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On Law Invariant Coherent Risk Measures

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1 Introduction

The idea of coherent risk measures has been introduced by Artzner, Delbaen, Eber and Heath [1]. We think of a special class of coherent risk measures and give a characterization of it. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We denote $L^\infty(\Omega, \mathcal{F}, P)$ by L^∞. Following [1], we give the following definition.

Definition 1 *We say that a map $\rho : L^\infty \to \mathbf{R}$ is a coherent risk measure if the following are satisfied.*
(1) *If $X \geq 0$, then $\rho(X) \leq 0$.*
(2) *Subadditivity : $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.*
(3) *Positive homogeneity : for $\lambda > 0$ we have $\rho(\lambda X) = \lambda\rho(X)$.*
(4) *For every constant c we have $\rho(X + c) = \rho(X) - c$.*

Then Delbaen [2] proved the following.

Theorem 2 *Let ρ be a coherent risk measure. Then the following conditions are equivalent.*
(1) *There is a (closed convex) set of probability measures $\mathcal{Q}$ such that any $Q \in \mathcal{Q}$ is absolutely continuous with respect to P and for $X \in L^\infty$*

$$\rho(X) = \sup\{E^Q[-X];\ Q \in \mathcal{Q}\}.$$

(2) *ρ satisfies the Fatou property, i.e., if $\{X_n\}_{n=1}^\infty \subset L^\infty$ are uniformly bounded and converging to X in probability, then*

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

(3) *If X_n is a uniformly bounded sequence that decreases to X, then $\rho(X_n)$ tends to $\rho(X)$.*

Now we introduce the following notion.

Definition 3 *We say that a map $\rho : L^\infty \to \mathbf{R}$ is law invariant, if $\rho(X) = \rho(Y)$ whenever $X, Y \in L^\infty$ have the same probability law.*

Our purpose is to characterize law invariant coherent risk measures with the Fatou property.

Let $\mathcal{D}$ be the set of probability distribution functions of bounded random variables, i.e., $\mathcal{D}$ is the set of non-decreasing right-continuous functions F on $\mathbf{R}$ such that there are $z_0, z_1 \in \mathbf{R}$ for which $F(z) = 0$, $z < z_0$ and $F(z) = 1$, $z \geq z_1$. Let us define $Z : [0,1) \times \mathcal{D} \to \mathbf{R}$ by

$$Z(x, F) = \inf\{z; F(z) > x\}, \qquad x \in [0,1),\ F \in \mathcal{D}.$$

Then $Z(\cdot, F) : [0,1) \to \mathbf{R}$ is non-decreasing and right continuous. We denote by F_X the probability distribution function of a random variable X.

For each $\alpha \in (0,1]$, let $\rho_\alpha : L^\infty \to \mathbf{R}$ be given by

$$\rho_\alpha(X) = \alpha^{-1} \int_{1-\alpha}^{1} Z(x, F_{-X})dx, \qquad X \in L^\infty.$$

Also, we define $\rho_0 : L^\infty \to \mathbf{R}$ by

$$\rho_0(X) = ess.sup(-X) \qquad X \in L^\infty.$$

Then it is easy to see that $\rho.(X) : [0,1] \to \mathbf{R}$ is a non-increasing continuous function for any $X \in L^\infty$.

We will show later that ρ_α, $\alpha \in [0,1]$, is a law invariant coherent risk measure with the Fatou property. Actually ρ_α is the same as WCM_α in [1].

From now on, we assume the following.

(Assumption) $(\Omega, \mathcal{F}, P)$ is a standard probability space and P is non-atomic.

Our main results are the following.

Theorem 4 *Let $\rho : L^\infty \to \mathbf{R}$. Then the following conditions are equivalent.*
(1) *There is a (compact convex) set $\mathcal{M}_0$ of probability measures on $[0,1]$ such that*

$$\rho(X) = \sup\{\int_0^1 \rho_\alpha(X)m(d\alpha);\ m \in \mathcal{M}_0\}, \qquad X \in L^\infty.$$

(2) *ρ is a law invariant coherent risk measure with the Fatou property.*

Theorem 5 *If m_1 and m_2 are probability measures on $[0,1]$, and if*

$$\int_0^1 \rho_\alpha(X)m_1(d\alpha) = \int_0^1 \rho_\alpha(X)m_2(d\alpha), \qquad \text{for all } X \in L^\infty,$$

then $m_1 = m_2$.

Definition 6 (1) *We say that a pair X and Y of random variables is comonotone, if*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \qquad P(d\omega) \otimes P(d\omega') - a.s.$$

(2) *We say that a map $\rho : L^\infty \to \mathbf{R}$ is comonotone, if*

$$\rho(X + Y) = \rho(X) + \rho(Y)$$

for any comonotone pair $X, Y \in L^\infty$.

Theorem 7 *Let $\rho : L^\infty \to \mathbf{R}$. Then the following conditions are equivalent.*
(1) *There is a probability measure m on $[0,1]$ such that for $X \in L^\infty$*

$$\rho(X) = \int_0^1 \rho_\alpha(X)m(d\alpha), \qquad X \in L^\infty.$$

(2) *ρ is a law invariant and comonotone coherent risk measure with the Fatou property.*

Definition 8 *We define* $VaR_\alpha : L^\infty \to \mathbf{R}$, $\alpha \in (0,1)$, *by*

$$VaR_\alpha(X) = \sup\{z \in \mathbf{R};\ F_{-X}(z) < 1-\alpha\}.$$

Theorem 9 *Let* $\alpha \in (0,1)$. *If* ρ *is law invariant coherent risk measure such that*

$$\rho(X) \geq VaR_\alpha(X), \qquad X \in L^\infty,$$

then we have

$$\rho(X) \geq \rho_\alpha(X), \qquad X \in L^\infty.$$

The author thanks Prof. Delbaen for useful discussions. In particular, Theorems 7 and 9 are suggested by him.

2 Key Lemma

Since we assume that $(\Omega, \mathcal{F})$ is a standard probability space and P be non-atomic, we may assume that our basic probability space $(\Omega, \mathcal{F}, P)$ is a Lebesgue space, i.e., $\Omega = [0,1)$, $\mathcal{F}$ is the Borel algebra over $[0,1)$, and P is the Lebesgue measure μ on $[0,1)$. Therefore we assume so throughout this paper.

Let $\mathcal{G}$ be the set of non-decreasing right-continuous probability density functions on $[0,1)$. In this section, we will prove the following.

Lemma 10 *Let* $\rho : L^\infty \to \mathbf{R}$. *Then the following conditions are equivalent.*
(1) *There is a subset* $\mathcal{G}_0$ *of* $\mathcal{G}$ *such that*

$$\rho(X) = \sup\{\int_0^1 Z(x, F_{-X})g(x)dx;\ g \in \mathcal{G}_0\}, \qquad X \in L^\infty.$$

(2) ρ *is a law invariant coherent risk measure with the Fatou property.*

Let $\mathcal{P}$ denote the set of probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to P. For any $Q \in \mathcal{P}$, Y_Q denotes the Radon-Nykodim density dQ/dP. Let $\mathcal{F}_n$, $n \geq 1$, be a sub-σ-algebra of $\mathcal{F}$ generated by $1_{[2^{-n}(k-1), 2^{-n}k)}, k = 1, \ldots, 2^n$. Let $\mathcal{X}$ be the set of all bounded random variables X such that X is $\mathcal{F}_n$-measurable for some n.

Then we have the following.

Lemma 11 *Let* $Q \in \mathcal{P}$ *and* $X \in \mathcal{X}$. *Then we have*

$$\begin{aligned}\int_0^1 Z(x, F_X)Z(x, F_{Y_Q})dx &= \sup\{E^Q[\tilde{X}]; \tilde{X} \in \mathcal{X}, F_{\tilde{X}} = F_X\} \\ &= \sup\{E^{\tilde{Q}}[X]; \tilde{Q} \in \mathcal{P}, F_{Y_{\tilde{Q}}} = F_{Y_Q}\}.\end{aligned}$$

We make some preparations before proving Lemma 11.

We easily see the following.

Proposition 12 *Let x_k, $k = 1, 2, \ldots, n$, be a sequence of numbers, and let y_k, $k = 1, 2, \ldots, n$, be a sequence of non-negative numbers. If $x_{i_1} \le x_{i_2} \le \ldots \le x_{i_n}$, $y_{j_1} \le y_{j_2} \le \ldots \le y_{j_n}$, and $\{i_1, i_2, \ldots, i_n\} = \{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$, then*

$$\sum_{k=1}^{n} x_k y_k \le \sum_{k=1}^{n} x_{i_k} y_{j_k}.$$

Also we have the following (see Williams [3] Chapters 3 and 17).

Proposition 13 (1) *For any $F \in \mathcal{D}$, the probability distribution function of the law of $Z(x, F)$ under $\mu(dx)$ is F.*
(2) *If $F_n \in \mathcal{D}$ converges to F weakly, then $Z(x, F_n)$ converges to $Z(x, F)$ for $\mu - a.s.x$.*

Now let us prove Lemma 11. Let $X \in \mathcal{X}$. Then X is $\mathcal{F}_n$-measurable for some $n \ge 1$. Let $Y_m = E[Y_Q|\mathcal{F}_m]$, $m \ge n$. Then for any $m \ge n$, we have

$$X(\omega) = \sum_{k=1}^{2^m} x_{m,k} 1_{[(k-1)2^{-m}, k2^{-m})}(\omega), \quad Y_m(\omega) = \sum_{k=1}^{2^m} y_{m,k} 1_{[(k-1)2^{-m}, k/2^{-m})}(\omega), \qquad P-a.s.,$$

where $x_{m,k} = 2^m E^P[X, [(k-1)2^{-m}, k2^{-m})]$ and $y_{m,k} = 2^m E^P[Y_Q, [(k-1)2^{-m}, k2^{-m})]$, $k = 1, 2, \ldots, 2^m$. Let σ_m and τ_m be a permutation on $\{1, 2, \ldots, 2^m\}$ such that

$$x_{m,\sigma_m(1)} \le x_{m,\sigma_m(2)} \le \cdots \le x_{m,\sigma_m(2^n)} \text{ and } y_{m,\tau_m(1)} \le y_{m,\tau_m(2)} \le \cdots \le y_{m,\tau_m(2^n)}.$$

Then one can easily obtain that

$$Z(x, F_X) = \sum_{k=1}^{2^m} x_{\sigma_m(k)} 1_{[(k-1)2^{-m}, k2^{-m})}(x), \qquad Z(x, F_{Y_m}) = \sum_{k=1}^{2^m} y_{\tau_m(k)} 1_{[(k-1)2^{-m}, k2^{-m})}(x),$$

and so

$$E^Q[X] = E[XY_Q] = E[XY_m] = 2^{-m} \sum_{k=1}^{2^m} x_{m,k} y_{m,k} \le 2^{-m} \sum_{k=1}^{2^m} x_{m,\sigma_m(k)} y_{m.,\tau_m(k)}$$

$$= \int_0^1 Z(x, F_X) Z(x, F_{Y_m}) dx.$$

Since $Y_m = E[Y_Q|\mathcal{F}_m]$ converges to Y P-a.s., we see by Proposition 13 that $Z(x, F_{Y_m})$ converges to $Z(x, F_{Y_Q})$ for μ-a.e.x. Since $\{Y_m\}_{m=n}^{\infty}$ are uniformly integrable, $\{Z(x, F_{Y_m})\}_{m=n}^{\infty}$ are also uniformly integrable by Proposition 13 (1). Therefore letting $m \to \infty$, we have

$$E^Q[X] \le \int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx \tag{1}$$

for any $X \in \mathcal{X}$. Let

$$\tilde{X}_m(\omega) = \sum_{k=1}^{2^m} x_{m,\sigma_m(\tau_m^{-1}(k))} 1_{[(k-1)2^{-m}, k2^{-m})}(\omega).$$

Then one can easily see that the probability distributions of X and $\tilde{X}_m$ under P are the same and $\tilde{X}_m \in \mathcal{X}$. Also, we have

$$E^Q[\tilde{X}_m] = 2^{-m} \sum_{k=1}^{2^m} x_{m,\sigma_m(\tau_m^{-1}(k))} y_{m,k}$$

$$= \int_0^1 Z(x, F_X) Z(x, F_{Y_m}) dx.$$

So letting $m \to \infty$, we have

$$\sup\{E^Q[\tilde{X}]; \tilde{X} \in \mathcal{X}, F_{\tilde{X}} = F_X\} \geq \int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx. \tag{2}$$

Let

$$\tilde{Y}_m(\omega) = \sum_{k=1}^{2^m} 1_{[((k-1)2^{-m}, k2^{-m})}(\omega) Y_Q(\omega - k2^{-m} + \tau_m(\sigma_m^{-1}(k))2^{-m}).$$

Then one can easily see that the probability distributions of Y_Q and $\tilde{Y}_m$ under P are the same. Let $\tilde{Q} = \tilde{Y}_m P$. Then we have

$$E^{\tilde{Q}}[X] = 2^{-m} \sum_{k=1}^{2^m} x_{m,k} y_{m,\tau_m(\sigma_m^{-1}(k))} = \int_0^1 Z(x, F_X) Z(x, F_{Y_m}) dx.$$

So letting $m \to \infty$, we have

$$\sup\{E^{\tilde{Q}}[X]; \tilde{Q} \in \mathcal{P}, F_{Y_{\tilde{Q}}} = F_{Y_Q}\} \geq \int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx. \tag{3}$$

We have Lemma 11 from Equations (1), (2) and (3).

This completes the proof of Lemma 11.

Proposition 14 *Let $Q \in \mathcal{P}$. Then for any $X \in L^\infty$, we have*

$$\int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx = \sup\{E^{\tilde{Q}}[X]; \tilde{Q} \in \mathcal{P}, F_{Y_{\tilde{Q}}} = F_{Y_Q}\}.$$

Proof. Let $\tilde{Y}$ be a random variable whose distribution is the same as that of Y_Q. Let $X_n = E[X|\mathcal{F}_n]$, $n \geq 1$. Then for any $m \geq 1$, we have

$$E[|X - X_n|\tilde{Y}] \leq E[|X - X_n|(\tilde{Y} \wedge m)] + \| X - X_n \|_\infty E[\tilde{Y}, \tilde{Y} > m]$$

$$\leq mE[|X - X_n|] + 2 \| X \|_\infty E[\tilde{Y}, \tilde{Y} > m].$$

So we have

$$\sup\{E^{\tilde{Q}}[|X - X_n|]; \tilde{Q} \in \mathcal{P}, F_{Y_{\tilde{Q}}} = F_{Y_Q}\} \to 0, \qquad n \to \infty.$$

By Proposition 13, we have

$$\int_0^1 Z(x, F_{X_n}) Z(x, F_{Y_Q}) dx \to \int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx, \qquad n \to \infty.$$

Therefore we have our assertion from Lemma 11. This completes the proof. ∎

Now let us prove Lemma 10.

Proof of Lemma 10. (1) ⇒ (2) Let $\mathcal{G}_0$ be a subset of $\mathcal{G}$, and $\rho : L^\infty \to \mathbf{R}$ be given by

$$\rho(X) = \sup\{\int_0^1 Z(x, F_{-X})g(x)dx;\ g \in \mathcal{G}_0\}, \qquad X \in L^\infty.$$

Then it is obvious that ρ is law invariant. So it is sufficient to prove that ρ is a coherent risk measure with the Fatou property. Let $\mathcal{Q}_0$ be the set of $Q \in \mathcal{P}$ such that $Z(\cdot, Y_Q) \in \mathcal{G}_0$. Then by Proposition 14, we have

$$\rho(X) = \sup\{E^Q[-X];\ Q \in \mathcal{Q}_0\}, \qquad X \in L^\infty.$$

So by Theorem 2, we see that ρ is a coherent risk measure with the Fatou property. This implies our assertion.

(2) ⇒ (1) Let ρ be a law invariant coherent risk measure with the Fatou property. Let $\mathcal{P}_0$ be the set of $Q \in \mathcal{P}$ such that $E^Q[-X] \le \rho(X)$ for all $X \in L^\infty$. Then by Theorem 2 we have

$$\rho(X) = \sup\{E^Q[-X];\ Q \in \mathcal{P}_0\}, \qquad X \in L^\infty.$$

Take a $Q \in \mathcal{P}_0$ and $X \in L^\infty$, and fix them for a while. Let $\tilde{X}(\omega) = Z(\omega; F_X)$, $\omega \in \Omega = [0, 1)$. Then we have $\rho(\tilde{X}) = \rho(X)$. Let U_n, $n \ge 1$, be random variables defined by

$$U_n = \begin{cases} \tilde{X}(\omega + 2^{-n}), & \omega \in [0, 1 - 2^{-n}), \\ \| X \|_\infty, & \omega \in [1 - 2^{-n}, 1) \end{cases}$$

Then we see that $U_n \downarrow \tilde{X}$, $P-a.s.$ Let $V_n = E^P[\tilde{X}|\mathcal{F}_n]$. Then we see that $V_n \le U_n$, $P-a.s.$ and that $V_n \to \tilde{X}$, $P-a.s.$ So by Theorem 2 we have

$$\liminf_{n\to\infty} \rho(V_n) \le \lim_{n\to\infty} \rho(U_n) = \rho(\tilde{X}).$$

On the other hand, by Lemma 11 and Proposition 14 we have

$$\begin{aligned} E^Q[-X] &\le \int_0^1 Z(x, F_{-\tilde{X}})Z(x, F_{Y_Q})dx \\ &= \lim_{n\to\infty} \int_0^1 Z(x, F_{-V_n})Z(x, F_{Y_Q})dx \\ &= \lim_{n\to\infty} \sup\{E^Q[-\tilde{V}]; \tilde{V} \in \mathcal{X},\ F_{\tilde{V}} = F_{V_n}\} \\ &\le \liminf_{n\to\infty} \rho(V_n) \le \rho(X). \end{aligned}$$

Thus letting $\mathcal{G}_0 = \{Z(\cdot, F_{Y_Q});\ Q \in \mathcal{P}_0\}$, we see that

$$\rho(X) = \sup\{\int_0^1 Z(x, F_{-X})g(x)dx;\ g \in \mathcal{G}_0\}.$$

This implies our assertion. This completes the proof of Lemma 10.

3 Proof of Theorem 4

In this section, we prove Theorem 4. Let $g \in \mathcal{G}$, and let $\tilde{g} : \mathbf{R} \to \mathbf{R}$ be given by $\tilde{g}(t) = 0, t < 0$, $\tilde{g}(t) = g(t), t \in [0,1)$, and $\tilde{g}(t) = g(1-), t \geq 1$. Then we have for any $X \in L^\infty$

$$\begin{aligned}\int_0^1 Z(x, F_{-X})g(x)dx &= \int_{[0,1)} (\int_x^1 Z(y; F_{-X})dy)d\tilde{g}(x) \\ &= \int_{[0,1)} \rho_{1-x}(X)(1-x)d\tilde{g}(x).\end{aligned}$$

Letting $X = -1$, we have

$$1 = \int_0^1 g(x)dx = \int_{[0,1)} (1-x)d\tilde{g}(x).$$

From this observation and Lemma 10, we have the following.

Proposition 15 *Let $\rho : L^\infty \to \mathbf{R}$. Then the following conditions are equivalent.*
(1) *There is a set $\mathcal{M}_0$ of probability measures on $(0,1]$ such that for $X \in L^\infty$*

$$\rho(X) = \sup\{\int_{(0,1]} \rho_\alpha(X)m(d\alpha);\ m \in \mathcal{M}_0\}.$$

(2) *ρ is a law invariant coherent risk measure with the Fatou property.*

Now we prove Theorem 4. For each probability measure m on $[0,1]$, let $\nu_n(m)$, $n \geq 1$, be a probability measure on $(0,1]$ given by

$$\nu_n(m)(A) = m(A \cap (0,1]) + m(\{0\})\delta_{1/n}(A), \qquad \text{for a Borel set in } [0,1].$$

Then we see that for any $X \in L^\infty$

$$\int_{[0,1]} \rho_\alpha(X)m(d\alpha) = \sup_n \int_{(0,1]} \rho_\alpha(X)\nu_n(m)(d\alpha).$$

This and Proposition 15 imply Theorem 4. This completes the proof of Theorem 4.

4 Proof of Theorem 5

We give some computation on ρ_α in this section.

Proposition 16 *Let $c \in (0,1]$ and $X_c(\omega) = 1_{[1-c,1)}(\omega)$, $\omega \in \Omega = [0,1)$.*
(1) *We have*

$$\rho_\alpha(-X_c) = 1 \wedge \frac{c}{\alpha} \qquad \alpha \in (0,1]$$

(2) *Let m be a probability measure on $[0,1]$ and let $f(s) = \int_{[0,1]} \rho_\alpha(-X_s)m(d\alpha)$, $s \in (0,1]$ Then $f(c)$ is differentiable at $s = c \in (0,1)$ such that $m(\{c\}) = 0$, and*

$$\frac{df}{ds}(c) = \int_{(c,1]} \frac{1}{\alpha} m(d\alpha).$$

Proof. Noting that $F_{X_c}(x) = 1_{[1-c,1)}(x)$, $x \in [0,1)$, we easily have the assertion (1). Then we have for $0 < s < t < 1$

$$\frac{f(t)-f(s)}{t-s} = \int_{(t,1]} \frac{1}{\alpha} m(d\alpha) + \frac{1}{t-s}\int_{(s,t]} \frac{\alpha - s}{\alpha} m(d\alpha).$$

This proves the assertion (2). ∎

Theorem 5 is an easy consequence of Proposition 16 (2).

5 Supporting measures and Proof of Theorem 9

Let $\mathcal{M}$ denote the set of all probability measures on $[0,1]$. Then $\mathcal{M}$ is a compact metric space with the Prohorov metric. Let ρ be a law invariant coherent risk measure with the Fatou property. Let

$$\mathcal{M}(\rho) = \{m \in \mathcal{M};\ \int_{[0,1]} \rho_\alpha(X) m(d\alpha) \le \rho(X) \text{ for all } X \in L^\infty\}.$$

Since $\rho_\alpha(X)$ is continuous in $\alpha \in [0,1]$, $\mathcal{M}(\rho)$ is a closed convex subset of $\mathcal{M}$. Then from Theorem 4 we have

$$\rho(X) = \sup\{\int_{[0,1]} \rho_\alpha(X) m(d\alpha);\ m \in \mathcal{M}(\rho)\}, \qquad X \in L^\infty.$$

For each $X \in L^\infty$ let

$$\tilde{\mathcal{M}}(X;\rho) = \{m \in \mathcal{M}(\rho);\ \int_{[0,1]} \rho_\alpha(X) m(d\alpha) = \rho(X)\}.$$

From the compactness of $\mathcal{M}(\rho)$ we see that $\tilde{\mathcal{M}}(X;\rho) \neq \emptyset$. It is obvious that $\tilde{\mathcal{M}}(X;\rho)$ depends only on the distribution F_X of X, and so we denote it by $\mathcal{M}(F_X;\rho)$.

Now we prove Theorem 9. Let ρ be a law invariant coherent risk measure such that

$$\rho(X) \ge \mathrm{VaR}_\alpha(X), \qquad X \in L^\infty.$$

Let $X_\varepsilon(\omega) = 1_{[1-\alpha-\varepsilon,1)}(\omega)$, $\omega \in \Omega = [0,1)$, $\varepsilon \in (0, 1-\alpha)$, and let $m_\varepsilon \in \tilde{\mathcal{M}}(X_\varepsilon;\rho)$. Then by Proposition 16 we see that

$$\rho(-X_\varepsilon) = \int_{[0,1]} (1 \wedge \frac{\alpha+\varepsilon}{s}) m_\varepsilon(ds).$$

On the other hand, we have $\mathrm{VaR}_\alpha(-X_\varepsilon) = 1$. So we see that $m_\varepsilon([0,\alpha+\varepsilon]) = 1$. Since $\mathcal{M}(\rho)$ is compact, we see that there is an $m \in \mathcal{M}(\rho)$ such that $m([0,\alpha]) = 1$. Therefore we see that $\rho_\alpha(X) \le \rho(X)$, $X \in L^\infty$.

This completes the proof of Theorem 9.

6 Proof of Theorem 7

Proposition 17 *Let X, Y be comonotone random variables and $a, b \in \mathbf{R}$. Then $\{X \geq a\} \subset \{Y \geq b\}$ $P-a.s.$ or $\{Y \geq b\} \subset \{X \geq a\}$ $P-a.s.$*

Proof. Let $C = \{X \geq a\} \cap \{Y < b\}$ and $D = \{X < a\} \cap \{Y \geq b\}$. Then for $(\omega, \omega') \in C \times D$,

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) < 0.$$

This implies that $P(C) = 0$ or $P(D) = 0$. So we have our assertion. ∎

As an immediate consequence, we have the following.

Corollary 18 *Let X, Y be comonotone random variables. Then we have*

$$P(X + Y \geq a + b) \geq P(X \geq a) \wedge P(Y \geq b), \qquad a, b \in \mathbf{R}.$$

Proposition 19 *Let $X, Y \in L^\infty$ be comonotone and $a, b \in \mathbf{R}$. Then*

$$Z(x, F_{X+Y}) = Z(x, F_X) + Z(x, F_Y), \qquad x \in [0,1)$$

Proof. By the definition of $Z(x, F_X)$ we have $F_X(Z(x, F_X)-) \leq x$. So we have $P(X \geq Z(x, F_X)) \geq 1 - x$. Similarly we have $P(Y \geq Z(x, F_Y)) \geq 1 - x$. Therefore by Corollary 18 we have

$$P(X + Y \geq Z(x, F_X) + Z(x, F_Y)) \geq 1 - x, \qquad x \in [0,1).$$

Note that Let $Z(x, F_{X+Y}) = \sup\{z \in \mathbf{R};\ F_{X+Y}(z) \leq x\}$, $x \in (0,1)$. So we see that $Z(x, F_X) + Z(x, F_Y) \leq Z(x, F_{X+Y})$, $x \in (0,1)$. On the other hand, we have

$$\int_{[0,1)} (Z(x, F_X) + Z(x, F_Y))\mu(dx) = E[X] + E[Y] = \int_{[0,1)} Z(x, F_{X+Y}X)\mu(dx).$$

So we see that

$$Z(x, F_X) + Z(x, F_Y) = Z(x, F_{X+Y}), \qquad \mu - a.e.x.$$

Since both sides are right continuous, we have our assertion. ∎

Proposition 20 *ρ_α, $\alpha \in [0,1]$, are comonotone.*

Proof. For each $\alpha \in (0,1]$, we see that ρ_α is comonotone from the definition of ρ_α and Proposition 19. Letting $\alpha \downarrow 0$, we see that ρ_0 is also comonotone. ∎

Proposition 21 *Let ρ be a comonotone law invariant coherent risk measure with the Fatou property. Then $\bigcap_{i=1}^n \mathcal{M}(F_i; \rho) \neq \emptyset$ for any $n \geq 1$ and $F_1, F_2, \ldots, F_n \in \mathcal{D}$.*

Proof. Let $X_i(\omega) = Z(\omega, F_i)$, $\omega \in \Omega = [0,1)$, $i = 1, \ldots, n$. Then $\sum_{i=1}^k X_i$ and X_{k+1} are comonotone for each $k = 1, \ldots, n-1$. Let $X = \sum_{i=1}^n X_i$. Then we have

$$\rho(X) = \sum_{i=1}^n \rho(X_i).$$

Let $m \in \tilde{\mathcal{M}}(X;\rho)$. Then we have

$$\sum_{i=1}^{n} \int_{[0,1]} \rho_\alpha(X_i) m(d\alpha) = \rho(X) = \sum_{i=1}^{n} \rho(X_i).$$

Also, we have

$$\int_{[0,1]} \rho_\alpha(X_i) m(d\alpha) \le \rho(X_i), \qquad i = 1, \dots, n.$$

So we have

$$\int_{[0,1]} \rho_\alpha(X_i) m(d\alpha) = \rho(X_i), \qquad i = 1, \dots, n.$$

This implies $m \in \tilde{\mathcal{M}}(X_i;\rho)$, $i = 1, \dots, n$. So we have our assertion. ∎

Now let us prove Theorem 7. Suppose that $m \in \mathcal{M}$ and $\rho : L^\infty \to \mathbf{R}$ is given by

$$\rho(X) = \int_{[0,1]} \rho_\alpha(X) m(d\alpha), \qquad X \in L^\infty.$$

Then by Theorem 4 and Proposition 20, we see that ρ is comonotone and law invariant.

On the other hand, suppose that ρ is a comonotone law invariant coherent risk measure with the Fatou property. Then by Proposition 21 and the fact that $\mathcal{M}$ is compact, we see that $\bigcap\{\mathcal{M}(F;\rho);\ F \in \mathcal{D}\} \neq \emptyset$. Let m be an element of this set. Then we see that

$$\rho(X) = \int_{[0,1]} \rho_\alpha(X) m(d\alpha), \qquad X \in L^\infty.$$

This completes the proof of Theorem 7.

7 A Remark

For each $\alpha \in (0,1]$ let $\varphi_\alpha : [0,1] \to [0,1]$ be given by

$$\varphi_\alpha(t) = \frac{t}{\alpha} \wedge 1, \qquad t \in [0,1].$$

Then we have the following.

Proposition 22 *For any $\alpha \in (0,1]$ and $X \in L^\infty$ satisfying $X \le 0, P-a.s.$, we have the following.*

$$\rho_\alpha(X) = \int_0^\infty \varphi_\alpha(P(-X > y))dy.$$

Proof. Let $\alpha \in (0,1)$ and $X \in L^\infty$ such that $X \le 0$ and X has a continuous strictly increasing distribution on $(ess.inf\ X, ess.sup\ X)$. Then we see that $Z(x, F_{-X}) = F_{-X}^{-1}(x)$, $x \in (0,1)$. Let $q_\alpha \in (0,\infty)$ be such that $F_{-X}(q_\alpha) = 1-\alpha$. Then we have

$$\rho_\alpha(X) = -\frac{1}{\alpha}\int_{q_\alpha}^\infty y d(1 - F_{-X}(y))$$

$$= -\frac{1}{\alpha}[y(1 - F_{-X}(y))]_{q_\alpha}^\infty + \frac{1}{\alpha}\int_{q_\alpha}^\infty (1 - F_{-X}(y))dy$$

$$= \int_0^\infty \varphi_\alpha(P(-X > y))dy.$$

Since any nonpositive random variables is approximated by such random variables in probability, we have our assertion for $\alpha \in (0,1)$. Letting $\alpha \uparrow 1$, we also have our assertion for $\alpha = 1$. This completes the proof. ∎

Let $m \in \mathcal{M}$, and let $\varphi(t;m) = \int_{(0,1]} \varphi_\alpha(t)m(d\alpha)$, $t \in [0,1]$. Then we see that $\varphi(\cdot, m) : [0,1] \to [0,1]$ is a continuous increasing concave function with $\varphi(0) = 0$, and $\varphi(1) = 1 - m(\{0\})$. We also see that

$$\frac{d}{dt}\varphi(t;m) = \int_t^1 \frac{1}{\alpha} m(d\alpha),$$

for any continuous point $t \in (0,1)$ of the measure m. So $\varphi(\cdot, m)$ determines m.

For any nonpositive $X \in L^\infty$ we have

$$\int_0^\infty \rho_\alpha(X)m(d\alpha) = m(\{0\})ess.sup(-X) + \int_0^\infty \varphi(P(-X > y); m)dy.$$

These observations imply the following.

Theorem 23 *Let $\rho : L^\infty \to \mathbf{R}$. Then the following are equivalent.*
(1)ρ is a law invariant and comonotone coherent risk measure with the Fatou property.
(2) There is a continuous nondecreasing concave function $\varphi : [0,1] \to [0,1]$ such that

$$\rho(X) = (1 - \varphi(1))ess.sup(-X) + \int_0^\infty \varphi(P(-X > y))dy$$

for any nonpositive $X \in L^\infty$.

References

[1] Artzner, Ph., F. Delbaen, J.-M. Eber, and D. Heath, Coherent Measures of Risk, Math. Finance 9(1999), 203-228.

[2] Delbaen, F., Coherent Risk Measures on General Probability Spaces, Preprint 1999.

[3] Williams, D., *Probability with Martingales* Cambridge University Press 1991, Cambridge.