

CENTRALIZING GROUP-LIKE OBJECTS IN TENSOR CATEGORIES AND THE INVARIANT χ

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In this note, we shall present a tensor-categorical interpretation of the invariant χ for subfactors, which can be applied to compute the invariant for group-subgroup subfactors.

Automorphisms in Subfactors

Given a subfactor $N \subset M$, set

$$Aut(M, N) = \{\theta \in Aut(M); \theta(N) = N\},$$

$$Int(M, N) = \{Adu \in Aut(M, N); u \text{ is a unitary in } N\}.$$

Each $\theta \in Aut(M, N)$ is inductively extended to automorphisms of the Jones tower

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

by

$$\theta(e_i) = e_i, \quad i = 1, 2, 3, \dots$$

and hence induces

$$Loi(\theta) = \text{the family of induced automorphisms on } N' \cap M \subset N' \cap M_1 \subset \cdots.$$

Remark. We can use automorphisms on $M' \cap M_1 \subset M' \cap M_1 \subset \cdots$ as well, which contains the equivalent information.

Theorem (Popa, Loi). *Let M, N be AFD II_1 -factors and $N \subset M$ be amenable. Then for $\theta \in Aut(M, N)$,*

- (i) *θ is centrally trivial iff θ is inner at some M_k , i.e., $\exists 0 \neq u \in M_k$ such that $\theta(x)u = ux$ for $x \in M$.*
- (ii) *$Loi(\theta) = 1$ iff $\theta \in \overline{Int(M, N)}$.*

According to Y. Kawahigashi, we define the group

$$\chi(M, N) = \frac{Cnt(M, N) \cap \overline{Int(M, N)}}{Int(M, N)}$$

as the χ -invariant for subfactors.

Theorem (Kawahigashi). *For subfactors of index < 4 ,*

$$\chi = \begin{cases} \mathbb{Z}_2 & \text{for } A_{2n+1} \ (n \geq 2) \text{ and } E_6, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } A_3, \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{for } D_4, \\ 0 & \text{otherwise.} \end{cases}$$

Interpretations with bimodules

For a factor N , the correspondance

$$\alpha \in \text{Aut}(N) \rightsquigarrow X_\alpha = {}_N L^2(N) \alpha_N$$

induces

$$\begin{aligned} X_\alpha^* &= X_{\alpha^{-1}} \\ X_\alpha \otimes^N X_\beta &= X_{\alpha\beta}. \end{aligned}$$

Here the right N -action in X_α is modified by α compared to the standard bimodule $L^2(N)$.

The bimodule X_α satisfies $X_\alpha \otimes X_\alpha^* \cong {}_N L^2(N)_N$. The converse is not always true: Let R be an AFD II_1 -factor, e be a non-trivial projection in R and $v : R \rightarrow eRe$ be an isomorphism. Then the bimodule $X = {}_{ReL^2(R)}_R$ gives an example, where the right action is induced from the isomorphism φ .

Theorem. *Let X be an N - N bimodule such that $X \otimes X^* \cong L^2(N)$ and consider one of the following cases.*

- (i) N is properly infinite.
- (ii) X is a descendent of an irreducible bimodule Z of finite index.

Then $\exists \alpha \in \text{Aut}(N)$ such that $X \cong X_\alpha$.

The bimodule $X_\alpha = {}_N L^2(N) \alpha_N$ is simply denoted by α in the following.

For a bimodule ${}_A X_B$ and $\alpha \in \text{Aut}(A)$, $\beta \in \text{Aut}(B)$, set

$$\alpha X \beta = \alpha \otimes X \otimes \beta$$

and

$$\text{Out}(A) \times_X \text{Out}(B) = \{([\alpha], [\beta]) \in \text{Out}(A) \times \text{Out}(B); \alpha X \cong X \beta\},$$

a subgroup of $\text{Out}(A) \times \text{Out}(B)$.

Theorem (Kosaki, Choda-Kosaki). *For $Z = {}_N L^2(M)_M$ with $N \subset M$ irreducible, we have*

- (i) $\text{Out}(N) \times_Z \text{Out}(M) \cong \text{Aut}(M, N) / \text{Int}(M, N)$.
- (ii) θ is inner at some M_k iff ${}_M L^2(M) \theta_M$ appears in $Z^* Z \cdots Z^* Z (= (Z^* Z)^k)$.

$\therefore$ (ii) Use $M_3 = \text{End}(ZZ^*Z_M)$ and the Frobenius reciprocity

$$\text{Hom}(\theta Z^* Z Z^*, Z^* Z Z^*) \cong \text{Hom}(L^2(M) \theta, (Z^* Z)^3)$$

for example. $\square$

Theorem (Goto). $Loi(\theta) = 1$ implies $[\theta] = ([\alpha], [\beta]) \in Aut(M, N)/Int(M, N)$ is in the center of fusion algebra, i.e.,

$$\alpha X \cong X\alpha, \quad \beta Y \cong Y\beta, \quad \alpha Z' \cong Z'\beta$$

for descendants ${}_NX_N$, ${}_MY_M$ and ${}_NZ'_M$ of Z .

Conversely the centrality in the fusion algebra forces the triviality of Loi invariant as long as the principal graph (or the dual principal graph) is multiplicity-free.

Theorem. The following are equivalent.

- (i) $Loi(\theta) = 1$.
- (ii) For each bimodule X in the tensor category generated by Z , we can find an isomorphism $I_X : X \rightarrow \theta X \theta^{-1}$ such that $I_{X^*} = \overline{I_X}$, $I_{XY} = I_X \otimes I_Y$, and the diagram

$$\begin{array}{ccc} X & \xrightarrow{I_X} & \theta X \theta^{-1} \\ T \downarrow & & \downarrow \theta T \theta^{-1} \\ Y & \xrightarrow{I_Y} & \theta Y \theta^{-1} \end{array}$$

commutes for $T \in Hom(X, Y)$.

Applications to $\chi(G, H)$

For a subgroup $H \subset G$ of a finite group G with an outer action on an AFD II_1 -factor, set

$$\chi(G, H) = \chi(R \rtimes G, R \rtimes H).$$

According to [KY], irreducible bimodules generated by ${}_{R \rtimes H} L^2(R \rtimes G)_{R \rtimes G}$ are parametrized by

$$\begin{aligned} R \rtimes G - R \rtimes G : & \quad \widehat{G} \\ R \rtimes H - R \rtimes G : & \quad \widehat{H} \\ R \rtimes H - R \rtimes H : & \quad \coprod_{a \in H \backslash G / H} \widehat{H \cap aHa^{-1}}. \end{aligned}$$

Note that the tensor category of $R \rtimes H - R \rtimes H$ bimodules contains the Tannaka dual of H as a subcategory. With this description, we can deduce

$$Cnt(M, N)/Int(M, N) \cong \Xi \times (N_G(H)/H),$$

where

$$\Xi = \{(\chi, \eta) \in H^* \times G^*; \chi = \eta|_H\}$$

and H^* and G^* refer to the group of 1-dimensional representations.

Taking the restriction of centralizing morphisms to the Tannaka dual of H , we can deduce the following.

Theorem. *We have*

$$\Xi \times Z(G)H/H \subset \chi(G, H) \subset \Xi \times \{\dot{c} \in C_G(H)H/H; \dot{c} \text{ acts trivially on } H \backslash G/H\},$$

where $C_G(H)$ denotes the centralizer of H in G and $\dot{a} \in N_G(H)/H$ acts on $H \backslash G/H$ by

$$HgH \mapsto H\dot{a}ga^{-1}H.$$

Corollary.

- (i) $\chi(G, \{e\}) \cong G^* \times Z(G)$.
- (ii) $\chi(A \rtimes H, H) \cong \{(\chi, \eta) \in A^* \times (A \rtimes H)^*; \chi = \eta|_H\} \times A^H$, where A is an abelian group and $A^H = \{a \in A; hah^{-1} = a, \text{ for all } h \in H\}$.
- (iii) $\chi(S_n, S_k) \cong \mathbb{Z}_2$.
- (iv) $\chi(A_n, A_k) = \{e\}$.

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