

Multiresolution Analysis with Lattice Basis

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概要

In this note, we consider the multiresolution analysis of $L^2(\mathbb{R}^n)$ with lattice basis and wavelet basis associated with it. Our main results are Theorem 1, characterizing orthonormal basis of V_j and Theorem 2, characterizing wavelet basis.

Let $A \in GL(n; \mathbb{R}^n)$ and define

$$\Gamma = \Gamma_A = \{Ak; k \in \mathbb{Z}^n\}.$$

Let $A = (\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$, where $\vec{a}_i (i = 1, 2, \dots, n)$ are column vectors in $\mathbb{R}^n$.

We call $\vec{a}_i$'s a basis of the lattice. Let $Q_n = [0, 1]^n$ and

$$\Omega_A = \sum_{j=1}^n t_j \vec{a}_j; (t_1, \dots, t_n) \in Q_n.$$

Let $A^* \in GL(n; \mathbb{R})$ be such that $A^t A^* = E_n$ and

$$\Gamma^* = \Gamma_{A^*} = \{A^* k; k \in \mathbb{Z}^n\}.$$

We call Γ^* the dual lattice of the lattice Γ_A .

Definition 1 A multiresolution analysis of $L^2(\mathbb{R}^n)$ is a collection of closed subspaces $V_j (j \in \mathbb{Z})$ of $L^2(\mathbb{R}^n)$ such that

- (1) V_j 's are increasing and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$
- (2) $f(x) \in V_j \iff f(2x) \in V_{j+1}$, where $x \in \mathbb{R}^n$
- (3) $f \in L^2(\mathbb{R}^n)$ belongs to V_0 if and only if $f(x-\gamma) \in V_0$ for any $\gamma \in \Gamma$
- (4) There exists $g \in V_0$ such that $\{g(x-\gamma); \gamma \in \Gamma\}$ is a Riesz basis of V_0 .

The above condition (4) means that there exist constant, C_1, C_2 , $0 < C_1 \leq C_2$ such that for any sequence of scalars $a(\gamma)$, $\gamma \in \Gamma$,

$$C_1 \sum_{\gamma \in \Gamma} |a(\gamma)|^2 \leq \left\| \sum_{\gamma \in \Gamma} a(\gamma) g(x-\gamma) \right\|^2 \leq C_2 \sum_{\gamma \in \Gamma} |a(\gamma)|^2.$$

The Fourier transform of $f(x-\gamma)$, $\gamma \in \Gamma$ is

$$\widehat{f(x-\gamma)}(\xi) = \exp(-\sqrt{-1}\xi \cdot \gamma) \widehat{f}(\xi)$$

For $\phi \in V_0$, let

$$\left(\frac{1}{2}\right)^{\frac{n}{2}} \phi\left(\frac{x}{2}\right) = \sum_{\gamma \in \Gamma} a(\gamma) \phi(x-\gamma).$$

Then its Fourier transform is

$$2^{\frac{n}{2}} \widehat{\phi(2\xi)} = \left[\sum_{\gamma \in \Gamma} a(\gamma) \exp(-\sqrt{-1}\xi \cdot \gamma) \right] \widehat{\phi}(\xi) \quad (\xi \in \mathbb{R}^n).$$

Define

$$m_0(\xi) = \sum_{\gamma \in \Gamma} a(\gamma) \exp(-\sqrt{-1}\xi \cdot \gamma) \quad (1)$$

The function $m_0(\xi)$ is $2\pi\Gamma^*$ -periodic and $\widehat{\phi(2\xi)} = m_0(\xi) \widehat{\phi}(\xi)$.

For $f_1(x), f_2(x) \in L^2(\mathbb{R}^n)$, and $\gamma_1, \gamma_2 \in \Gamma$, we have a formula

$$\langle f_1(x-\gamma_1), f_2(x-\gamma_2) \rangle = \left(\frac{1}{2\pi}\right)^n \langle \exp(-\sqrt{-1}\xi \cdot (\gamma_1 - \gamma_2)) \hat{f}_1, \hat{f}_2 \rangle$$

Lemma 1 For $\gamma \in \Gamma$, we have a formula

$$\left(\frac{1}{2\pi}\right)^n \langle \exp(-\sqrt{-1}\xi \cdot \gamma) \hat{f}_1, \hat{f}_2 \rangle = \frac{1}{|\det(A)|} \int_{\mathbb{R}^n/\mathbb{Z}^n} \exp(-2\pi\sqrt{-1}\xi \cdot k) C(f_1, f_2)(\xi) d\xi$$

, where $\gamma = Ak$, $k \in \mathbb{Z}^n$, and

$$C(f_1, f_2)(\xi) = \sum_{\gamma^* \in \Gamma^*} f_1(2\pi \widehat{A^* \xi} + 2\pi \gamma^*) \overline{f_2(2\pi \widehat{A^* \xi} + 2\pi \gamma^*)}.$$

Proof. We have a formula

$$\begin{aligned} \langle \exp(-\sqrt{-1}\xi \cdot Ak) \hat{f}_1, \hat{f}_2 \rangle &= \int_{\mathbb{R}^n} \exp(-\sqrt{-1}A^t \xi \cdot k) \widehat{f_1(\xi)} \overline{\widehat{f_2(\xi)}} d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} \exp(-2\pi\sqrt{-1}\xi \cdot Ak) \widehat{f_1(2\pi\xi)} \overline{\widehat{f_2(2\pi\xi)}} d\xi \\ &= \frac{1}{|\det(A)|} \int_{\mathbb{R}^n/\mathbb{Z}^n} \exp(-2\pi\sqrt{-1}\xi \cdot k) C(f_1, f_2)(\xi) d\xi \quad \square \end{aligned}$$

Note that the function $\widetilde{C(f_1, f_2)}(\xi) = C(f_1, f_2)\left(\frac{A^t \xi}{2\pi}\right)$ is $2\pi\Gamma^*$ - periodic.

Theorem 1 Let $\varphi(x) \in L^2(\mathbb{R}^n)$. Then a system

$$\{2^{\frac{n_j}{2}} \varphi(2^j x - \gamma); \gamma \in \Gamma\}$$

is an orthonormal basis of $V_j (j \in \mathbb{Z})$ if and only if

$$\widetilde{C(f_1, f_2)}(\xi) = |\det(A)| \quad \text{a.a. } \xi \in \mathbb{R}^n.$$

Proof. It is sufficient to prove for V_0 . We have a formula,

$$\langle \varphi(x - \gamma_1), \varphi(x - \gamma_2) \rangle = \left(\frac{1}{2\pi}\right)^n \langle \exp(-\sqrt{-1}\xi \cdot (\gamma_1 - \gamma_2)) \hat{\varphi}, \hat{\varphi} \rangle \quad (3)$$

With $\gamma_j = Ak_j (j = 1, 2)$, by Lemma 1, the right hand side of equation (3) is equal to

$$\frac{1}{|\det(A)|} \int_{\mathbb{R}^n/\mathbb{Z}^n} \exp(-2\pi\sqrt{-1}\xi \cdot (k_1 - k_2)) \widetilde{C(\varphi, \varphi)}(2\pi A^* \xi) d\xi.$$

If $\widetilde{C(\varphi, \varphi)}(\xi) = |\det(A)|$ a.a., then the left hand side of equation (3) is equal to $\delta(\gamma_1, \gamma_2)$.

Conversely, let the left hand side of equation (3) $= \delta(\gamma_1, \gamma_2)$.

Put

$$\widetilde{C(\varphi, \varphi)}(2\pi A^* \xi) = \sum_{l \in \mathbb{Z}^n} a(l) \exp(2\pi \xi \cdot l)$$

then the right hand side of equation (3)

$$= \frac{1}{|\det(A)|} \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}^n / \mathbb{Z}^n} \exp(-2\pi \sqrt{-1} \xi \cdot (k_1 - k_2 - l)) d\xi = \frac{a(k_1 - k_2)}{|\det(A)|}.$$

Hence, we get $a(0) = |\det(A)|$, and $a(l) = 0$, $l \neq 0$,

i.e. $\widetilde{C(\varphi, \varphi)}(\xi) = |\det(A)|$, a.a. $\xi \in \mathbb{R}^n$. $\square$

Corollary 1 When a system $\{\varphi(x - \gamma); \gamma \in \Gamma\}$ is an orthonormal basis of V_0 , we have a formula

$$\sum_{\eta \in E} |m_0(\xi + \pi A^* \eta)|^2 = 1, \text{ for almost all } \xi \in \mathbb{R}^n, \quad (4)$$

, where $E = \{0, 1\}^n$.

Proof.

$$\begin{aligned} \widetilde{C(\varphi, \varphi)}(2\xi) &= |\det(A)| \\ &= \sum_{\gamma^* \in \Gamma^*} |\widehat{\varphi(2\xi + 2\pi\gamma^*)}|^2 \\ &= \sum_{\gamma^* \in \Gamma^*} |m_0(\xi + \pi\gamma^*)|^2 |\widehat{\varphi(\xi + \pi\gamma^*)}|^2 \\ &= \sum_{k \in \mathbb{Z}^n} |m_0(\xi + \pi A^* k)|^2 |\widehat{\varphi(\xi + \pi A^* k)}|^2 \\ &= \sum_{\eta \in E} |m_0(\xi + \pi A^* \eta)|^2 |\det(A)|. \end{aligned}$$

$\square$

Now let $\{g(x - \gamma); \gamma\}$ be a Riesz basis of V_0 .

$$C_1 |\det(A)| \leq \widetilde{C(g, g)}(\xi) \leq C_2 |\det(A)|, \text{ a.a. } \xi \in \mathbb{R}^n$$

,with $0 < C_1 \leq C_2$.

Define $\varphi(x)$ as

$$\widehat{\varphi(\xi)} = \sqrt{|\det(A)|} \frac{\widehat{g(\xi)}}{\sqrt{C(g,g)(\xi)}}.$$

Then, the system $\langle 2^{\frac{n}{2}j} \varphi(2^j x - \gamma) \mid \gamma \in \Gamma \rangle$ is an orthonormal basis of $V_j (j \in \mathbb{Z})$ by Theorem 1.

Our aim is to decompose V_{j+1} as $V_{j+1} = V_j \oplus W_j (j \in \mathbb{Z})$.

At first we consider the decomposition $V_1 = V_0 \oplus W_0$.

Let $\psi_\varepsilon(x) \in V_1$, $\varepsilon \in E$ and put

$$\widehat{\psi_\varepsilon(2\xi)} = m_\varepsilon(\xi) \widehat{\varphi(\xi)}.$$

, where $\psi_{(0,\dots,0)}(x) = \varphi(x)$ and $m_\varepsilon(\xi) (\varepsilon \in E)$ are $2\pi\Gamma^*$ -periodic.

Put

$$m_\varepsilon(\xi) = \sum_{\eta \in E} \exp(-\sqrt{-1}\xi \cdot A\eta) m_{\varepsilon,\eta}(2\xi)$$

With these notations, we have

Theorem 2 *The system $\langle \psi_\varepsilon(x - \gamma) \mid \gamma \in \Gamma, \varepsilon \in E \rangle$ is an orthonormal basis if and only if the matrix*

$$U(\xi) = 2^{\frac{n}{2}} m_{\varepsilon,\eta}(\xi)_{(\varepsilon,\eta) \in E^2} \quad (5)$$

is unitary for a.a. $\xi \in \mathbb{R}^n$.

In this case, define

$$W_{(0,\eta)} = \overline{\langle \psi_\eta(x - \gamma); \gamma \in \Gamma \rangle} \quad (\eta \in E),$$

and

$$W_0 = \bigoplus_{\eta \in E \setminus \{0\}} W_{(0,\eta)}.$$

Proof. By the Plancherel formula, for $\psi_\varepsilon(x - \gamma_1), \psi_\eta(x - \gamma_2)$ we have a formula

$$\begin{aligned} \langle \psi_\varepsilon(x - \gamma_1), \psi_\eta(x - \gamma_2) \rangle &= \left(\frac{1}{2\pi}\right)^n \langle \exp(-\sqrt{-1}\xi \cdot (\gamma_1 - \gamma_2)) \hat{\psi}_\varepsilon, \hat{\psi}_\eta \rangle \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \langle \exp(-\sqrt{-1}\xi \cdot (\gamma_1 - \gamma_2)) m_\varepsilon\left(\frac{\xi}{2}\right) \overline{m_\eta\left(\frac{\xi}{2}\right)} \widehat{|\varphi\left(\frac{\xi}{2}\right)|^2} d\xi \\ &= 2^n |\det(A)| \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n/2\pi\Gamma^*} \exp(-i\xi \cdot 2(\gamma_1 - \gamma_2)) m_\varepsilon(\xi) \overline{m_\eta(\xi)} d\xi \end{aligned}$$

Thus it is sufficient to consider the integral $I := \int_{\mathbb{R}^n/2\pi\Gamma^*} m_\varepsilon(\xi) \overline{m_\eta(\xi)} d\xi$

Now, for the integral I , we have

$$\begin{aligned} I &= \sum_{\varepsilon'} \sum_{\eta'} \int_{\mathbb{R}^n/2\pi\Gamma^*} \exp(-i\xi \cdot A(\varepsilon' - \eta')) [m_{(\varepsilon, \varepsilon')}(2\xi) \overline{m_{(\eta, \eta')}(2\xi)}] d\xi \\ &= \sum_{\varepsilon'} \sum_{\eta'} \frac{1}{|\det(A)|} \int_{\mathbb{R}^n/2\pi\mathbb{Z}^n} \exp(-i\xi \cdot (\varepsilon' - \eta')) [m_{(\varepsilon, \varepsilon')}(2A^*\xi) \overline{m_{(\eta, \eta')}(2A^*\xi)}] d\xi \end{aligned}$$

, whence we get the result. $\square$

On the other hand, we have a formula

$$m_\varepsilon(\xi + \pi A^* \eta) = \sum_{\eta' \in E} \exp(-i\xi \cdot -i\pi \eta \cdot \eta') m_{(\varepsilon, \eta')}(2\xi),$$

where $\eta \in E$. Define the matrix Λ ,

$$\Lambda = ((\exp(-\sqrt{-1}\pi \varepsilon \cdot \eta))),$$

then we have the identity $\Lambda \Lambda^* = 2^n$.

We have also the equation

$$U(\xi) = 2^{\frac{n}{2}} \text{diag}((\exp(\sqrt{-1}\xi \cdot A\varepsilon')) \Lambda^{-1}((m_\varepsilon(\xi + \pi A^* \eta))).$$

Thus, we get

Corollary 2 *The system $\langle \psi_\varepsilon(x - \gamma); \gamma \in \Gamma, \varepsilon \in E \rangle$ is an orthonormal basis of V_1 if and only if the matrix*

$$(m_\varepsilon(\xi + \pi A^* \eta))_{(\varepsilon, \eta) \in E^2}$$

is unitary for almost all $\xi \in \mathbb{R}^n$.

Example Let $A = (a_{kj}) \in GL(n; \mathbb{R})$ and $\vec{a_j} = (a_{kj})_{1 \leq k \leq n}$ ($j = 1, \dots, n$). Let $g(x)$ be the characteristic function of the fundamental domain Ω_A of the associated lattice Γ_A .

We have its Fourier transform $\widehat{g(\xi)}$;

$$\widehat{g(\xi)} = |\det(A)| \exp\left(-\frac{1}{2}\sqrt{-1} \sum_{k=1}^n \xi_k \left(\sum_{j=1}^n a_{jk}\right)\right) \prod_{j=1}^n \frac{\sin\left(\frac{\sum_{k=1}^n a_{jk}\xi_k}{2}\right)}{\frac{\sum_{k=1}^n a_{jk}\xi_k}{2}}.$$

Let $\varphi(x) = \frac{g(x)}{\sqrt{|\det(A)|}}$, then the system $\langle \varphi(x - \gamma) \mid \gamma \in \Gamma \rangle$ is an orthonormal basis of V_0 . Here we have the expression

$$m_0(\xi) = \exp\left(-\frac{1}{2}\sqrt{-1} \sum_{k=1}^n \xi_k \left(\sum_{j=1}^n a_{jk}\right)\right) \prod_{j=1}^n \cos\left(\frac{\sum_{k=1}^n a_{jk}\xi_k}{2}\right).$$

The factor $m_u(\xi) = \prod_{j=1}^n \cos\left(\frac{\sum_{k=1}^n a_{jk}\xi_k}{2}\right)$ corresponds to the characteristic function of the domain obtained translating Ω_A to the origin $\delta_A/2$, where δ_A is the diagonal of Ω_A .

Let $n = 2$, and

$$\eta_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \eta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Put $\xi = (\xi_1, \xi_2)^t$ and define

$$\begin{aligned} m_0(\xi) &= m_0(\xi + \pi A^* \eta_0) \\ m_1(\xi) &= \sqrt{-1} \exp(-\sqrt{-1} \xi \cdot A \eta_3) m_0(\xi + \pi A^* \eta_1) \\ m_2(\xi) &= \sqrt{-1} \exp(-\sqrt{-1} \xi \cdot A \eta_2) m_0(\xi + \pi A^* \eta_2) \\ m_3(\xi) &= -\exp(-\sqrt{-1} \xi \cdot A \eta_1) m_0(\xi + \pi A^* \eta_3) \end{aligned}$$

Then the system

$$\langle \psi_0(x - \gamma), \psi_1(x - \gamma), \psi_2(x - \gamma), \psi_3(x - \gamma); \gamma \in \Gamma, x \in \mathbb{R}^2 \rangle$$

defined as follows, is an orthonormal basis of V_1 , where $\psi_0(x) = \varphi(x)$ and

$$\langle \psi_1(x - \gamma), \psi_2(x - \gamma), \psi_3(x - \gamma); \gamma \in \Gamma, x \in \mathbb{R}^2 \rangle$$

is an orthonormal basis of W_0 ; for $j = 0, 1, 2, 3$

$$\widehat{\psi_j(2\xi)} = m_j(\xi)\widehat{\varphi(\xi)}.$$

Remark : The factors $\exp(-\sqrt{-1}\xi \cdot A\eta_3)$, $\exp(-\sqrt{-1}\xi \cdot A\eta_2)$, and $\exp(-\sqrt{-1}\xi \cdot A\eta_1)$ in $m_1(\xi)$, $m_2(\xi)$, and $m_3(\xi)$ could be of forms

$$\exp(-\sqrt{-1}\xi \cdot Ak_1), \exp(-\sqrt{-1}\xi \cdot Ak_2), \text{ and } \exp(-\sqrt{-1}\xi \cdot Ak_3)$$

(,respectively) as long as column vectors $k_j \in \mathbb{Z}^2$ ($j = 1, 2, 3$) .

With the form $m_j(\xi) = \exp(-\frac{\xi \cdot A\eta_3}{2})m_{u,j}(\xi)$ ($j = 0, 1, 2, 3$) , we describe some examples ;

Type B_2 :

$$m_{u,0}(\xi) = \cos(\frac{\xi_1 - \xi_2}{2}) \cos(\frac{\xi_2}{2})$$

$$m_{u,1}(\xi) = -\sqrt{-1} \exp(-\sqrt{-1}\xi_2) \cos(\frac{\xi_1 - \xi_2}{2}) \sin(\frac{\xi_2}{2})$$

$$m_{u,2}(\xi) = -\sqrt{-1} \exp(-\sqrt{-1}(\xi_1 - \xi_2)) \sin(\frac{\xi_1 - \xi_2}{2}) \cos(\frac{\xi_2}{2})$$

$$m_{u,3}(\xi) = -\exp(-\sqrt{-1}\xi_1) \sin(\frac{\xi_1 - \xi_2}{2}) \sin(\frac{\xi_2}{2})$$

Type C_2 :

$$m_{u,0}(\xi) = \cos(\frac{\xi_1 - \xi_2}{2}) \cos(\xi_2)$$

$$m_{u,1}(\xi) = -\sqrt{-1} \exp(-\sqrt{-1}(\xi_1 + \xi_2)) \cos(\frac{\xi_1 - \xi_2}{2}) \sin(\xi_2)$$

$$m_{u,2}(\xi) = -\sqrt{-1} \exp(-\sqrt{-1}(\xi_1 - \xi_2)) \sin(\frac{\xi_1 - \xi_2}{2}) \cos(\xi_2)$$

$$m_{u,3}(\xi) = -\exp(-2\sqrt{-1}\xi_2) \sin(\frac{\xi_1 - \xi_2}{2}) \sin(\xi_2)$$

Type D_2 :

$$m_{u,0}(\xi) = \cos(\frac{\xi_1 - \xi_2}{2}) \cos(\frac{\xi_1 + \xi_2}{2})$$

$$m_{u,1}(\xi) = -\sqrt{-1} \exp(-2\sqrt{-1}\xi_1) \cos(\frac{\xi_1 - \xi_2}{2}) \sin(\frac{\xi_1 + \xi_2}{2})$$

$$m_{u,2}(\xi) = -\sqrt{-1} \exp(-\sqrt{-1}(\xi_1 - \xi_2)) \sin(\frac{\xi_1 - \xi_2}{2}) \cos(\frac{\xi_1 + \xi_2}{2})$$

$$m_{u,3}(\xi) = -\exp(-\sqrt{-1}(\xi_1 + \xi_2)) \sin(\frac{\xi_1 - \xi_2}{2}) \sin(\frac{\xi_1 + \xi_2}{2})$$

Type G_2 :

$$\begin{aligned}
 m_{u,0}(\xi) &= \cos\left(\frac{\xi_1}{2}\right) \cos\left(\frac{3\xi_1 - \sqrt{3}\xi_2}{4}\right) \\
 m_{u,1}(\xi) &= \sqrt{-1} \exp\left(\sqrt{-1} \frac{\xi_1 - \sqrt{3}\xi_2}{2}\right) \cos\left(\frac{\xi_1}{2}\right) \sin\left(\frac{3\xi_1 - \sqrt{3}\xi_2}{4}\right) \\
 m_{u,2}(\xi) &= -\sqrt{-1} \exp(-\sqrt{-1}\xi_1) \sin\left(\frac{\xi_1}{2}\right) \cos\left(\frac{3\xi_1 - \sqrt{3}\xi_2}{4}\right) \\
 m_{u,3}(\xi) &= \exp\left(\sqrt{-1} \frac{3\xi_1 - \sqrt{3}\xi_2}{2}\right) \sin\left(\frac{\xi_1}{2}\right) \sin\left(\frac{3\xi_1 - \sqrt{3}\xi_2}{4}\right)
 \end{aligned}$$

□

Let

$$\begin{aligned}
 V_0 &= \overline{\langle \psi_0(x - \gamma); \gamma \in \Gamma \rangle} \\
 W_{(0,j)} &= \overline{\langle \psi_j(x - \gamma); \gamma \in \Gamma \rangle} \quad (j = 1, 2, 3) \\
 W_0 &= \bigoplus_{j=1}^3 W_{(0,j)}
 \end{aligned}$$

Then

$$V_1 = V_0 \oplus W_0.$$

Now, in general , with the notation in Corollary 2, for $\eta \in \tilde{E} : E \setminus (0, \dots, 0)$ define

$$W_{(j,\eta)} = \overline{\langle 2^{\frac{n_j}{2}} \psi_\eta(2^j x - \gamma); \gamma \in \Gamma \rangle} \quad (j \in \mathbb{Z}).$$

Then we have an orthogonal decomposition

$$\begin{aligned}
 V_{j+1} &= V_j \bigoplus_{\eta \in \tilde{E}} W_{(j,\eta)} \\
 L^2(\mathbb{R}^n) &= V_0 \bigoplus_{(j \geq 1, \eta \in \tilde{E})} W_{(j,\eta)} \\
 L^2(\mathbb{R}^n) &= \bigoplus_{(j \in \mathbb{Z}, \eta \in \tilde{E})} W_{(j,\eta)}
 \end{aligned}$$

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