

CR immersions from S^{2n+1} to S^{4n+1}

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Introduction

Elie Cartan's contribution to CR geometry is well known. In his fundamental papers, he solved the problem of equivalence for a piece of real hypersurface in $\mathbb{C}^2$ up to biholomorphism [Ca1]. It was Chern and Moser, among others, who vastly generalized this work to higher dimensions [ChM].

Recently, Huang and Ji gave a classification of the proper holomorphic maps from the unit ball $\mathbf{B}^{n+1} \subset \mathbb{C}^{n+1}$ to $\mathbf{B}^{2n+1}$, $n \geq 2$, [HJ]. We wish to show in this note that the corresponding local CR analogue is true. Let us denote $\Sigma^n = \partial\mathbf{B}^{n+1} = S^{2n+1}$.

Theorem. *Let $f : \Sigma^n \hookrightarrow \Sigma^{2n}$ be a C^3 , local CR immersion, $n \geq 2$. Then up to automorphisms of the spheres, either f is linear, or f is locally equivalent to the boundary of Whitney map.*

When the codimension is small, CR Gauß equation puts the second fundamental form of a CR immersion into a simple normal form. Our idea is to explore the successive derivatives of this relation.

The computation involved is reminiscent of Cartan's local isometric embedding of Hyperbolic space $\mathbb{H}^n$ in Euclidean space $\mathbb{E}^{2n-1}$ via exteriorly orthogonal symmetric bilinear forms [Ca2]. Overdetermined nature of CR geometry forces the structure equation to close up rather than become involutive.

Local CR immersions $\Sigma^1 \hookrightarrow \Sigma^2$ have been classified by Faran [Fa]. In contrast to $n \geq 2$ cases, there exist four inequivalent such immersions.

Corollary [HJ]. *Let $F : B^{n+1} \rightarrow B^{2n+1}$ be a proper holomorphic map which is C^3 up to the boundary, $n \geq 2$. Then up to automorphisms of the unit balls, either F is linear, or F is equivalent to Whitney map.*

I'd like to thank Prof. Morimoto and Prof. Miyaoka for the invitation and hospitality.

1 CR immersion

We first set up the basic structure equations for CR immersions in spheres. For general reference in CR geometry, [ChM][EHZ].

Let $\mathbb{C}^{N+1,1}$ be the complex vector space with coordinates $z = (z^0, z^A, z^{N+1})$, $1 \leq A \leq N$, and a Hermitian scalar product

$$\langle z, \bar{z} \rangle = z^A \bar{z}^A + i(z^0 \bar{z}^{N+1} - z^{N+1} \bar{z}^0).$$

Let Σ^N be the set of equivalence classes up to scale of null vectors with respect to this product. Let $SU(N+1, 1)$ be the group of unimodular linear transformations that leave the form $\langle z, \bar{z} \rangle$ invariant. Then $SU(N+1, 1)$ acts transitively on Σ^N , and

$$p : SU(N+1, 1) \rightarrow \Sigma^N = SU(N+1, 1)/P$$

for an appropriate subgroup P [ChM].

Explicitly, consider an element $Z = (Z_0, Z_A, Z_{N+1}) \in SU(N+1, 1)$ as an ordered set of $(N+2)$ -column vectors in $\mathbb{C}^{N+1,1}$ such that $\det(Z) = 1$, and that

$$\langle Z_A, \bar{Z}_B \rangle = \delta_{AB}, \quad \langle Z_0, \bar{Z}_{N+1} \rangle = -\langle Z_{N+1}, \bar{Z}_0 \rangle = i, \quad (1)$$

while all other scalar products are zero. We define $p(Z) = [Z_0]$, where $[Z_0]$ is the equivalence class of null vectors represented by Z_0 . The left invariant Maurer-Cartan form π of $SU(N+1, 1)$ is defined by the equation

$$dZ = Z\pi,$$

which is in coordinates

$$d(Z_0, Z_A, Z_{N+1}) = (Z_0, Z_B, Z_{N+1}) \begin{pmatrix} \pi_0^0 & \pi_A^0 & \pi_{N+1}^0 \\ \pi_0^B & \pi_A^B & \pi_{N+1}^B \\ \pi_0^{N+1} & \pi_A^{N+1} & \pi_{N+1}^{N+1} \end{pmatrix}. \quad (2)$$

Coefficients of π are subject to the relations obtained from differentiating (1) which are

$$\begin{aligned} \pi_0^0 + \bar{\pi}_{N+1}^{N+1} &= 0 \\ \pi_0^{N+1} &= \bar{\pi}_0^{N+1}, \quad \pi_{N+1}^0 = \bar{\pi}_{N+1}^0 \\ \pi_A^{N+1} &= -i\bar{\pi}_0^A, \quad \pi_{N+1}^A = i\bar{\pi}_A^0 \\ \pi_B^A + \bar{\pi}_A^B &= 0 \\ \text{tr } \pi &= 0, \end{aligned}$$

and π satisfies the structure equation

$$-d\pi = \pi \wedge \pi. \quad (3)$$

It is well known that the $\text{SU}(N+1, 1)$ -invariant CR structure on $\Sigma^N \subset \mathbb{C}P^{N+1}$ as a real hypersurface is biholomorphically equivalent to the standard CR structure on $S^{2N+1} = \partial B^{N+1}$, where $B^{N+1} \subset \mathbb{C}^{N+1}$ is the unit ball. The structure equation (2) shows that for any local section $s : \Sigma^N \rightarrow \text{SU}(N+1, 1)$, this CR structure is defined by the hyperplane fields $(s^*\pi_0^{N+1})^\perp = \mathcal{H}$ and the set of $(1,0)$ -forms $\{s^*\pi_0^A\}$.

Definition. Let M be a CR manifold of hypersurface type with CR hyperplane fields $\mathcal{H}^M$ equipped with a complex structure. An immersion $f : M \hookrightarrow \Sigma^N$ is a *CR immersion* if $f_* : \mathcal{H}^M \rightarrow \mathcal{H}$ is complex linear.

Consider $f^*\text{SU}(N+1, 1) \rightarrow M$. From the definition, we may arrange so that $\pi_0^a = 0$ for $n+1 \leq a \leq m = N-n$ on this bundle. Differentiating this, we get

$$\pi_i^a \wedge \pi_0^i + \pi_{N+1}^a \wedge \pi_0^{N+1} = 0.$$

By Cartan's lemma,

$$\pi_i^a \equiv h_{ij}^a \pi_0^j \pmod{\pi_0^{N+1}}, \quad (4)$$

for coefficients $h_{ij}^a = h_{ji}^a$. h_{ij}^a represents the second fundamental form of f [EHZ].

Example [Whitney immersion]. Let $\xi = (\xi^0, \xi^i, \xi^{n+1})$, $\mu = (\mu^0, \mu^i, \mu^{n+i}, \mu^{2n+1})$, $1 \leq i \leq n$, be the coordinates of $\mathbb{C}^{n+1,1}$, and $\mathbb{C}^{2n+1,1}$ respectively. *Whitney immersion* $\Gamma_n : \Sigma^n \rightarrow \Sigma^{2n}$ is induced from the quadratic map $\widehat{\Gamma}_n : \mathbb{C}^{n+1,1} \rightarrow \mathbb{C}^{2n+1,1}$ defined as

$$\begin{aligned} \mu^0 &= 2\xi^0 \xi^{n+1}, \\ \mu^{2n+1} &= (\xi^{n+1})^2 - (\xi^0)^2, \\ \mu^i &= \xi^i (i\xi^0 + \xi^{n+1}), \\ \mu^{n+i} &= \xi^i (-i\xi^0 + \xi^{n+1}). \end{aligned}$$

$\widehat{\Gamma}_n^* \langle \mu, \bar{\mu} \rangle = 2(\xi^0 \bar{\xi}^0 + \xi^{n+1} \bar{\xi}^{n+1}) \langle \xi, \bar{\xi} \rangle$, and the induced map Γ_n is well defined.

It is easy to check Γ_n is CR-equivalent to the boundary map $\partial\mathcal{W}_n : S^{2n+1} \rightarrow S^{4n+1}$ of the following *Whitney map* $\mathcal{W}_n : \mathbb{B}^{n+1} \rightarrow \mathbb{B}^{2n+1}$. Here (z^0, z^i) is a coordinate of $\mathbb{C}^{n+1}$.

$$\mathcal{W}_n(z^0, z^i) = ((z^0)^2, z^0 z^i, z^i). \quad (5)$$

This equivalence is via the isomorphism $\Sigma^n \simeq S^{2n+1}$ given in coordinates

$$\begin{aligned} z^0 &= \frac{i\xi^0 + \xi^{n+1}}{-i\xi^0 + \xi^{n+1}} \\ z^i &= \frac{\sqrt{2}\xi^i}{-i\xi^0 + \xi^{n+1}} \end{aligned}$$

Set $\Sigma_0^n = \{[\xi] \in \Sigma^n \mid \xi^i = 0, \forall i\}$ and $\Sigma_s^n = \{[\xi] \in \Sigma^n \mid i\xi^0 + \xi^{n+1} = 0\}$. Then Γ_n is an immersion which is 1 to 1 on $\Sigma^n - \Sigma_0^n$, 2 to 1 on Σ_0^n , and the second fundamental form vanishes along Σ_s^n .

2 Proof of theorem

Our proof of Theorem is based on the following algebraic Lemma due to Iwatani on the asymptotic subspace of the second fundamental form of a Bochner-Kähler submanifold [Iw][Br]. Let $V = \mathbb{C}^n$, $W = \mathbb{C}^n$ with the standard Hermitian scalar product. Let $\{z^i\}$ be a unitary $(1,0)$ -basis for V^* , and $\{w_a\}$ be a unitary basis for W . Let $S^{p,q}$ denote the space of polynomials of type (p,q) on V .

Lemma [Iw]. Suppose $H = h_{ij}^a z^i z^j \otimes w_a \in S^{2,0} \otimes W$ satisfies

$$\gamma(H, H) = h_{ij}^a \bar{h}_{ki}^a z^i z^j \otimes \bar{z}^k \bar{z}^l = (z^k \bar{z}^k) h \in S^{2,2}, \quad h \in S^{1,1},$$

or simply $\gamma(H, H)$ is Bochner-flat [Br]. Then the asymptotic vectors $\{v \in V \mid H(v, v) = 0\}$ form a subspace of V of codimension at most one.

Up to a unitary transformation on V , we may thus arrange

$$h_{ij}^a = h_i^a \delta_{jn} + h_j^a \delta_{in},$$

for coefficients h_i^a . Set $\nu_i = h_i^a w_a \in W$. A computation shows $\gamma(H, H)$ is Bochner-flat when $\langle \nu_i, \nu_j \rangle = 0$ for $i \neq j$, and $\langle \nu_i, \nu_i \rangle = \langle \nu_j, \nu_j \rangle$ for all i, j . Up to a unitary transformation of W , we may set

$$\nu_i = r w_i,$$

for some $r \geq 0$.

Let $f : \Sigma^n \hookrightarrow \Sigma^{2n}$ be a local CR immersion. Let $\mathcal{H}$ be the CR hyperplane fields on Σ^{2n} . Since Σ^n is CR flat, after identifying $V = f_* T\Sigma^n \cap \mathcal{H} \simeq \mathbb{C}^n$ and $W = V^\perp \simeq \mathbb{C}^n \subset \mathcal{H}$, the second fundamental form of f is Bochner-flat [EHZ]. From (4) and the argument above, we may write

$$\begin{aligned} \pi_q^{n+i} &\equiv r \delta_{iq} \pi_0^n \quad \text{mod } \pi_0^{2n+1}, \quad \text{for } q < n, \\ \pi_n^{n+i} &\equiv r(1 + \delta_{in}) \pi_0^i \quad \text{mod } \pi_0^{2n+1}. \end{aligned}$$

Assume f is not linear, $H \neq 0$, and we may scale $r = 1$ using the group action by $\text{Re } \pi_0^0$. We obtain the following normalized structure equation for a nonlinear local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{2n}$,

$$\begin{aligned}\pi_q^{n+i} &= \delta_{iq} \omega^n + h_q^i \pi_0^{2n+1} \quad \text{for } q < n, \\ \pi_n^{n+i} &= (1 + \delta_{in}) \omega^i + h_n^i \pi_0^{2n+1},\end{aligned}\tag{6}$$

for coefficients h_j^i .

Theorem is now obtained by successive application of Maurer-Cartan equation (3) to this structure equation. We assume $n \geq 3$ for simplicity for the rest of this section, as $n = 2$ case can be treated in a similar way. We shall agree on the index range $1 \leq p, q, s, t \leq n - 1$, and denote $p' = n + p$, $n' = n + n$. We denote $\theta = \pi_0^{2n+1}$, $-d\theta \equiv i\pi_0^k \wedge \bar{\pi}_0^k = i\varpi \pmod{\theta}$, and $\pi_0^i = \omega^i$ for the sake of notation.

Step 1. Differentiating $\pi_s^{n'} = h_s^n \theta \pmod{\theta}$, we get

$$i h_s^n \varpi \equiv (\pi_{s'}^{n'} - 2\pi_s^n) \wedge \omega^n + \pi_{2n+1}^{n'} \wedge (-i\bar{\omega}^s) \pmod{\theta}.$$

Since $n - 1 \geq 2$, this implies $h_s^n = 0$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{n'} - 2\pi_s^n \\ \pi_{2n+1}^{n'} \end{pmatrix} \equiv \begin{pmatrix} 2c_s & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ -i\bar{\omega}^s \end{pmatrix} \pmod{\theta}$$

for coefficients c_s, u .

Step 2. Differentiating $\pi_s^{t'} = h_s^t \theta \pmod{\theta}$ for $t \neq s$, we get

$$i h_s^t \varpi \equiv (\pi_{s'}^{t'} - \pi_s^t) \wedge \omega^n - \pi_s^n \wedge \omega^t + \pi_{2n+1}^{t'} \wedge (-i\bar{\omega}^s) \pmod{\theta}.$$

Since $n - 1 \geq 3$, this implies $h_s^t = 0$ for $t \neq s$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{t'} - \pi_s^t \\ -\pi_s^n \\ \pi_{2n+1}^{t'} \end{pmatrix} \equiv \begin{pmatrix} 0 & b_s & -i\bar{b}_t \\ b_s & 0 & e \\ -i\bar{b}_t & e & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ \omega^t \\ -i\bar{\omega}^s \end{pmatrix} \pmod{\theta}$$

for coefficients b_s, e . Since $\pi_s^{t'} - \pi_s^t$ is skew Hermitian, it cannot have any ω^n -term.

Step 3. Differentiating $\pi_t^{t'} = \omega^n + i h_t^t \theta \pmod{\theta}$ and collecting terms, we get

$$h_t^t \varpi \equiv (\pi_t^{t'} - \pi_t^t + \pi_0^0 - \pi_n^n) \wedge \omega^n + (b_p \omega^n - i e \bar{\omega}^p) \wedge \omega^p + (-b_t \omega^t + \bar{b}_t \bar{\omega}^t) \wedge \omega^n$$

$\pmod{\theta}$. Since $n-1 \geq 2$, this implies $h_t^t = e$, and

$$\begin{aligned} \Delta_t &= \pi_t^{t'} - \pi_t^t + \pi_0^0 - \pi_n^n \\ &= a_t \omega^n - i e \bar{\omega}^n + (b_t \omega^t - \bar{b}_t \bar{\omega}^t) + \sum_p b_p \omega^p - A_t \theta \end{aligned}$$

for coefficients a_t, A_t .

Step 4. From *Step 2*, we may use the group action by π_{2n+1}^n to translate $e = 0$, which we assume from now on. We also translate $h_n^t = 0$ by π_{2n+1}^t . Differentiating $\pi_n^{t'} = \omega^t \pmod{\theta}$ with these relations, we get

$$0 \equiv \bar{b}_t \left(\sum_p \omega^p \wedge \bar{\omega}^p \right) + (2\bar{c}_t - 3\bar{b}_t) \omega^n \wedge \bar{\omega}^n + (a_t + 2i\bar{u}) \omega^n \wedge \omega^t \pmod{\theta}.$$

Thus $b_t = c_t = 0$, $a_t = -2i\bar{u}$.

Step 5. Differentiating $\pi_n^{n'} = 2\omega^n + h_n^n \theta \pmod{\theta}$ and collecting terms, we get

$$i h_n^n \varpi \equiv 2(\pi_n^{n'} - \pi_n^n + \pi_0^0 - \pi_n^n) \wedge \omega^n + i u \omega^p \wedge \bar{\omega}^p - i u \omega^n \wedge \bar{\omega}^n \pmod{\theta}.$$

This implies $h_n^n = u$, and

$$\begin{aligned} \Delta_n &= \pi_n^{n'} - \pi_n^n + \pi_0^0 - \pi_n^n \\ &= a_n \omega^n - i u \bar{\omega}^n - A_n \theta \end{aligned}$$

for coefficients a_n, A_n . But $\Delta_t - \Delta_n$ is purely imaginary, and comparing with *Step 3*, $a_n = -3i\bar{u}$.

Step 6. Now by considering θ -terms in *Step 1, 2, 3, 4, 5* and the fact $\pi_i^a \wedge \omega^i + \pi_{2n+1}^a \wedge \theta = 0$, we obtain the following simple structure equations. We omit the details of computations.

$$\begin{pmatrix} \pi_{s'}^{n'} - 2\pi_s^n \\ \pi_{2n+1}^{n'} \end{pmatrix} = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ -i\bar{\omega}^s \end{pmatrix},$$

$$\pi_{s'}^{t'} = \pi_s^t, \quad t \neq s,$$

$$\Delta_t = -2i\bar{u}\omega^n - A\theta,$$

$$\begin{pmatrix} -\pi_s^n \\ \pi_{2n+1}^{t'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\pi_{2n+1}^t = (A - iu\bar{u})\omega^t + B_t\theta$$

$$\pi_{2n+1}^n = A\omega^n + B_n\theta$$

Step 7. Differentiating $\pi_{s'}^{t'} - \pi_s^t = 0$, $t \neq s$, $\pi_s^n = 0$, we get first $B_s = 0$, $B = 0$, and

$$A - \bar{A} = i(u\bar{u} - 1). \quad (7)$$

Step 8. Differentiating $\pi_n^{n'} = 2\omega^n + u\theta$ and $\pi_{2n+1}^{n'} = u\omega^n$,

$$du = u(\pi_{2n+1}^{2n+1} - \pi_0^0 + \pi_n^n - \pi_n^{n'}) + 2(A - A_n)\omega^n - 2uA\theta. \quad (8)$$

Step 9. Differentiating $\pi_{s'}^{n'} = -iu\bar{\omega}^s$ using (8) and collecting terms in $\theta \wedge \bar{\omega}^s$,

$$A_n = 2A - \bar{A}.$$

Step 10. Differentiating $\pi_{2n+1}^t = (\bar{A} - i)\omega^t$, $\pi_{2n+1}^n = A\omega^n$, we get

$$dA = A(\pi_{2n+1}^{2n+1} - \pi_0^0) + \pi_{2n+1}^0 + 2(u\bar{\omega}^n - \bar{u}\omega^n) + (u\bar{u} - A^2)\theta. \quad (9)$$

We normalize $A = i\alpha$ for a real number α using group action by π_{2n+1}^0 . Since $\pi_i^i + \bar{\pi}_i^i = 0$, $\Delta_t + \bar{\Delta}_t = \pi_0^0 + \bar{\pi}_0^0$, and (9) is now reduced to

$$d\alpha = 2i(\alpha + 1)(\bar{u}\omega^n - u\bar{\omega}^n) \quad (10)$$

$$\pi_{2n+1}^0 = -(\alpha + 1)^2\theta.$$

When $u \neq 0$, we may also rotate u to be a positive number, in which case it is determined by (7)

$$2\alpha + 1 = u\bar{u}. \quad (11)$$

At this stage, note that the only independent coefficients in the structure equations are α , u , and that the expression for their derivatives does not involve any new coefficients. The structure equations for local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{2n}$ thus *close up* as follows.

$$\begin{pmatrix} \pi_q^{p'} & \pi_n^{p'} \\ \pi_q^{n'} & \pi_n^{n'} \end{pmatrix} = \begin{pmatrix} \delta_{pq} \omega^n & \omega^p \\ 0 & 2\omega^n + u\theta \end{pmatrix} \quad (12)$$

$$\begin{pmatrix} \pi_{s'}^{n'} - 2\pi_s^n \\ \pi_{2n+1}^{n'} \end{pmatrix} = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \omega^n \\ -i\bar{\omega}^s \end{pmatrix}$$

$$\pi_{s'}^{t'} = \pi_s^t, \quad t \neq s,$$

$$\begin{pmatrix} -\pi_s^n \\ \pi_{2n+1}^{t'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\pi_{2n+1}^t = (A - iu\bar{u})\omega^t$$

$$\pi_{2n+1}^n = A\omega^n$$

$$\Delta_t = -2i\bar{u}\omega^n - A\theta$$

$$\Delta_n = -3i\bar{u}\omega^n - iu\bar{\omega}^n - A_n\theta$$

$$du = u(\pi_{2n+1}^{2n+1} - \pi_0^0 + \pi_n^n - \pi_n^{n'}) + 2(A - A_n)\omega^n - 2uA\theta,$$

$$d\alpha = 2i(\alpha + 1)(\bar{u}\omega^n - u\bar{\omega}^n)$$

$$\pi_{2n+1}^0 = -(\alpha + 1)^2\theta$$

where $A = i\alpha$, $A_n = 3i\alpha$, and α, u satisfy the relation (11). Moreover, a long but direct computation shows that this structure equation is compatible, i.e., $d^2 = 0$ is a formal identity of the structure equation.

Remark. The structure equation closes up at order 3. This implies a C^3 , nonlinear CR immersion $f : \Sigma^n \rightarrow \Sigma^{2n}$ is real analytic.

The remaining step in the proof of **Theorem** consists of identifying this as the structure equation of Whitney immersion. We make one useful observation. Let $\Sigma^* = \Sigma^n - \Sigma_s^n$, which is a connected set. Note also the structure equation (12) implies that the set of points where $u = 0$, or equivalently $\alpha = -\frac{1}{2}$, cannot have any interior on Σ^* . We claim the invariant α takes any value $> -\frac{1}{2}$ on Σ^* .

Suppose $\alpha_+ = \sup_{\Sigma^*} \alpha > -\frac{1}{2}$ is finite. Applying the existence part of Cartan's generalization of Lie's third fundamental theorem on closed structure equations, [Br3], there exists for any $p_0 \in \Sigma^n$ a neighborhood $U \subset \Sigma^n$ of p_0 and a CR immersion $g : U \rightarrow \Sigma^{2n}$ with invariant $\alpha|_{p_0} = \alpha_+$, hence necessarily $u_{p_0} = \sqrt{2\alpha_0 + 1}$. From (12), $d\alpha|_{p_0} \neq 0$ and let $p_- \in U$ be a point with $-\frac{1}{2} < \alpha|_{p_-} < \alpha_+$. Then by uniqueness part of Cartan's theorem, there exists a neighborhood $U' \subset U$ of p_- on which g agrees with Whitney immersion Γ_n up to automorphisms of Σ^n and Σ^{2n} . Since g and Γ_n satisfy the closed set of structure equations, they are real analytic. Thus g is a part of Γ_n . But $d\alpha|_{p_0} \neq 0$, and there exists a point $p_+ \in U \subset \Sigma^*$ such that $\alpha|_{p_+} > \alpha_+$, a contradiction. By similar argument, $\inf_{\Sigma^*} \alpha = -\frac{1}{2}$, and the claim follows for Σ^* is connected.

Proof of Theorem. Since the set of points $\alpha = -\frac{1}{2}$ cannot have any interior, let $p \in \Sigma$ be a point with $\alpha|_p > -\frac{1}{2}$. From the results above and the uniqueness part of aforementioned Cartan's theorem, $f = \Gamma_n$ on a neighborhood U of p up to automorphisms of the spheres. The theorem follows for both f and Γ_n are real analytic. $\square$

Proof of Corollary. By the regularity theorem [HJX], F is real analytic up to ∂F . Since the CR structure on $S^{2n+1} = \Sigma^n$ is definite, the set of points where ∂F has holomorphic rank n is a dense open subset. Assume F is not linear. There exists a point $p \in \Sigma^n$ where the second fundamental form does not vanish either. By **Theorem**, ∂F agrees with Whitney immersion Γ_n in a neighborhood of p up to automorphisms of

the spheres. The real analyticity then implies $\partial F = \Gamma_n$ on Σ^n , and hence $F = \mathcal{W}_n$ on $\mathbf{B}^{n+1}$. $\square$

We may apply the existence part of Cartan's generalization of Lie's third fundamental theorem and show that Whitney immersion gives an example of a deformable CR-submanifold. Take a point $p \in \Sigma^n$, and an analytic one parameter family of real numbers $\alpha_t > -\frac{1}{2}$, and set $u_t = \sqrt{2\alpha_t + 1}$. Then by the existence part of Cartan's theorem, there exists a neighborhood U of p and a one parameter family of CR immersions $f_t : U \rightarrow \Sigma^{2n}$ with the induced structure equations (12) such that the invariants α, u have the prescribed values α_t, u_t at p . This deformation of course is *tangential* and does not actually deform the submanifold. It is due to an intrinsic CR symmetry of Σ^n that cannot be extended to a symmetry of the ambient Σ^{2n} along Whitney immersion.

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