

Shape Optimization Problem on the Lateral Boundary for Thermodynamical Phase Separation

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1. Formulation of an optimization problem

This paper is concerned with an optimization problem on the lateral boundary $\partial\Omega$ for a thermodynamical phase separation model formulated in a domain Ω .

Ω is a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with smooth boundary $\partial\Omega$ and T is a fixed positive number. Our state problem $SP(\Gamma)$ is of the form

$$\left\{ \begin{array}{l} \rho(u)_t + \lambda(w)_t - \Delta u = f \quad \text{in } Q := (0, T) \times \Omega, \\ w_t - \Delta \{-\mu \Delta w_t - \kappa \Delta w + \xi + g(w) - \lambda'(w)u\} = 0 \quad \text{in } Q, \\ \xi \in \beta(w) \quad \text{in } Q, \\ u = h_D \quad \text{on } \Sigma_D := (0, T) \times \Gamma, \\ \frac{\partial u}{\partial n} + n_0 u = h_N \quad \text{on } \Sigma_N := (0, T) \times \Gamma', \quad \Gamma' := \partial\Omega \setminus \Gamma, \\ \frac{\partial w}{\partial n} = 0, \quad \frac{\partial}{\partial n} \{-\mu \Delta w_t - \kappa \Delta w + \xi + g(w) - \lambda'(w)u\} = 0 \quad \text{on } \Sigma := (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0, w(0, \cdot) = w_0 \quad \text{in } \Omega. \end{array} \right.$$

Throughout this paper, we use the following notation.

For a general (real) Banach space Y , we denote by $|\cdot|_Y$ the norm in Y and by Y^* the dual of Y . Also, for a positive finite number T , we denote by $C_w([0, T]; Y)$ the space of all weakly continuous functions $u : [0, T] \rightarrow Y$, and by definition " $u_n \rightarrow u$ in $C_w([0, T]; Y)$ as $n \rightarrow +\infty$ " means that for each $z^* \in Y^*$, $\langle z^*, u_n(t) \rangle_{Y^*, Y}$ converges to $\langle z^*, u(t) \rangle_{Y^*, Y}$ uniformly in $t \in [0, T]$ as $n \rightarrow +\infty$, where $\langle \cdot, \cdot \rangle_{Y^*, Y}$ is the duality pairing between Y^* and Y .

For simplicity we put

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad H_0 := \{v \in H; \int_{\Omega} v dx = 0\}, \quad V_0 := V \cap H_0,$$

and

$$\Pi := \{\Gamma \subset \partial\Omega; \Gamma \text{ is compact in } \partial\Omega, \sigma(\Gamma) > 0\}.$$

For each $\Gamma \in \Pi$, we put

$$V(\Gamma) := \{z \in V; z = 0 \text{ a.e. on } \Gamma\}$$

which is a closed subspace of V , and

$$\begin{aligned} (v, w) &:= \int_{\Omega} v w dx && \text{for } v, w \in H, \\ (v, w)_{\partial\Omega} &:= \int_{\partial\Omega} v w d\sigma && \text{for } v, w \in L^2(\partial\Omega), \\ a(v, w) &:= \int_{\Omega} \nabla v \cdot \nabla w dx && \text{for } v, w \in V. \end{aligned}$$

In general, given a subset E of $\overline{\Omega}$, χ_E denotes the characteristic function of E defined on $\overline{\Omega}$.

We now introduce a notion of convergence in Π . By definition, a sequence $\{\Gamma_n\} \subset \Pi$ converges to $\Gamma \in \Pi$, denoted by $\Gamma_n \rightarrow \Gamma$ in Π as $n \rightarrow +\infty$, if the following conditions (C1) – (C3) are satisfied:

- (C1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in V(\Gamma_{n_k})$ and $z_k \rightarrow z$ weakly in V as $k \rightarrow +\infty$, then $z \in V(\Gamma)$.
- (C2) For any $z \in V(\Gamma)$, there is a sequence $\{z_n\} \subset V$ such that $z_n \in V(\Gamma_n)$, $n = 1, 2, \dots$, and $z_n \rightarrow z$ in V as $n \rightarrow +\infty$.
- (C3) $\chi_{\Gamma_n} \rightarrow \chi_{\Gamma}$ in $L^1(\partial\Omega)$ as $n \rightarrow +\infty$.

Also, a subset Π' of Π is said to have property (C), if Π' is compact in the sense of (C1) – (C3), namely, any sequence $\{\Gamma_n\}$ of Π' contains a subsequence convergent to a certain $\Gamma \in \Pi'$.

We suppose precise assumptions on the data as follows.

- (H1) ρ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ whose domain $D(\rho)$ and range $R(\rho)$ are open in $\mathbb{R}$, and it is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$, and there are constants $A_0 > 0$ and α with $1 \leq \alpha < 2$ such that

$$|\rho(r_1) - \rho(r_2)| \geq \frac{A_0 |r_1 - r_2|}{|r_1 r_2|^\alpha + 1} \quad \text{for all } r_1, r_2 \in D(\rho).$$

- (H2) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $\overline{D(\beta)} = [\sigma_*, \sigma^*]$ for constants σ_* , σ^* with $-\infty < \sigma_* < \sigma^* < +\infty$.

- (H3) λ is a C^2 -function from $\mathbb{R}$ into itself and g is a C^1 -function from $\mathbb{R}$ into itself; λ' is the derivative of λ .

- (H4) (i) $f \in W^{1,2}(0, T; H)$;
(ii) $h_D \in W^{1,2}(0, T; H^{1/2}(\partial\Omega))$ such that there is a function $\tilde{h}_D \in W^{1,2}(0, T; V)$ with $\rho(\tilde{h}_D) \in W^{1,2}(0, T; V)$;

(iii) $h_N \in W^{1,2}(0, T; L^2(\partial\Omega)) \cap L^\infty(\Sigma)$ such that

$$n_0 \inf D(\rho) \leq h_N(t, x) \leq n_0 \sup D(\rho) \quad \text{for a.e. } (t, x) \in \Sigma$$

and there are positive constants A_1 and A'_1 such that

$$\rho(r)(n_0 r - h_N(t, x)) \geq -A_1|r| - A'_1 \quad \text{for all } r \in D(\rho) \text{ and a.e. } (t, x) \in \Sigma.$$

(H5) (i) $u_0 \in V$ such that $\rho(u_0) \in H$ and $u_0 = h_D(0, \cdot)$ a.e. on $\partial\Omega$;

(ii) $w_0 \in H^2(\Omega)$ such that

$$\sigma_* < \frac{1}{|\Omega|} \int_{\Omega} w_0 dx =: m < \sigma^*$$

and $\frac{\partial w_0}{\partial n} = 0$ a.e. on $\partial\Omega$ and there is $\xi_0 \in H$ satisfying

$$\xi_0 \in \beta(w_0) \quad \text{a.e. in } \Omega, \quad -\kappa \Delta w_0 + \xi_0 \in V.$$

Corresponding to functions h_D , h_N and $\Gamma \in \Pi$, we consider the function $h_\Gamma : [0, T] \rightarrow V$ given by

$$\begin{cases} h_\Gamma(t) = h_D(t) & \text{a.e. on } \Gamma, \\ a(h_\Gamma(t), z) + (n_0 h_\Gamma(t) - h_N(t), z)_{\partial\Omega} = 0 & \text{for all } z \in V(\Gamma); \end{cases}$$

note under condition (H4) and $\sigma(\Gamma) \geq \sigma_0$ for a positive constant σ_0 that such a function h_Γ exists in $W^{1,2}(0, T; V)$ and $|h_\Gamma|_{W^{1,2}(0, T; V)} \leq K$ for a certain constant K depending only on quantities in (H4) and σ_0 . Moreover, if $\Gamma_n \rightarrow \Gamma$ in Π as $n \rightarrow +\infty$, then $h_{\Gamma_n} \rightarrow h_\Gamma$ in $C([0, T]; V)$ as $n \rightarrow +\infty$ (cf. [6]).

We now give the weak formulation for state problem $SP(\Gamma)$ for each $\Gamma \in \Pi$.

Definition 1.1. A couple $\{u, w\}$ of functions $u : [0, T] \rightarrow V$ and $w : [0, T] \rightarrow H^2(\Omega)$ is called a (weak) solution of $SP(\Gamma)$, if the following properties (w1) – (w4) are fulfilled:

(w1) $u - h_\Gamma \in C_w([0, T]; V(\Gamma))$, $\rho(u) \in C_w([0, T]; H)$, $\rho(u)' \in L^2(0, T; V(\Gamma)^*)$,

$w \in C_w([0, T]; H^2(\Omega))$ with $\frac{\partial w(t)}{\partial n} = 0$ a.e. on $\partial\Omega$ for all $t \in [0, T]$, and $w' \in L^2(0, T; H)$.

(w2) $u(0) = u_0$ and $w(0) = w_0$.

(w3) For all $z \in V(\Gamma)$ and a.e. $t \in [0, T]$,

$$\frac{d}{dt}(\rho(u)(t) + \lambda(w)(t), z) + a(u(t), z) + n_0(u(t) - h_\Gamma(t), z)_{\partial\Omega} = (f(t), z).$$

(w4) There exists a function $\xi \in L^2(0, T; H)$ such that $\xi \in \beta(w)$ a.e. in Q and

$$\begin{aligned} \frac{d}{dt}(w(t), \eta - \mu \Delta \eta) + \kappa(\Delta w(t), \Delta \eta) - (g(w(t)) + \xi(t) - \lambda'(w(t))u(t), \Delta \eta) &= 0 \\ \text{for all } \eta \in H^2(\Omega) \text{ with } \frac{\partial \eta}{\partial n} &= 0 \text{ a.e. on } \partial\Omega \text{ and a.e. } t \in [0, T]. \end{aligned}$$

According to a result [5, Theorem 2.2], problem $SP(\Gamma)$ has an unique solution $\{u, w\}$ for each $\Gamma \in \Pi$. Based on the solvability of $SP(\Gamma)$, we now propose an optimization problem.

For a given non-empty subset Π_c of Π having property (C), our optimization problem, denoted by $P(\Pi_c)$, is to find a set $\Gamma_* \in \Pi_c$ such that

$$J(\Gamma_*) = \inf_{\Gamma \in \Pi_c} J(\Gamma),$$

where

$$J(\Gamma) := A \int_Q |u_\Gamma - u_d|^2 dxdt + B |w_\Gamma - w_d|_{C(\bar{Q})}^2 + C \int_{\Sigma(\Gamma)} |h_d|^2 d\sigma dt \quad \Gamma \in \Pi_c,$$

A, B, C are positive constants, u_d, w_d, h_d are given in $L^2(Q), C(\bar{Q}), L^2(\Sigma)$, respectively, and $\{u_\Gamma, w_\Gamma\}$ is the solution of state problem $SP(\Gamma)$; $d\sigma$ stands for the surface element on $\partial\Omega$.

Our main results are stated as follows.

Theorem 1.1. *Let Π_c be a non-empty subset of Π having property (C). Then, optimization problem $P(\Pi_c)$ has at least one solution $\Gamma_* \in \Pi_c$.*

The above existence result is obtained from the following theorem on the continuous dependence of the solution $\{u_\Gamma, w_\Gamma\}$ of $SP(\Gamma)$ upon $\Gamma \in \Pi$.

Theorem 1.2. *Let $\{\Gamma_n\}$ be a sequence in Π such that $\Gamma_n \rightarrow \Gamma$ in Π as $n \rightarrow +\infty$, and $\{u_n, w_n\}$ and $\{u, w\}$ be the solutions of $SP(\Gamma_n)$ and $SP(\Gamma)$, respectively. Then*

$$u_n \rightarrow u \text{ in } C_w([0, T]; V), \quad w_n \rightarrow w \text{ in } C_w([0, T]; H^2(\Omega))$$

as $n \rightarrow +\infty$.

For a detailed proofs, see a forthcoming paper [3].

It is easily seen from Theorem 1.2 that any minimizing sequence $\{\Gamma_n\} \subset \Pi_c$ of the cost functional $J(\cdot)$ on Π_c contains a subsequence convergent to a solution of $P(\Pi_c)$.

2. Regular approximation for $P(\Pi_c)$

In this section, from the numerical point of view we discuss regular approximation of $SP(\Gamma)$ and $P(\Pi_c)$.

At first, we introduce the approximation $\rho^\nu, \beta^\varepsilon$ and χ_Γ^τ for ρ, β and χ_Γ , respectively, which are defined below.

(a) Let $D(\rho) := (r_*, r^*)$ for $-\infty \leq r_* < r^* \leq +\infty$, and choose two families $\{a_\nu; 0 < \nu \leq 1\}$ and $\{b_\nu; 0 < \nu \leq 1\}$ in $D(\rho)$ such that

$$r_* < a_\nu < a_{\nu'} < a_1 < b_1 < b_{\nu'} < b_\nu < r^* \quad \text{if } 0 < \nu < \nu' < 1$$

and

$$a_\nu \downarrow r_*, \quad b_\nu \uparrow r^* \quad \text{as } \nu \downarrow 0.$$

Then, $\rho^\nu : \mathbb{R} \rightarrow \mathbb{R}$ is defined for each $\nu \in (0, 1]$ by

$$\rho^\nu(r) := \begin{cases} \rho(b_\nu) + r - b_\nu & \text{for } r > b_\nu, \\ \rho(r) & \text{for } a_\nu \leq r \leq b_\nu, \\ \rho(a_\nu) + r - a_\nu & \text{for } r < a_\nu. \end{cases}$$

(b) For each $0 < \varepsilon \leq 1$, β^ε is the Yosida-approximation of β , namely,

$$\beta^\varepsilon(r) := \frac{r - (I + \varepsilon\beta)^{-1}r}{\varepsilon}, \quad r \in \mathbb{R}.$$

(c) Let $\{\chi_\Gamma^\tau\} := \{\chi_\Gamma^\tau; 0 < \tau \leq 1, \Gamma \in \Pi_c\}$ be a family of smooth functions on $\partial\Omega$ and suppose that it satisfies the following properties ($\chi 1$) – ($\chi 3$):

($\chi 1$) $0 \leq \chi_\Gamma \leq \chi_\Gamma^\tau \leq 1$; $\text{supp}(\chi_\Gamma^\tau) \subset \{x \in \partial\Omega; \text{dist}(x, \Gamma) \leq \tau\}$ for all $\tau \in (0, 1]$ and $\Gamma \in \Pi_c$.

($\chi 2$) For each $\tau \in (0, 1]$, $\{\chi_\Gamma^\tau; \Gamma \in \Pi_c\}$ is compact in $L^1(\partial\Omega)$.

($\chi 3$) Let $V(\tau, \Gamma) := \{z \in V; \chi_\Gamma^\tau z = 0 \text{ a.e. on } \Gamma\}$ for each $\tau \in (0, 1]$ and $\Gamma \in \Pi_c$. If $\tau_n \downarrow 0$ and $\Gamma_n \in \Pi_c$, then there are a subsequence $\{n_k\}$ of $\{n\}$ and $\Gamma \in \Pi_c$ such that $\chi_{\Gamma_{n_k}}^{\tau_{n_k}} \rightarrow \chi_\Gamma$ in $L^1(\partial\Omega)$ as $k \rightarrow \infty$, and $V(\tau_{n_k}, \Gamma_{n_k}) \rightarrow V(\Gamma)$ in V as $k \rightarrow \infty$ in the sense of Mosco [6].

Now we propose a regular approximation for $SP(\Gamma)$, referred as $SP(\Gamma)^{\nu\varepsilon\tau\delta}$, $\nu, \varepsilon, \tau, \delta \in (0, 1]$, by the penalty method:

$$\begin{cases} \rho^\nu(u)_t + \lambda(w)_t - \Delta u = f & \text{in } Q, \\ w_t - \Delta(-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) = 0 & \text{in } Q, \\ \frac{\partial u}{\partial n} = -\frac{\chi_\Gamma^\tau}{\delta}(u - h_D) + (1 - \chi_\Gamma^\tau)(h_N - n_0 u) & \text{on } \Sigma, \\ \frac{\partial w}{\partial n} = 0, \quad \frac{\partial}{\partial n}(-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) = 0 & \text{on } \Sigma, \\ u(0) = u_{0\nu} := \min\{\max\{u_0, a_\nu\}, b_\nu\}, \quad w(0) = w_0 & \text{in } \Omega. \end{cases}$$

The notion of a weak solution of $SP(\Gamma)^{\nu\varepsilon\tau\delta}$ is given below.

Definition 2.1. A couple $\{u, w\}$ of functions $u : [0, T] \rightarrow V$ and $w : [0, T] \rightarrow H^2(\Omega)$ is called a solution of $SP(\Gamma)^{\nu\varepsilon\tau\delta}$, if the following conditions (w1)' – (w4)' are satisfied:

(w1)' $u \in W^{1,2}(0, T; H) \cap C([0, T]; V)$,

$w \in W^{1,2}(0, T; H) \cap C_w([0, T]; H^2(\Omega))$ with $\frac{\partial w(t)}{\partial n} = 0$ a.e. on $\partial\Omega$ for all $t \in [0, T]$.

(w2)' $u(0) = u_{0\nu}$, $w(0) = w_0$.

(w3)' For all $z \in V$ and a.e. $t \in [0, T]$,

$$\begin{aligned} &(\rho^\nu(u)'(t) + \lambda(w)'(t), z) + a(u(t), z) \\ &+ \left(\frac{\chi_\Gamma^\tau}{\delta}(u(t) - h_D(t)) - (1 - \chi_\Gamma^\tau)(h_N(t) - n_0 u(t)), z\right)_{\partial\Omega} = (f(t), z). \end{aligned}$$

(w4)' For all $\eta \in H^2(\Omega)$ with $\frac{\partial \eta}{\partial n} = 0$ a.e. on $\partial\Omega$ and a.e. $t \in [0, T]$,

$$(w'(t), \eta - \mu\Delta\eta) + \kappa(\Delta w(t), \Delta\eta) - (g(w(t)) + \beta^\varepsilon(w(t)) - \lambda'(w(t))u(t), \Delta\eta) = 0.$$

According to a result in [4], $SP(\Gamma)^{\nu\varepsilon\tau\delta}$ has a unique solution $\{u, w\}$. Our regular approximate optimization problem $P(\Pi_c)^{\nu\varepsilon\tau\delta}$ is to find $\Gamma_*^{\nu\varepsilon\tau\delta} \in \Pi_c$ such that

$$J^{\nu\varepsilon\tau\delta}(\Gamma_*^{\nu\varepsilon\tau\delta}) = \inf_{\Gamma \in \Pi_c} J^{\nu\varepsilon\tau\delta}(\Gamma),$$

where

$$J^{\nu\varepsilon\tau\delta}(\Gamma) := A \int_Q |u - u_d|^2 dx dt + B |w - w_d|_{C(\bar{Q})}^2 + C \int_{\Sigma} (1 - \chi_{\Gamma}^{\tau}) |h_d|^2 d\sigma dt,$$

$\{u, w\}$ is the solution of $SP(\Gamma)^{\nu\varepsilon\tau\delta}$.

Finally, we show a convergence result.

Theorem 2.1. *Let Π_c , $\{\rho^{\nu}\}$, $\{\beta^{\varepsilon}\}$, $\{\chi_{\Gamma}^{\tau}\}$ be as above. Then:*

- (1) *For $\nu, \varepsilon, \tau, \delta \in (0, 1]$, $P(\Pi_c)^{\nu\varepsilon\tau\delta}$ has at least one solution $\Gamma_*^{\nu\varepsilon\tau\delta} \in \Pi_c$.*
- (2) *Let $\{\nu_n\}$, $\{\varepsilon_n\}$, $\{\tau_n\}$ and $\{\delta_n\}$ be any null sequences and let $\{\Gamma_n := \Gamma_*^{\nu_n \varepsilon_n \tau_n \delta_n}\}$ be a sequence of solutions of $P(\Pi_c)^{\nu_n \varepsilon_n \tau_n \delta_n}$. Then, $\{\Gamma_n\}$ contains a subsequence convergent in Π and any limit Γ_* is a solution of $P(\Pi_c)$.*

For a detailed proof, see a forthcoming paper [3].

References

- [1] H. W. Alt and I. Pawlow, Existence of solutions for non-isothermal phase separation, Adv. Math. Soc. Appl. 1(1992), 319 - 409.
- [2] H. Brézis, M. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets, Comm. Pure Appl. Math. 23(1970), 123 - 144.
- [3] A. Kadoya, N. Kato, N. Kenmochi and J. Shirohzu, Optimal control for a class of phase separation models, in preparation.
- [4] N. Kenmochi and M. Niezgódka, Nonlinear System for non-isothermal diffusive phase Separation, J. Math Anal. Appl. 188(1994), 651 - 679.
- [5] N. Kenmochi and M. Niezgódka, Viscosity approach for modelling non-isothermal diffusive phase separation, Japan J. Industrial Appl. Math., (1996), to appear.
- [6] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, Advances Math., 3(1969), 510-585.
- [7] O. Penrose and P. C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transitions, Physica D, 13(1990), 44-62.
- [8] J. Shirohzu, N. Sato and N. Kenmochi, Asymptotic convergence in models for phase change problems, in Nonlinear Analysis and Applications, pp.361-385, GAKUTO Inter. Ser. Math. Sci. Appl., Vol. 7, Gakkōtoshō, Tokyo, 1995.