

A NOTE ON SPACES OF RATIONAL FUNCTIONS

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§1. INTRODUCTION

Spaces of holomorphic maps between complex manifolds have played an important role in topology and mathematical physics. In this note, we shall consider the topology of spaces of holomorphic self-maps of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ and announce the main results of [GKMY].

For a non-negative integer d , let Hol_d be the space of all holomorphic maps from S^2 to S^2 of degree d . Let Hol_d^* be the subspace of Hol_d consisting of all maps which preserve a basepoint of S^2 . The corresponding space of continuous maps of degree d is denoted by Map_d , and the subspace of based maps by Map_d^* . We call the map $f : X \rightarrow Y$ a *homotopy equivalence up to dimension d* if the induced homomorphism $f_* : \pi_j(X) \rightarrow \pi_j(Y)$ is bijective for all $j < d$ and surjective for $j = d$. It is an elementary fact that Hol_d and Hol_d^* are connected spaces. For $d = 1$, it is easy to see that $\text{Hol}_1 \cong \text{PSL}_2(\mathbb{C})$ and $\text{Hol}_1^* \cong \mathbb{C}^* \times \mathbb{C}$. The following general results were obtained by Epshtein, J.D.S Jones and Segal:

Theorem 0 ([Ep], [Se]). *Let d be a positive integer.*

- (1) $\pi_1(\text{Hol}_d) = \mathbb{Z}/2d$.
- (2) $\pi_1(\text{Hol}_d^*) = \mathbb{Z}$.
- (3) *The natural inclusion maps*

$$i_d : \text{Hol}_d^* \rightarrow \text{Map}_d^* \text{ and } j_d : \text{Hol}_d \rightarrow \text{Map}_d$$

are homotopy equivalences up to dimension d . $\square$

The stable homotopy type of Hol_d^* was studied in [CCMM] and in this note we shall extend the above result by determining some further homotopy groups of the spaces Hol_d and Hol_d^* . Our main results are as follows:

Theorem 1.

(1) For $k \geq 2$,

$$\pi_k(\text{Hol}_d) = \begin{cases} \pi_k(S^3) & d = 1 \\ \pi_k(S^3) \oplus \pi_k(S^2) & d = 2 \\ \mathbb{Z}/2 & d \geq 3, k = 2 \end{cases}$$

(2) If $k \geq 3$ and $d \geq 3$, then $\pi_k(\text{Hol}_d) = \pi_k(\text{Hol}_d^*) \oplus \pi_k(S^3)$.

(3) In particular, if $d > k \geq 3$, then $\pi_k(\text{Hol}_d) = \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.

Theorem 2. The space Hol_2 may be identified with a homogeneous space of the form $(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/H$, where H is isomorphic to $\mathbb{C}^* \rtimes \mathbb{Z}/4$. In this semi-direct product, the action of $\mathbb{Z}/4 = \langle \sigma : \sigma^4 = 1 \rangle$ is given by $\sigma \cdot \alpha = \alpha^{-1}$ for $\alpha \in \mathbb{C}^*$. In particular, Hol_2 is homotopy equivalent to $(S^3 \times S^3)/(S^1 \rtimes \mathbb{Z}/4)$.

Theorem 3.

(1) The universal cover of Hol_2^* is homotopy equivalent to S^2 .

(2) The universal cover of Hol_2 may be identified with a homogeneous space of the form $(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/D$, where D is isomorphic to $\mathbb{C}^*$. In particular, it is homotopy equivalent to $S^3 \times S^2$.

In this note, we shall discuss only Theorem 2 and (1) of Theorem 3, referring to [GKMY] for the remaining results.

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§2. SKETCH PROOF OF THEOREM 2

From now on, we identify Hol_d with the space of functions $f = p_1/p_2$, where p_1, p_2 are coprime polynomials such that $\max\{\deg(p_1), \deg(p_2)\} = d$. The group Hol_1 acts on Hol_d by pre- and post-compositions: for $(A, B) \in \text{Hol}_1 \times \text{Hol}_1$ and $f \in \text{Hol}_d$ we have

$$(A, B) \cdot f(z) = A(f(B^{-1}(z))).$$

The following proposition is well known:

Proposition 2.1. The group $\text{Hol}_1 \times \text{Hol}_1$ acts transitively on Hol_2 . $\square$

It is well known that the map

$$SL_2(\mathbb{C}) \rightarrow \text{Hol}_1, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d},$$

is a double covering and induces an isomorphism $PSL_2(\mathbb{C}) \cong \text{Hol}_1$ of Lie groups. Thus the group $G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ acts (transitively) on Hol_2 .

Lemma 2.2. Let H denote the isotropy subgroup of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ at $z^2 \in \text{Hol}_2$. Then

$$H = \left\{ \left(\pm \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right), \left(\pm \begin{pmatrix} 0 & i\alpha^2 \\ i\alpha^{-2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \right) : \alpha \in \mathbb{C}^* \right\}.$$

Proof. This follows by direct calculation. $\square$

It is also easy to establish:

Lemma 2.3. Let $K = \mathbb{C}^* \rtimes \mathbb{Z}/4$ be the group defined by the action of $\mathbb{Z}/4 = \langle \sigma : \sigma^4 = 1 \rangle$ on $\mathbb{C}^*$ by $\sigma \cdot \alpha = \alpha^{-1}$ for $\alpha \in \mathbb{C}^*$. Then H and K are isomorphic Lie groups. $\square$

Proof of Theorem 2. The first part of the Theorem follows easily from Proposition 2.1, Lemma 2.2 and Lemma 2.3. The inclusion map of the maximal compact subgroup $SU(2) = S^3$ of $SL_2(\mathbb{C})$ induces the homotopy equivalence $(S^3 \times S^3)/(S^1 \rtimes \mathbb{Z}/4) \simeq \text{Hol}_2$. $\square$

§3. THE UNIVERSAL COVER OF Hol_2^*

In this section, we shall show that the universal cover of Hol_2^* is homotopy equivalent to S^2 .

Proposition 3.1 ([CS]). There is a fibration $S^1 \rightarrow \text{Hol}_2^* \xrightarrow{\pi} \mathbb{R}P^2$.

Remark. R.Cohen and D.Shimamoto ([CS]) deduce this from results of Donaldson ([D]) and Atiyah and Hitchin ([AH]) on monopoles. Although we give an elementary proof using the homogeneous structure of Hol_2 in [GKMY], we shall give here an alternative direct proof.

Proof. We may identify Hol_2^* with the space of all holomorphic self maps f of $S^2 = \mathbb{C} \cup \{\infty\}$ of degree 2 with basepoint condition $f(\infty) = 0$. Then any $f \in \text{Hol}_2^*$ is of the form

$$f(z) = (az + b)/(z^2 + cz + d)$$

where the polynomials $az + b$ and $z^2 + cz + d$ are coprime, $(a, b) \neq (0, 0)$ and $a, b, c, d \in \mathbb{C}$.

For each pair (U, V) of subspaces of S^2 , let $Q_{m,n}(U, V)$ be the space of all disjoint positive divisors of $Sp^m(U) \times Sp^n(V)$,

$$Q_{m,n}(U, V) = \{(\xi, \eta) \in Sp^m(U) \times Sp^n(V) : \xi \cap \eta = \emptyset\},$$

where $Sp^k(X)$ denotes the k -th symmetric product of X , $Sp^k(X) = X^k / \Sigma_k$. (Σ_k is the symmetric group on k letters.)

The action of $\mathbb{C}^*$ on $S^2 = \mathbb{C} \cup \{\infty\}$ (by multiplication) induces a free action on Hol_2^* and the quotient space $\text{Hol}_2^*/\mathbb{C}^*$ may be identified with $Q_{1,2}(S^2, \mathbb{C})$. It suffices to show that $Q_{1,2}(S^2, \mathbb{C}) \simeq \mathbb{R}P^2$.

Observe that

$$Q_{1,2}(S^2, \mathbb{C}) \simeq Q_{1,2}(S^2, D_-) = Q_{1,2}(\overline{D}_+, D_-) \cup Q_{1,2}(\overline{D}_-, D_-),$$

where $D_{\pm}$ are the open northern and southern hemispheres of $S^2 = \{z \in \mathbb{R}^3 : \|z\| = 1\}$.

The map $u : Q_{1,2}(\overline{D}_+, D_-) \rightarrow Q_{1,2}(\overline{D}_+, \{0\}) \cong \overline{D}_+$ given by

$$(\alpha, \beta_1 + \beta_2) \mapsto (\alpha, 0 + 0)$$

is a homotopy equivalence.

The map $v : Q_{1,2}(\overline{D}_-, D_-) \rightarrow \mathbb{C}^*$ given by

$$(\alpha, \beta_1 + \beta_2) \mapsto (\alpha - \beta_1)(\alpha - \beta_2)$$

is also a homotopy equivalence. Regarding the intersection

$$Q_{1,2}(\overline{D}_+, D_-) \cap Q_{1,2}(\overline{D}_-, D_-) = Q_{1,2}(S^1, D_-),$$

we have the homotopy commutative diagram

$$\begin{array}{ccc} Q_{1,2}(S^1, D_-) & \xrightarrow{=} & Q_{1,2}(S^1, D_-) \\ u \downarrow & & v \downarrow \\ Q_{1,2}(S^1, \{0\}) & \xrightarrow{f} & \mathbb{C}^* \end{array}$$

It follows from the definition of the map v that the bottom map f is given by the map

$$Q_{1,2}(S^1, \{0\}) \cong S^1 \rightarrow \mathbb{C}^*; S^1 \ni z \mapsto z^2 \in \mathbb{C}^*.$$

Hence

$$Q_{1,2}(S^2, \mathbb{C}) \simeq Q_{1,2}(S^2, D_-) \simeq \mathbb{C}^* \cup_{f \circ u} Q_{1,2}(\overline{D}_+, D_-) \simeq S^1 \cup_2 e^2 \simeq \mathbb{R}P^2. \quad \square$$

Proof of (1) of Theorem 3.

Consider the fibration $S^1 \rightarrow \text{Hol}_2^* \xrightarrow{\pi} \mathbb{R}P^2$. Let $p : S^2 \rightarrow \mathbb{R}P^2$ and $q : \tilde{\text{Hol}}_2^* \rightarrow \text{Hol}_2^*$ be the universal coverings. Since $\tilde{\text{Hol}}_2^*$ is simply connected, there is a lift $\theta : \tilde{\text{Hol}}_2^* \rightarrow S^2$ such that $p \circ \theta = \pi \circ q$. It follows from diagram chasing that $\theta_* : \pi_j(\tilde{\text{Hol}}_2^*) \rightarrow \pi_j(S^2)$ is an isomorphism for all $j \geq 2$. Since both spaces are simply connected, θ is a homotopy equivalence. $\square$

Remark. Using Theorem 3, we can prove that the C_2 -operad structure on $\text{Hol}^* = \coprod_{d \geq 0} \text{Hol}_d^*$ given by [BM] is not compatible with that on $\Omega^2 S^2$ up to homotopy. See [GKMY] for the details. $\square$

REFERENCES

- [AH] M.F. Atiyah and N.J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton Univ. Press, 1988.
- [BM] C.P. Boyer and B.M. Mann, *Monopoles, non-linear σ models and two-fold loop spaces*, Commun. Math. Phys. **115** (1988), 571–594.
- [CCMM] F.R. Cohen, R.L. Cohen, B.M. Mann and R.J. Milgram, *The topology of rational functions and divisors of surfaces*, Acta Math. **166** (1991), 163–221.
- [CS] R.L. Cohen and D.H. Shimamoto, *Rational functions, labelled configurations and Hilbert schemes*, J. Lond. Math. Soc. **43** (1991), 509–528.
- [D] S.K. Donaldson, *Nahm's equations and the classification of monopoles*, Commun. Math. Phys. **96** (1984), 387–407.
- [E] S.I. Epshtein, *Fundamental groups of spaces of coprime polynomials*, Functional Analysis and its Applications **7** (1973), 82–83.
- [GKMY] M.A. Guest, A. Kozłowski, M. Murayama and K. Yamaguchi, *The homotopy type of spaces of rational functions*, preprint (1992).
- [S] G.B. Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), 39–72.