \|u_\lambda\|_\infty^2 (C_2 + U_\lambda).$$

Here,

$$\begin{aligned} C_2 &:= 2 \int_0^1 \frac{1 - s^2}{\sqrt{1 - s^2 - 2(1 - s^{p+1})/(p+1)}} ds, \\ U_\lambda &:= 2 \int_0^1 \frac{(1 - s^2)(S(s) - R_\lambda(s))}{\sqrt{R_\lambda(s)} \sqrt{S(s)} (\sqrt{R_\lambda(s)} + \sqrt{S(s)})} ds. \end{aligned}$$

Proof. For $0 \leq t \leq 1$,

$$\frac{d}{dt} \left[\frac{1}{2} u'_\lambda(t)^2 - \frac{1}{p+1} u_\lambda(t)^{p+1} + \frac{1}{2} \lambda u_\lambda(t)^2 \right] = 0.$$

Then

$$\frac{1}{2} u'_\lambda(t)^2 - \frac{1}{p+1} u_\lambda(t)^{p+1} + \frac{1}{2} \lambda u_\lambda(t)^2 = \text{constant} = -\frac{1}{p+1} \|u_\lambda\|_\infty^{p+1} + \frac{1}{2} \lambda \|u_\lambda\|_\infty^2.$$

We put

$$M_\lambda(\theta) := \lambda(\|u_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1} (\|u_\lambda\|_\infty^{p+1} - \theta^{p+1}).$$

Then for $0 \leq t \leq 1/2$,

$$(2.1) \quad u'_\lambda(t) = \sqrt{M_\lambda(u_\lambda(t))}.$$

Then

$$\|u_\lambda\|_\infty^2 - \|u_\lambda\|_2^2 = 2 \int_0^{1/2} (\|u_\lambda\|_\infty^2 - u_\lambda^2(t)) \frac{u'_\lambda(t)}{\sqrt{M_\lambda(u_\lambda(t))}} dt$$

$$\begin{aligned}
&= 2 \int_0^{\|u_\lambda\|_\infty} (\|u_\lambda\|_\infty^2 - \theta^2) \frac{1}{\sqrt{M_\lambda(\theta)}} d\theta \\
&= 2\lambda^{-1/2} \|u_\lambda\|_\infty^2 \int_0^1 \frac{1-s^2}{\sqrt{R_\lambda(s)}} ds \\
&= \lambda^{-1/2} \|u_\lambda\|_\infty^2 \left(2 \int_0^1 \frac{1-s^2}{\sqrt{S(s)}} ds + U_\lambda \right) \\
&= \lambda^{-1/2} \|u_\lambda\|_\infty^2 (C_2 + U_\lambda).
\end{aligned}$$

Thus the proof is complete. ■

Lemma 2.2. For $\lambda \gg 1$

$$|U_\lambda| \leq C\lambda^{-1/2} e^{-\sqrt{(p-1)(1+o(1))}/(2\sqrt{\lambda})}.$$

The proof of Lemma 2.2 is long and tedious. So we omit the proof here. By using Lemmas 2.1 and 2.2, we easily obtain Theorem 1.

3 New example

In this section, we consider new example of $f(u)$. Let $f(u) = u^p e^u$ ($p > 1$).

Theorem 4. Assume that $f(u) = u^p e^u$ ($p > 1$) in (1.1). Then as $\alpha \rightarrow \infty$

$$\lambda(\alpha) = \alpha^{p-1} e^\alpha + \frac{\pi}{4} \alpha^{(p+1)/2} e^{\alpha/2} (1 + o(1)).$$

To prove Theorem 4, we begin with the fundamental properties of $\lambda(\alpha)$. We know

$$\begin{aligned}
\frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} &\leq \lambda(\alpha) \leq \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + \pi^2, \\
u_\alpha(t) &= \|u_\alpha\|_\infty (1 + o(1)) = \alpha (1 + o(1)), \quad t \in I, \\
u_\alpha(t) &= u_\alpha(1-t), \quad 0 \leq t \leq 1, \\
u_\alpha\left(\frac{1}{2}\right) &= \max_{0 \leq t \leq 1} u_\alpha(t) = \|u_\alpha\|_\infty, \\
u'_\alpha(t) &> 0, \quad 0 \leq t < \frac{1}{2}.
\end{aligned}$$

Lemma 3.1. For $\alpha \gg 1$

$$\|u_\alpha\|_\infty^2 - \alpha^2 = \frac{\pi}{2}(1 + o(1)) \sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}} \|u_\alpha\|_\infty^2.$$

Proof. Put

$$F(u) := \int_0^u f(s) ds.$$

Then for $0 \leq t \leq 1$,

$$\frac{d}{dt} \left[\frac{1}{2} u'_\alpha(t)^2 - F(u_\alpha(t)) + \frac{1}{2} \lambda(\alpha) u_\alpha(t)^2 \right] = 0.$$

Therefore, for $0 \leq t \leq 1$,

$$\frac{1}{2} u'_\alpha(t)^2 - F(u_\alpha(t)) + \frac{1}{2} \lambda(\alpha) u_\alpha(t)^2 = \text{constant} = -F(\|u_\alpha\|_\infty) + \frac{1}{2} \lambda(\alpha) \|u_\alpha\|_\infty^2.$$

We put

$$\begin{aligned} M_\alpha(\theta) &:= \lambda(\alpha)(\|u_\alpha\|_\infty - \theta^2) - 2(F(\|u_\alpha\|_\infty) - F(\theta)), \\ Q_\alpha(s) &:= \lambda(\alpha)\|u_\alpha\|_\infty^2(1 - s^2) - 2(F(\|u_\alpha\|_\infty) - F(s\|u_\alpha\|_\infty)). \end{aligned}$$

Then for $0 \leq t \leq 1/2$

$$(3.1) \quad u'_\alpha(t) = \sqrt{M_\alpha(u_\alpha(t))}.$$

By putting $\theta := u_\alpha(t)$, $s = \theta/\|u_\alpha\|_\infty$

$$\begin{aligned} \|u_\alpha\|_\infty^2 - \alpha^2 &= 2 \int_0^{1/2} (\|u_\alpha\|_\infty^2 - u_\alpha^2(t)) \frac{u'_\alpha(t)}{\sqrt{M_\alpha(u_\alpha(t))}} dt \\ &= 2 \int_0^{\|u_\alpha\|_\infty} (\|u_\alpha\|_\infty^2 - \theta^2) \frac{1}{\sqrt{M_\alpha(\theta)}} d\theta \\ &= 2 \frac{\|u_\alpha\|_\infty^2}{\sqrt{\lambda(\alpha)}} \int_0^1 \frac{1 - s^2}{\sqrt{Q_\alpha(s)/(\lambda(\alpha)\|u_\alpha\|_\infty^2)}} ds. \end{aligned}$$

Then we can show that as $\alpha \rightarrow \infty$

$$\int_0^1 \frac{1 - s^2}{\sqrt{Q_\alpha(s)/(\lambda(\alpha)\|u_\alpha\|_\infty^2)}} ds \rightarrow \int_0^1 \sqrt{1 - s^2} ds = \frac{\pi}{4}.$$

■

Lemma 3.2. For $\alpha \gg 1$

$$(3.2) \quad \|u_\alpha\|_\infty - \alpha = \frac{\pi}{4}(1 + o(1)) \sqrt{\frac{\alpha}{f(\alpha)}} \alpha.$$

Proof. By Lemma 3.1, for $\alpha \gg 1$,

$$\|u_\alpha\|_\infty^2 \left(1 - \frac{\pi}{2}(1 + o(1)) \sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}} \right) = \alpha^2.$$

By this and Taylor expansion, for $\alpha \gg 1$

$$\begin{aligned} \|u_\alpha\|_\infty &= \alpha \left(1 - \frac{\pi}{2}(1 + o(1)) \sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}} \right)^{-1/2} \\ &= \alpha \left(1 + \frac{\pi}{4}(1 + o(1)) \sqrt{\frac{\|u_\alpha\|_\infty}{f(\|u_\alpha\|_\infty)}} \right) \\ &= \alpha \left(1 + \frac{\pi}{4}(1 + o(1)) \sqrt{\frac{\alpha}{f(\|u_\alpha\|_\infty)}} \right). \end{aligned}$$

For $\alpha \gg 1$, we can show

$$(3.3) \quad f(\|u_\alpha\|_\infty) = f(\alpha)(1 + o(1))$$

■

Lemma 3.3. For $\alpha \gg 1$

$$f(\|u_\alpha\|_\infty) - f(\alpha) = \frac{\pi}{4} f'(\alpha) \alpha \sqrt{\frac{\alpha}{f(\alpha)}} (1 + o(1)).$$

Proof. For $\alpha \gg 1$, we have

$$(3.4) \quad f'(\|u_\alpha\|_\infty) = f'(\alpha)(1 + o(1))$$

Then

$$\begin{aligned} f(\|u_\alpha\|_\infty) - f(\alpha) &= f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha) \\ &\leq f'(\|u_\alpha\|_\infty)(\|u_\alpha\|_\infty - \alpha) \\ &= \frac{\pi}{4}(1 + o(1)) f'(\alpha) \alpha \sqrt{\frac{\alpha}{f(\alpha)}}, \\ f(\|u_\alpha\|_\infty) - f(\alpha) &= f'(\alpha_1)(\|u_\alpha\|_\infty - \alpha) \\ &\geq f'(\alpha)(\|u_\alpha\|_\infty - \alpha) \\ &= \frac{\pi}{4}(1 + o(1)) f'(\alpha) \alpha \sqrt{\frac{\alpha}{f(\alpha)}}. \end{aligned}$$

■

Proof of Theorem 4. By Taylor expansion, for $\alpha \gg 1$

$$\begin{aligned}\lambda(\alpha) &= \frac{f(\|u_\alpha\|_\infty)}{\|u_\alpha\|_\infty} + O(1) \\ &= \frac{f(\alpha) + \frac{\pi}{4}f'(\alpha)\alpha\sqrt{\alpha/f(\alpha)}(1+o(1))}{\alpha(1 + \frac{\pi}{4}\sqrt{\alpha/f(\alpha)}(1+o(1)))} + O(1) \\ &= \frac{1}{\alpha} \left(f(\alpha) + \frac{\pi}{4}f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1)) \right) \left(1 - \frac{\pi}{4}\sqrt{\frac{\alpha}{f(\alpha)}}(1+o(1)) \right) + O(1).\end{aligned}$$

Since for $\alpha \gg 1$,

$$f(\alpha)\sqrt{\frac{\alpha}{f(\alpha)}} \ll f'(\alpha)\alpha\sqrt{\frac{\alpha}{f(\alpha)}},$$

by this, we obtain Theorem 4. ■

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