

A Hierarchy of the Fragments of the System of Inductive Definition (Preliminary Report)

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1 introduction

Gentzen [7] proved the consistency of PA (Peano Arithmetic) by using the transfinite induction up to the first *epsilon* number ϵ_0 . Here ϵ_0 is $\lim_k \omega_k$, where $\omega_0 = 0$ and $\omega_{k+1} = \omega^{\omega_k}$. Later in [8] he proved that the accessibility (i.e., transfinite induction) proof up to any ordinal less than ϵ_0 , eg., ω_k for any natural number k , is provable in PA .

In his [8] the nestedness complexity of implications used in the accessibility proof increases by one while the accessibility of one higher ω -tower ω_{k+1} is proved from the accessibility of ω_k . Hence by considering Gentzen's work [7, 8] a natural question arises; does the hierarchy of ω -towers, $\{\omega_k\}_{k=1,2,\dots}$, correspond exactly to a certain hierarchy of fragments of PA ?

Mints [10] answered this question by estimating the least upper bounds of accessibility ordinals for the fragments of PA , where the fragments are defined by means of the number of alternations of quantifiers, using one quantifier system developed in his former paper [9]. (Shirai [13] also gave a similar result by means of the number of quantifiers.)

The purpose of our paper is to investigate in a similar correspondence (between the hierarchy of critical ordinals and the hierarchy of fragment systems) for the system of ξ -iterated Inductive Definition ID_ξ [6]. We first analyze in Section 2 Arai's optimal accessibility proof for ID_ξ ([3]) to obtain a hierarchy of accessible ordinals for the fragments of intuitionistic ID_ξ , where the fragments are defined in terms of the nestedness complexity of implications. Then we show in Section 3 the least upper bounds of accessible ordinals (i.e., the critical ordinals) for those fragments, by analyzing Takeuti-Arai's consistency proofs of ID_ξ ([3]). In fact, for the upper bounds proof we use the fragments of classical ID_ξ in terms of the nestedness complexity of classical negations. Since the fragments of ID_ξ obtained by means of the number of alternations of quantifiers (in a prenex normal form) are also characterized by the nestedness complexity of negations with the help of universal quantifiers (by representing an existential quantifier $\exists$ by means of $\neg\forall\neg$), our result for ID_ξ corresponds to Mints' ([10]) for PA .

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2 Provability of transfinite inductions on $\omega(\xi, k, 0)$ in subsystems of $S_k(ID_\xi^i(\mathcal{U}_0))$

Let $(I, \prec)$ be the well ordered system whose order type is ordinal $\xi + 1$. Arai [1] proved the well ordering of Takeuti's system of ordinal diagram $O(\xi + 1, 1)$ in the system ID_ξ^i (the intuitionistic system of ξ -times iterated inductive definition).

In this chapter we introduce a hierarchy of fragments $S_k(ID_\xi^i)$ of ID_ξ^i based on the nestedness complexity of implications, and observe Arai's well ordering proof of [1] on these fragments.

Now we recall the definitions of $ID_\xi^i(\mathcal{U})$ and ID_ξ^i of Feferman [6].

Definition 1 (System $ID_\xi^i(\mathcal{U})$ and ID_ξ^i , cf. Feferman [6])

For any positive operator form $\mathcal{U}$, $ID_\xi^i(\mathcal{U})$ is obtained from PA by adding the following axiom schemata.

$$(P_\xi.1) \quad \rightarrow \forall x \prec \xi (A(P_x^\mathcal{U}, P_{\prec x}^\mathcal{U}, x) \subseteq P_x^\mathcal{U})$$

$$(P_\xi.2) \quad \rightarrow \forall x \prec \xi (\mathcal{U}(V, P_{\prec x}^\mathcal{U}, x) \subseteq V \supset P_x^\mathcal{U} \subseteq V)$$

$$(TI)_\xi \quad \text{Prog}[I, \prec, V] \rightarrow (I \subseteq V)$$

$$\text{where } P_{\prec a}^\mathcal{U} := \{x, y\} (x \prec a \wedge P^\mathcal{U} xy)$$

$$ID^i := \bigcup \{ID_\xi^i(\mathcal{U}) \mid \mathcal{U} \text{ is a positive operator form}\}$$

The starting point of Arai's well ordering proof is to define the notion of accessibility with respect to $\prec_i$ for $i \prec \xi$ (cf. §26 [14]) by using the set constants A_i which is definable in $ID_\xi^i(\mathcal{U}_0)$ with the following $\mathcal{U}_0$;

$$(A.1)_\xi \quad \forall i \prec \xi \text{Prog}[F_i, \prec_i, A_i]$$

$$(A.2)_\xi \quad \forall i \prec \xi (\text{Prog}[F_i, \prec_i, V] \rightarrow A_i \subseteq V) \quad \text{for each abstract } V \text{ in } ID_\xi^i(\mathcal{U}_0)$$

where $\mathcal{U}_0$ is a X -positive operator form defined as $\mathcal{U}_0(X, Y, i, \mu) := \mathcal{F}(i, \mu, Y) \wedge \forall \nu \prec_i \mu (\mathcal{F}(i, \nu, Y) \rightarrow X(\nu))$ where $\mathcal{F}(i, \mu, Y) := \forall k \prec i \forall \rho \subseteq_k \mu Y(k, \rho)$, $\text{Prog}[\alpha, \gamma, \beta] := \forall x (\alpha(x) \wedge \forall y (\gamma(y, x) \wedge \alpha(y) \rightarrow \beta(y)) \rightarrow \beta(x))$, and $F_i(\mu) := \forall j \prec i \forall \nu \subseteq_j \mu A_j(\nu)$ (the intended meaning of $F_i(\mu)$ is that μ is an i -fan (cf. Definition 26.16 [14])).

Remember that $ID_\xi^i(\mathcal{U})$ has the mathematical induction of the following form;

$$(VJ) \quad V(0), \forall x (V(x) \rightarrow V(x')) \rightarrow V(t)$$

The above $ID_\xi^i(\mathcal{U}_0)$ is the specific subsystem of the system ID_ξ^i of Inductive Definition in which the induction schemata are used only for the accessibility predicate A_i of ordinals.

We consider the subsystem $S_k(ID_\xi^i(\mathcal{U}_0))$ of $ID_\xi^i(\mathcal{U}_0)$ where each abstract V in $(A.2)_\xi$, $(TI)_\xi$ and (VJ) is restricted to that of level $lv(V) \leq k$; where $lv(V)$ is defined by the definition below.

We introduce the notion of level of A ($lv(A)$) for a formula A to express, roughly speaking, the implicational complexity of A . We assume that the language contains only $\forall$, $\supset$ and $\wedge$ for the logical connectives in this section.

We first recall the degree d of a formula in the language of $ID_\xi^i(\mathcal{U})$ defined in Arai [3], which intends to indicate how many times inductive definition is applied.

Definition 2 (cf. Def 2.4 in Arai [3])

- $d(t = s) = 0$ for all term t, s and predicate variable X .

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$$d(P^\mathcal{U} ts) = \begin{cases} i \oplus 1 & \text{if } t \text{ is a closed term whose value is } i \prec \xi. \\ \xi & \text{otherwise} \end{cases}$$

$$d(t_1 \prec s \wedge P^{\mathcal{U}} t_2 r) = \begin{cases} i & \text{if } s \text{ is a closed term whose value is } i \prec \xi \text{ and } t_1 \text{ is a} \\ & \text{closed term representing the same numeral as } t_2. \\ \xi & \text{otherwise} \end{cases}$$

Definition 3 (level $lv(A)$ of formula A in the language of $ID_{\xi}^i(\mathcal{U})$) For the formula A in the language of $ID_{\xi}^i(\mathcal{U})$, the level $lv(A)$ of the formula A is defined inductively as follows:

$$\begin{aligned} lv(P) &:= 0 \text{ for any atom of the language of } PA. \\ lv(A \wedge B) &:= \max\{lv(A), lv(B)\} \\ lv(\forall x A) &:= \begin{cases} \max\{2, lv(A)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases} \\ lv(A \supset B) &:= \begin{cases} \max\{lv(A) + 1, lv(B)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases} \\ lv(P^{\mathcal{U}} t) &:= \begin{cases} 1 & \text{if } d(P^{\mathcal{U}} t) = \xi \\ 0 & \text{otherwise} \end{cases} \\ lv(t \prec s \wedge P_t^{\mathcal{U}}) &:= \begin{cases} 1 & \text{if } d(P_t^{\mathcal{U}}) = \xi \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The subsystems $S_k(ID_{\xi}^i(\mathcal{U}))$ and $S_k(ID_{\xi}^i)$ of $ID_{\xi}^i(\mathcal{U})$ and ID_{ξ}^i are defined in terms of level lv as follows;

Definition 4 (the subsystem $S_k(ID_{\xi}^i(\mathcal{U}))$ of $ID_{\xi}^i(\mathcal{U})$) $S_k(ID_{\xi}^i(\mathcal{U}))$ is $ID_{\xi}^i(\mathcal{U})$ except that for every abstract V in $(A.2)_{\xi}$, $(TI)_{\xi}$ and (VJ) , $lv(V) \leq k$ holds.
 $S_k(ID_{\xi}^i) := \bigcup \{S_k(ID_{\xi}^i(\mathcal{U})) \mid \mathcal{U} \text{ is a positive operator form}\}$

The following notation is introduced;

Notation 1 Let $TI[\alpha, \gamma, \mu]$ denote the schema defined as $TI[\alpha, \gamma, \mu] := \alpha(\mu) \wedge (Prog[\alpha, \gamma, V] \rightarrow \forall \nu (\gamma(\mu, \nu) \wedge \alpha(\nu) \rightarrow V(\nu)))$. And $TI[\alpha, \gamma, \mu]_Q$ is the result of $TI[\alpha, \gamma, \mu]$ by substituting Q for V

Notation 2 $\omega(\xi, 0, \alpha) := \alpha$ and $\omega(\xi, n+1, \alpha) := (\xi, \omega(\xi, n, \alpha))$.

Then by checking Arai's well ordering proof of $O(\xi+1, 1)$ [1] carefully, Proposition 1 is easily observed.

Proposition 1 For a formula Q with $lv(Q) \leq 2$ and $k > 2$, $TI[F_0, <_0, \omega(\xi, k, 0)]$ is provable in $S_k(ID_{\xi}^i(\mathcal{U}_0))$. Namely, the ordinal $\omega(\xi, k, 0)$ is accessible in $S_k(ID_{\xi}^i(\mathcal{U}_0))$ with respect to $<_0$.

Proof.

We follow Arai's [1].

We only consider the case in which ξ is a limit. (See Remark after Proposition 2 for the successor ξ case.) Let $\bigcap_{k \prec i} A_k := \{\mu\} \forall k \prec i A_k(\mu)$. In Lemma 3 of [1] $(TI)_{\xi}$ is used with the abstract $\{i\} Prog[F_i, <_i, \bigcap_{k \prec i} A_k] := \{i\} \forall x (F_i(x) \wedge \forall y <_i x (F_i(y) \rightarrow \bigcap_{k \prec i} A_k(y)) \rightarrow \bigcap_{k \prec i} A_k(x))$, here $lv(Prog[F_i, <_i, \bigcap_{k \prec i} A_k(\mu)]) = 3$. Let $\bar{A} := \bigcap_{j \prec \xi} A_j$ and $R_i(\nu) := \forall \mu <_{\xi} (i, \nu) (F_{\xi}(\mu) \rightarrow \bar{A}(\mu))$. In Lemma 4 of [1] $(A.2)_{\xi}$ is used with the abstract $\{x\} R_i(x) := \forall \mu <_{\xi} (i, x) (F_{\xi}(\mu) \rightarrow \bar{A}(\mu))$ (with $lv(R_i(x)) = 2$) and $(TI)_{\xi}$ is used with the abstract $\{i\} R_i(0) := \forall \mu <_{\xi} (i, 0) (F_{\xi}(\mu) \rightarrow \bar{A}(\mu))$ (with $lv(R_i(0)) = 2$).

Then in Lemma 5 of [1] it is shown that $TI[F_{\xi}, <_{\xi}, (\xi, 0)]_Q$ is provable in $ID_{\xi}^i(\mathcal{U}_0)$ for each unary predicate $Q(x)$ in $ID_{\xi}^i(\mathcal{U})$; In the case where $\lim(\xi)$, $(A.2)_{\xi}$ are used

with the abstract $\{x\}(x \prec_\xi (i, 0) \rightarrow Q(x))$ for all $i \prec \xi$ (with level $lv(Q)$). In the case where $Suc(\xi)$, $(A.2)_\xi$ is used with the abstract $\{x\}(x <_\xi (\xi, 0) \rightarrow Q(x))$ (with level $lv(Q)$).

Hence until now it is observed that

$$(I) \quad S_{Max(3, lv(Q))}(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, (\xi, 0)]_Q.$$

From (I) it is derived in the way familiar by Gentzen [8] that

$$(II) \quad S_{k+3}(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, k+3, 0)]_Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.$$

Let us observe the proof of (II). In Lemma 7 of [1] it is shown that $Prog[F_\xi, <_\xi, Q] \rightarrow Prog[F_\xi, <_\xi, s[Q]]$, where $s[Q]$ is a jump operator defined as $s[Q](\mu) := \forall \rho (F_\xi(\rho) \rightarrow \forall \nu <_\xi \rho (F_\xi(\nu) \rightarrow Q(\nu)) \rightarrow \forall \nu <_\xi \rho + (\xi, \mu)^\xi (F_\xi(\nu) \rightarrow Q(\nu)))$, where $\lambda \nu \mu. \mu + \nu^\xi$ is a primitive recursive function which is a generalization of $\lambda \nu \mu. \nu + \omega^\mu$ of Gentzen [8] and defined in [1] as follows;

- If $\mu = 0$, then $\mu + \nu^\xi = \nu + \mu^\xi = \nu$
- Suppose $\mu \neq 0$ and $\nu \neq 0$ and
 $\mu \equiv \mu_1 \# \dots \# \mu_m$ with $\mu_1 \geq_\xi \dots \geq_\xi \mu_m \neq 0$
 $\nu \equiv \nu_1 \# \dots \# \nu_n$ with $\nu_1 \geq_\xi \dots \geq_\xi \nu_n \neq 0$
 Let l be the number such that $0 \leq l \leq m$ and $\mu_l \leq_\xi \nu_1 <_\xi \mu_{l+1}$,
 then $\mu + \nu^\xi := \mu_1 \# \dots \# \mu_l \# \nu_1 \# \dots \# \nu_n$

Note that $lv(s^n[Q]) = n + Max(2, lv(Q))$ with $n \geq 1$, where $s^n[Q] := \overbrace{s[\dots s[Q]\dots]}^{n\text{-times}}$.

Let us sketch the proof of $Prog[F_\xi, <_\xi, Q] \rightarrow Prog[F_\xi, <_\xi, s[Q]]$ due to Gentzen [8], where a mathematical induction of the level $\leq lv(Q)$ is used;

Assume

$$Prog[F_\xi, <_\xi, Q] \quad \dots (1)$$

$$F_\xi(x) \wedge \forall y <_\xi x (F_\xi(y) \rightarrow s[Q](y)) \quad \dots (2)$$

We have to show $s[Q](x)$. So assume further

$$F_\xi(\rho) \quad \dots (3)$$

$$\forall \nu <_\xi \rho (F_\xi(\nu) \rightarrow Q(\nu)) \quad \dots (4)$$

$$\nu <_\xi \rho \oplus (\xi, x)^\xi \wedge F_\xi(\nu) \quad \dots (5)$$

Under the above assumptions (1) $\sim$ (5), we have to show $Q(\nu)$.

Consider the case where $x \neq 0$. Since $\nu <_\xi \rho \oplus (\xi, x)^\xi$, there exists primitive recursive functions f and g such that $\nu <_\xi \rho \oplus (\xi, f(x, \nu, \rho)) \cdot g(x, \nu, \rho)$ with $f(x, \nu, \rho) <_\xi x$ and $F_\xi(f(x, \nu, \rho))$. From (2), $s[Q](f(x, \nu, \rho))$ holds. Then a universal instantiation with $\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n$ (note that $\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n <_\xi \rho \oplus (\xi, x)^\xi$) for an arbitrary n allows the following:

$$F_\xi(\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n) \rightarrow \forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n (F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <_\xi (\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n) \oplus (\xi, f(x, \nu, \rho)^\xi) (F_\xi(\eta) \rightarrow Q(\eta)) \dots (6)$$

From $F_\xi(\rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n)$ (from (5)) and the property of Suc , the following holds;

$$\forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n (F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot Suc(n) (F_\xi(\eta) \rightarrow Q(\eta)) \dots (7)$$

Then mathematical induction with abstract $\{n\}(\forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot n (F_\xi(\eta) \rightarrow Q(\eta)))$, whose level is $Max(2, lv(Q))$, implies (with (4)) $\forall \eta <_\xi \rho \oplus (\xi, f(x, \nu, \rho)^\xi) \cdot g(x, \nu, \rho) (F_\xi(\eta) \rightarrow Q(\eta))$. Hence from (5), $Q(\nu)$ holds.

Consider the case where $x = 0$. For each formula Q , $s[Q]$ denotes the formula of the following form; $s[Q](\mu) := \forall \rho (F_\xi(\rho) \rightarrow \forall \nu <_\xi \rho (F_\xi(\nu) \rightarrow Q(\nu)) \rightarrow \forall \nu <_\xi \rho + \mu^\xi (F_\xi(\nu) \rightarrow Q(\nu)))$. Then we can prove without $(A.1)_\xi$, $(A.2)_\xi$, TI_ξ and the mathematical induction that $Prog[F_\xi, <_\xi, Q] \rightarrow Prog[F_\xi, <_\xi, s[Q]]$. As is shown

above, in Lemma 7 of [1] all the mathematical inductions used are restricted to those of level $\leq \text{Max}(2, \text{lv}(Q))$.

From now we assume $\text{lv}(Q) \leq 2$. With the help of $\text{Prog}[F_\xi, <_\xi, Q] \rightarrow \text{Prog}[F_\xi, <_\xi, s[Q]]$ and $\text{Prog}[F_\xi, <_\xi, s[Q]] \rightarrow \text{Prog}[F_\xi, <_\xi, s^2[Q]]$, in which proof all mathematical inductions are restricted to those of level ≤ 3 , (I) implies the following (II)₀;

$$(II)_0 \quad S_3(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, 3, 0)]_Q$$

By replying this methode, the above (II) is obtained.

Then following Arai [1], the next proposition is derived from (II).

$$S_{k+3}(ID_\xi^i(\mathcal{U}_0)) \vdash TI[F_0, <_0, \omega(\xi, k+3, 0)]_Q \text{ with } \text{lv}(Q) \leq 2 \text{ and } k \geq 0.$$

Hence the proposition holds.

□

Using the above, Proposition 2 follows;

Proposition 2 *For $k > 2$, the ordinal up to $\omega(\xi, k+1, 0)$ is accessible in $S_k(ID_\xi^i(\mathcal{U}_0))$ with respect to $<_0$.*

Remark 1;

From the case in which ξ is a successor ordinal, the transfinite induction formula $\{i\}\text{Prog}[F_i, <_i, \bigcap_{k < i} A_k]$ at the beginning of the proof of Proposition 1 above is replaced by $\{i\}\text{Prog}[F_i, <_i, A_i]$, which has level 2, instead of 3. Hence, the Propositions 1 and 2 hold for $k > 1$.

3 Unprovability of the transfinite induction up to $\omega(\xi, k+1, 0)$ in system $S_k(AI_\xi^-)$

Our aim in this chapter is to prove the estimation we have observed in previous chapter is sharp one;

$$S_k(ID_\xi) \not\vdash TI[F_0, <_0, \omega(\xi, k+1, 0)] \text{ for } k > 2$$

On the whole segment of $ID_\xi = \bigcup_n S_n(ID_\xi)$, Arai [3] proves that $ID_\xi \not\vdash TI[F_0, <_0, O(\xi+1, 1)]$. Note that $O(\xi+1, 1) := \bigcup_k \omega(\xi, k, 0)$. He shows that the consistency of ID_ξ is provable using tranfinite induction up to $O(\xi+1, 1)$ by the proof reduction method which is originally due to Gentzen-Takeuti. In this section we modify his consistency proof in more delicate manner and prove the following by the cut elimination (proof reduction) method;

$$TI[F_0, <_0, \omega(\xi, k+1, 0)] \vdash \text{Cons}(S_k(ID_\xi)) \text{ for } k > 2$$

Our crucial point is to introduce a η -height h_η for each $\eta \preceq \xi$ (Definition 11) and consider a ordinal assignment to a proof $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$ with ξ -sort of height (Definition 13).

For the Gentzen-Takeuti cut elimination procedure to work, Arai [3] formalises his system AI_ξ^- of ξ -times iterated inductive definition in the form of iterated comprehension axiom by using second order free variables. System AI_ξ^- is defined by adding the following principles based on PA .

Definition 5 (System AI_{ξ}^{-} , cf. Arai [3])

For any arithmetical form $\mathcal{B}$, the following axioms schemata are added.

$$(Q^{\mathcal{B}} : right) \quad \frac{\Gamma \rightarrow \Delta, \mathcal{B}(X, Q_{\prec t}^{\mathcal{B}}, t, s)}{\Gamma \rightarrow \Delta, Q^{\mathcal{B}}ts} \quad \text{where } Q_{\prec t}^{\mathcal{B}} := \{x, y\}(x \prec t \wedge Q^{\mathcal{B}}xy)$$

$$(Q^{\mathcal{B}} : left) \quad t \prec \xi, Q^{\mathcal{B}}ts \rightarrow \mathcal{B}(V, Q_{\prec t}^{\mathcal{B}}, t, s)$$

We assume that the language contains only $\forall, \neg$ and $\wedge$ for the logical connectives. Then, the definition of lv in the previous section is modified as follows;

Definition 6 (η -level $lv_{\eta}(A)$ of a formula A with $\eta \preceq \xi$) For the formula A in the language of AI_{ξ}^{-} and an ordinal $\eta \preceq \xi$, the η -level $lv_{\eta}(A)$ of the formula A is defined inductively as follows, where d is defined in Definition 2 of previous section with using $Q^{\mathcal{B}}$ instead of $P^{\mathcal{U}}$ and $d(Xt) := 0$ (for X a predicate variable):

$$\begin{aligned} lv_{\eta}(P) &:= 0 \text{ for any atom of } L_{PA}. \\ lv_{\eta}(A \wedge B) &:= \max\{lv_{\eta}(A), lv_{\eta}(B)\} \\ lv_{\eta}(\forall x A) &:= \begin{cases} \max\{2, lv_{\eta}(A)\} & \text{if } lv_{\eta}(A) \geq 1 \\ 0 & \text{if } lv_{\eta}(A) = 0 \end{cases} \\ lv_{\eta}(\neg A) &:= \begin{cases} lv_{\eta}(A) + 1 & \text{if } lv_{\eta}(A) \geq 1 \\ 0 & \text{if } lv_{\eta}(A) = 0 \end{cases} \\ lv_{\eta}(Q_t^{\mathcal{B}}) &:= \begin{cases} 1 & \text{if } d(Q_t^{\mathcal{B}}) = \eta \\ 0 & \text{otherwise} \end{cases} \\ lv_{\eta}(t \prec s \wedge Q_t^{\mathcal{B}}) &:= \begin{cases} 1 & \text{if } d(t \prec s \wedge Q_t^{\mathcal{B}}) = \eta \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that lv_{η} for $\eta = \xi$ is the same as lv of the previous section (with using $Q^{\mathcal{B}}$ instead of $P^{\mathcal{U}}$ in the definition of lv of the previous section with replacing $\supset$ by $\neg$.)

We can define the fragments $S_k(AI_{\xi}^{-})$ in the same manner as $S_k(ID_{\xi})$ as follows.

Definition 7 (the subsystem $S_k(AI_{\xi}^{-})$ of AI_{ξ}^{-}) $S_k(AI_{\xi}^{-})$ is AI_{ξ}^{-} except that for every abstract V in $Q^{\mathcal{B}}:left$ and (VJ) , $lv_{\xi}(V) \leq k$ holds.

ID_{ξ} is obtained from ID_{ξ}^i in the previous section by changing the underlying logic from the intuitionistic to the classical. For each formula F of the language of ID_{ξ} , we define a formula F^* of the language of AI_{ξ} by substituting $Q^{\mathcal{B}}$ for all occurrences of $P^{\mathcal{U}}$, where

$$\mathcal{B}(X, Y, c_0, c_1) := \forall y(U(X, Y, c_0, y) \rightarrow Xy) \rightarrow Xc_1.$$

It is well known that by this $*$, ID_{ξ} is embeddable into AI_{ξ} (cf. [3]). Obviously $lv(F) = lv_{\xi}(F^*)$ holds i.e., ξ -level of a formula remains the same through the above interpretation.

Until the end of this section, we assume that all formulas occurring in a proof figure of AI_{ξ}^{-} are of the following normal form:

Lemma 1 (the normal form of a formula in AI_{ξ}^{-}) For arbitrary formula A of the language of AI_{ξ}^{-} , there exists a formula of the following form, called a normal formula, which is equivalent to A (in LK);

$$\forall \vec{x}_1 \neg \dots \forall \vec{x}_n \forall \neg \quad \forall \vec{y} D[Q^{\mathcal{B}}_{t_1 s_1}, \dots, Q^{\mathcal{B}}_{t_m s_m}]$$

where $D[*_1, \dots, *_m]$ is a context of the language of PA , and no quantifier occurring in D bounds any $*_i$ ($1 \leq i \leq m$) and $lv_{\eta}(D[Q^{\mathcal{B}}_{t_1 s_1}, \dots, Q^{\mathcal{B}}_{t_m s_m}]) \leq 2$ for any $\eta \preceq \xi$.

Definition 8 (normal proofs) Let S be a sequent of normal formulas. A normal proof of S is a proof in which $\forall\neg$ -left rules are used, instead of $\forall$ -left rules in a proof;

$$\frac{\Gamma \rightarrow \Delta, A(t_1, \dots, t_n)}{\forall x_1 \dots x_n \neg A(x_1, \dots, x_n), \Gamma \rightarrow \Delta} \forall\neg\text{-left}$$

Note that the original $\neg$ -left rule may also appear in a normal proof.

Lemma 2 Any provable sequent of normal formulas has a normal proof.

From now on we assume any $S_k(Al_\xi^-)$ -proof to be normal by virtue of the above two lemmata.

Definition 9 For each formula A , $\eta(A) \preceq \xi$ is defined as $\eta(A) := \text{Max}\{\eta \mid lv_\eta(A) \neq 0\}$.

Definition 10 ($g_\eta(A)$ with $\eta \prec \xi$)

$$g_\eta(A) := \begin{cases} g(A) & \text{if } \eta(A) \geq \eta \\ 0 & \text{if } \eta(A) < \eta \end{cases}$$

where $g(A)$ denotes the number of logical symbols in A .

We modify the notion of proof with degree $\langle P, d \rangle$ of Arai [3] into $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$ by introducing ξ -sort of height $\{h_\eta\}_{\eta \preceq \xi}$, as follows:

Definition 11 (A proof with ξ -sort of height $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$) A proof $\langle P, d \rangle$ (with degree d) is called a proof with ξ -sort of height $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$ if for each sequent S of P and each ordinal $\eta \preceq \xi$, a natural number $h_\eta(S)$ satisfying the following condition is assigned. We call h_η a η -height.

0. $h_\eta(S) = 0$ for every $\eta \preceq \xi$ if S is the end sequent of P .

For the last inference I of the form

$$I \quad \frac{S}{S'}$$

1. $h_\eta(S) = 0$ for every $\eta \preceq \xi$ if I is a substitution.
2. $h_\eta(S) = h_\eta(S')$ for every $\eta \preceq \xi$ if I is an inference except substitution, induction and cut.
3. $\begin{cases} 1 & h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} \text{ for } \eta \prec \xi \\ 2 & h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} \end{cases}$
if I is a cut, where D is the cut formula of the inference I .
4. $\begin{cases} 1 & h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} + 1 \text{ for } \eta \prec \xi \\ 2 & h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} + 1 \end{cases}$
if I is an induction.

Definition 12 For each sequent S of $\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$, $\eta(S) \preceq \xi$ is defined as $\eta(S) := \begin{cases} d(I) & \text{if } S \text{ is the upper sequent of the substitution } I \\ \text{Max}\{\eta \mid h_\eta(S) \neq 0\} & \text{otherwise} \end{cases}$

The following is an immediate consequence from Definition 12.

Lemma 3 For any proof with ξ -sort of height $< P, \{h_\eta\}_{\eta \leq \xi}, d >$ and for any inference I (with a lower sequent S' and an upper sequent S) in $< P, \{h_\eta\}_{\eta \leq \xi}, d >$,

$$\eta(S) \geq \eta(S')$$

holds.

Notation 3 For $i \leq \xi$ and an ordinal diagram α , an ordinal diagram $\omega(i, n, \alpha)$ is defined inductively as follows.

- $\omega(i, 0, \alpha) := \alpha$
- $\omega(i, n + 1, \alpha) := (i, \omega(i, n, \alpha))$

Definition 13 (ordinal assignment) Let I be an inference of the form

$$I \quad \frac{S_1 \quad S_2}{S}$$

Then $O(S)$ is defined as follows:

1. When I is a cut,

$$O(S) := \omega(\eta(S), k - h_{\eta(S)}(S), c[\omega(\eta(S_1), h_{\eta(S_1)}(S_1), O(S_1) \# O(S_2))])$$

Here $k := \text{Max}\{h_{\eta(S)}(T) \mid T \text{ is above } I\}$ and $c[*] := \omega(\gamma_1, k_1, \omega(\gamma_2, k_2, \dots, \omega(\gamma_n, k_n, *)))$, where $\{\gamma_1, \dots, \gamma_n\} := \{\gamma \mid \eta(S) < \gamma < \eta(S_1) \text{ and } h_\gamma(T) \neq 0 \text{ for some } T \text{ above } I\}$ with $\gamma_1 < \dots < \gamma_n$ and $k_i := \text{Max}\{h_{\gamma_i}(T) \mid T \text{ is above } I\}$.¹

2. When I is a logical inference,

$$O(S) := O(S_1) \# O(S_2) \# 0$$

3. When I is a structural inference,

$$O(S) := O(S_1) \# O(S_2)$$

4. When I is a substitution,

$$O(S) := (d(I), O(S_1))$$

Theorem 1 The transfinite induction on $\omega(\xi, k + 1, 0)$ is unprovable in $S_k(AI_\xi^-)$ for $k > 2$.

Proof.

We refine the proof reduction process of Arai [3] to define the reduction process for $S_k(AI_\xi^-)$ ($k > 2$), and show that the well-orderness of $\omega(\xi, k + 1, 0)$ implies the termination of the reduction process, hence the consistency of $S_k(AI_\xi^-)$. Then the above theorem follows from Gödel's incompleteness theorem.

(preparation)

Without loss of generality, we assume that all logical initial sequents of the form $p \rightarrow p$ where p is an atomic and that there exists no free variables which is not used as an eigenvariable.

(elimination of initial sequents in the end-piece) As usual.

(elimination of weakening) elimination of weakening known in the usual way (cf. Takeuti [14]) does work not only for a weakening in end-piece but also for a more general weakening with such a weakening formula D as the bundle I (cf. p78 of [14]) which begins with D ends with a cut formula D and no logical inference affect I .

¹In the case where $\eta(S) = \eta(S_1)$, $c[*]$ is $*$ and $O(S) := \omega(\eta(S), k + h_{\eta(S)}(S_1) - h_{\eta(S)}(S), O(S_1) \# O(S_2))$.

(elimination of the mathematical induction rule) As usual.

Then from sublemma 12.9 of [14], there exists a suitable cut J in the end piece of $\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$. Let I_1 and I_2 be boundary logical inferences whose principal formulas are ancestors of left and right cut formulas of J .

We shall demonstrate following three essential cases both for limit ordinal ξ and for successor ordinal ξ ;

(Case 1) The case where the cut formula $C := A \wedge B$ with $\eta(C) \prec \xi$:

Let K (whose lower sequent is T and whose upper sequent is T_1) denotes the uppermost inference below J such that either (i) or (ii) holds;

$$\begin{aligned} \eta(T) &= \eta(A) \wedge (h_{\eta(A)}(S_1) > h_{\eta(A)}(T)) \cdots (i) \\ \eta(T) &< \eta(A) \cdots (ii) \end{aligned}$$

where A is the auxiliary formula of I_1 and I_2

$\langle P, \{h_\eta\}_{\eta \leq \xi}, d \rangle$ is as follows:

$$\begin{array}{c} \frac{\frac{S_1^{I_1}}{S_1^{I_1}} \quad \frac{S_2^{I_1}}{S_2^{I_1}} \quad I_1 \quad \frac{S_1^{I_2}}{S_1^{I_2}} \quad \frac{S_2^{I_2}}{S_2^{I_2}} \quad I_2}{S_1^J \quad S_2^J} J \\ \frac{T_1}{T} K \\ \vdots \end{array} \quad \begin{array}{l} S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, A_1 \\ S_2^{I_1} : \Gamma_2 \rightarrow \Delta_2, B_1 \\ S_1^{I_2} : \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, A_1 \wedge B_1 \\ S_1^{I_2} : A_3, \Pi_3 \rightarrow \Lambda_3 \\ S_2^{I_2} : A_3 \wedge B_3, \Pi_3 \rightarrow \Lambda_3 \\ S_1^J : \Gamma \rightarrow \Delta, A \wedge B \\ S_2^J : A \wedge B, \Pi \rightarrow \Lambda \\ S_1^J : \Gamma, \Pi \rightarrow \Delta, \Lambda \\ T : \Phi \rightarrow \Psi \end{array}$$

$\langle P', \{h'_\eta\}_{\eta \leq \xi}, d' \rangle$ is as follows, where $\tilde{I}_1$ and $\tilde{I}_2$ are weakening-right (with a weakening formula A_1) and weakening-left (with a weakening formula A_3) respectively;

$$\begin{array}{c} \frac{\frac{S_1^{I_1}}{S_1^{I_1}} \quad \tilde{I}_1 \quad \frac{S_2^{I_2}}{S_2^{I_2}} \quad \tilde{I}_2}{S_1^* \quad S_2^*} J \quad \frac{\frac{S_1^{I_1}}{S_1^{I_1}} \quad \tilde{I}_1 \quad \frac{S_2^{I_2}}{S_2^{I_2}} \quad \tilde{I}_2}{S_1^* \quad S_2^*} J \\ \frac{T_1^*}{U_1} K \quad \frac{T_2^*}{U_2} K \\ \frac{T^*}{I} \end{array} \quad \begin{array}{l} S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, A_1, A_1 \wedge B_1 \\ S_1^* : \Gamma \rightarrow \Delta, A, A \wedge B \\ S_2^{I_2} : A_3, A_3 \wedge B_3, \Pi_3 \rightarrow \Lambda_3 \\ S_2^* : A \wedge B, A, \Pi \rightarrow \Lambda \\ U_1 : \Phi \rightarrow \Psi, A \\ U_2 : A, \Phi \rightarrow \Psi \\ T^* : \Phi, \Phi \rightarrow \Psi, \Psi \end{array}$$

(case 1.1): The case where (i) holds. Then for any sequent T' between S_1 and T , $\eta(T') \geq \eta(A)$ holds.

(case 1.1.1) $\eta(T_1) = \eta(T)$

$O_{P'}(T^*) <_0 O_P(T)$ is checked as usual way.

(case 1.1.2) $\eta(T_1) > \eta(T)$

special case of (case 1.2)

(Case 1.2): The case where (ii) holds. Then $\eta(T) < \eta(A) \leq \eta(T_1)$ holds. We assign

$$h'_\eta(U_1) := \begin{cases} h_\eta(T) & \text{if } \eta < \eta(T) \\ g(A) & \text{if } \eta = \eta(A) \text{ , and } h'_\eta(T_1^*) := h'_\eta(T_1^{**}) := h_\eta(T_1) \text{ for all } \eta \leq \xi. \\ 0 & \text{otherwise} \end{cases}$$

Hence $\eta(U_1) = \eta(A)$ holds. On the other hand, there exist contexts a and b such

that $O_P(T) = \omega(\eta(T), k - h_{\eta(T)}(T), a[\omega(\eta(A), k_i, b[\alpha_1 \# \alpha_2])])$,
 $O_{P'}(U_1) = \omega(\eta(U_1), m - h_{\eta(U_1)}(U_1), b[\alpha'_1 \# \alpha_2]) = \omega(\eta(A), m - g(A), b[\alpha'_1 \# \alpha_2])$ and
 $O_{P'}(T^*) = \omega(\eta(T), k' - h_{\eta(T)}(T), a[\omega(\eta(U_1), g(A), O_{P'}(U_1) \# O_{P'}(U_2))])$
 Since $\omega(\eta(A), k_i, b[\alpha_1 \# \alpha_2]) > \omega(\eta(U_1), g(A), O_{P'}(U_1) \# O_{P'}(U_2))$, $O_{P'}(T^*) <_0 O_P(T)$
 holds.

(Case 2) The case where cut formula is $\forall \vec{x} \neg B(\vec{x})$:

$\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$ is as follows; here I_2 is $\forall \neg$ -left.

$$\begin{array}{c}
 \frac{S_1^{I_1}}{S_1^{I_1}} I_1 \quad \frac{S_1^{I_2}}{S_1^{I_2}} I_2 \\
 \vdots \quad \vdots \\
 \frac{S_1^J}{S_1^J} \quad \frac{S_2^J}{S_2^J} J \\
 \hline
 S^J \\
 \vdots \\
 \overline{T} \quad K \\
 \vdots
 \end{array}
 \quad
 \begin{array}{l}
 S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, \neg B(\vec{x}) \\
 S_1^{I_1} : \Gamma_1 \rightarrow \Delta_1, \forall \vec{x} \neg B(\vec{x}) \\
 S_1^{I_2} : \Pi_1 \rightarrow \Lambda_1, \neg B(\vec{t}) \\
 S_1^{I_2} : \forall \vec{x} \neg B(\vec{x}), \Pi_1 \rightarrow \Lambda_1 \\
 S_1^J : \Gamma \rightarrow \Delta, \forall \vec{x} \neg B(\vec{x}) \\
 S_2^J : \forall \vec{x} \neg B(\vec{x}), \Pi \rightarrow \Lambda \\
 T : \Phi \rightarrow \Psi
 \end{array}$$

$\langle P', \{h'_\eta\}_{\eta \preceq \xi}, d' \rangle$ is as follows, where $\tilde{I}_1$ and $\tilde{I}_2$ are weakening-right and weakening-left (respectively) with weakening formulas $\forall \vec{x} \neg B(\vec{x})$. Note that by virtue of (preparation) and (elimination of weakening), any formula of the form $\neg B(\vec{x})$ which is an ancestor of the auxiliary formula of I_1 is a descendant of principal formulas of an inference $\neg$ -right. Hence the following $S_1^{I_1}(\vec{x})$ can be obtained.

$$\begin{array}{c}
 \frac{S_1^{I_1}(t)}{S_1^{I_1}(t)} \tilde{I}_1 \quad \frac{\quad}{\quad} I_2 \\
 \vdots \quad \vdots \\
 \frac{S_1^J * (t)}{S_1^J * (t)} \quad \frac{S_2^J}{S_2^J} J \\
 \hline
 S_1^*(t)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\quad}{\quad} I_1 \quad \frac{S_1^{I_2}}{S_1^{I_2}} \tilde{I}_2 \\
 \vdots \quad \vdots \\
 \frac{S_1^J}{S_1^J} \quad \frac{S_2^J *}{S_2^J *} J \\
 \hline
 S_2^*
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{\frac{\quad}{U_1(t)} K \quad \frac{\quad}{U_2} K}{T^*} N}{\quad} \\
 \vdots
 \end{array}$$

$$\begin{array}{ll}
 S_1^{I_1}(\vec{x}) : & B(\vec{x}), \Gamma_1 \rightarrow \Delta_1 \\
 S_1^{I_1}(\vec{x}) : & B(\vec{x}), \Gamma_1 \rightarrow \Delta_1, \forall \vec{x} \neg B(\vec{x}) \\
 S_1^J * (\vec{x}) : & B(\vec{x}), \Gamma \rightarrow \Delta, \forall \vec{x} \neg B(\vec{x}) \\
 S_1^*(\vec{x}) : & B(\vec{x}), \Gamma, \Pi \rightarrow \Delta, \Lambda \\
 S_2^J * : & \forall \vec{x} \neg B(\vec{x}), \Pi \rightarrow \Lambda, B(\vec{t}) \\
 S_2^* : & \Gamma, \Pi \rightarrow \Delta, \Lambda, B(\vec{t}) \\
 S_1^{I_2} : & \Pi_1 \rightarrow \Lambda_1, B(\vec{t}) \\
 S_1^{I_2} : & \forall \vec{x} \neg B(\vec{x}), \Pi_1 \rightarrow \Lambda_1, B(\vec{t}) \\
 U_1(\vec{x}) : & B(\vec{x}), \Phi \rightarrow \Psi \\
 U_2 : & \Phi \rightarrow \Psi, B(\vec{t}) \\
 T^* : & \Phi, \Phi \rightarrow \Psi, \Psi
 \end{array}$$

Since $lv_{\eta(B(\vec{x}))}(\forall \vec{x} \neg B(\vec{x})) > lv_{\eta(B(\vec{x}))}(B(\vec{x}))$ holds, $O(P') <_0 O(P)$ is checked as the usual way.

(Case 3) The case where the cut formula of J is $Q^{\mathbf{B}}t_s$:

$\langle P, \{h_\eta\}_{\eta \preceq \xi}, d \rangle$ is as follows, where K (with the lower sequent T) denotes the upper most inference below J such that $\eta(T) \preceq d(B(X, Q_{\prec t}, t, s)) := i$;

Let T_1 denote such upper sequent of K that is below J .

$$\begin{array}{c}
 \vdots \\
 \frac{S_1^{I_1}}{S_1^{I_1}} I_1 \quad S \\
 \vdots \\
 \frac{S_1^J}{S_1^J} \quad S_2^J \quad J \\
 \vdots \\
 \frac{T_1 \quad (T_2)}{T} K \\
 \vdots
 \end{array}
 \quad
 \begin{array}{l}
 S : \quad t_2 \prec \xi, Q t_2 s_2 \rightarrow B(V, Q_{\prec t_2}, t_2, s_2) \\
 S_1^{I_1} : \quad \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 S_1^{J_1} : \quad \Gamma_1 \rightarrow \Delta_1, Q t_1 s_1 \\
 S_1^J : \quad \Gamma_2 \rightarrow \Delta_2, Q t s \\
 S_2^J : \quad Q t s, \Pi \rightarrow \Lambda \\
 S^J : \quad \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2 \\
 T_1 : \quad \Phi_1 \rightarrow \Psi_1 \\
 T_2 : \quad \Phi \rightarrow \Psi
 \end{array}$$

$$O_P(T) = \omega(\eta(T), k - h_{\eta(T)}(T), c[\omega(\eta(T_1), h_{\eta(T_1)}(T_1), O_P(T_1) \# O_P(T_2))]).$$

$\langle P', \{h'_\eta\}_{\eta \preceq \xi}, d' \rangle$ is as follows, where $\tilde{I}_1$ is weakening-right with a weakening formula $Q^B t_1 s_1$;

$$\begin{array}{c}
 \vdots \\
 \frac{S_1^{I_1}}{S_1^{I_1}} \tilde{I}_1 \quad S \\
 \vdots \\
 \frac{S_1^{J*}}{S_1^{J*}} \quad S_2^J \quad J \\
 \vdots \\
 \frac{T_1^* \quad (T_2)}{T^*} \\
 \vdots \\
 \frac{T^*}{\tilde{T}^*} \text{ sub}
 \end{array}
 \quad
 \begin{array}{l}
 S_1^{I_1} : \quad \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 S_1^{J_1} : \quad \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1), Q t_1 s_1 \\
 S_1^{J*} : \quad \Gamma_2 \rightarrow \Delta_2, Q t s, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 S_2^J : \quad \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 T_1^* : \quad \Phi_1 \rightarrow \Psi_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 T^* : \quad \Phi \rightarrow \Psi, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
 \tilde{T}^* : \quad \Phi \rightarrow \Psi, \mathcal{B}(V, Q_{\prec t_1}, t_1, s_1)
 \end{array}$$

We assign $\{h'_\eta\}_{\eta \preceq \xi}$ as follows:

- $h'_\eta(\tilde{T}^*) := h_\eta(S)$ for all $\eta \preceq \xi$
- $h'_\eta(T_1^*) := h_\eta(T_1)$ for all $\eta \preceq \xi$.
- $h'_\eta(T^*) := \begin{cases} h_\eta(T) & \text{if } \eta \leq \eta(T) \\ 0 & \text{otherwise} \end{cases}$

$$O_{P'}(\tilde{T}^*) = (i, O_P(T^*))$$

$O_{P'}(T^*) = \omega(\eta(T^*), l - h'_{\eta(T^*)}(T^*), d[\omega(\eta(T_1^*), h_{\eta(T_1^*)}(T_1^*), O_{P'}(T_1^*) \# O_{P'}(T_2^*))])$
 Note that $\eta(T^*) = i$. Obviously $k = l$ from the figure of P' . And from the above assignment $h', c[*] = \omega(\gamma_1, k_1, \dots, \omega(\gamma_s, k_s, d[*]))$ with $\gamma_s < i = \gamma_{s+1}$. Hence $O(P') <_0 O(P)$ holds.

□

The following Corollary is immediate from the above theorem and the fact that $S_k(ID_\xi^i)$ is a subsystem of $S_k(Al_\xi^-)$ under the interpretation $*$ (cf. the paragraph after Definition 7).

Corollary 1 *The transfinite induction on $\omega(\xi, k+1, 0)$ is unprovable in $S_k(ID_\xi^i)$ for $k > 2$.*

Proof. As remarked after Definition 7, ξ -level does not change under the interpretation of an Al_ξ -formula to an ID_ξ -formula. Hence the Corollary is obvious.

□

Theorem 2 (Main Theorem)

$$|S_k(ID_\xi^i(\mathcal{U}_0))| = |S_k(ID_\xi^i)| = |S_k(Al_\xi^-)| = |\omega(\xi, k+1, 0)|_{<_0} \text{ with } k > 2.$$

Remark 2: Our system $S_k(ID_\xi)$ can be reformulated by means of the alternation complexity of quantifiers when we include $\exists$ in our language. Here, a normal formula is of the form $Q_1 \bar{x}_1 \bar{Q}_1 \bar{y}_1 \dots Q_n \bar{x}_n \bar{Q}_n \bar{y}_n \forall \bar{y} D[P^{\mathcal{U}} t_1 s_1, \dots, P^{\mathcal{U}} t_m s_m]$, where $D[*_1, \dots, *_m]$ is a context of the language of PA with no quantifier occurring in D bounds any $*_i$ ($1 \leq i \leq m$), and $\{Q_j, \bar{Q}_j\} = \{\forall, \exists\}$ ($j = 1, \dots, m$). lv is essentially the same as lv_ξ except that we measure the alternation complexity of quantifiers instead of nestedness complexity of negations; namely,

$$lv(D[P^{\mathcal{U}} t_1 s_1, \dots, P^{\mathcal{U}} t_m s_m]) := \begin{cases} 1 & \text{if all } P^{\mathcal{U}} t_i s_i \text{ } (i = 1, \dots, m) \text{ is positive in } D \\ 2 & \text{otherwise} \end{cases}$$

Then the lv of above normal formula is $n+i$ if $\bar{Q}_n = \forall$ and $n+1+i$ if $\bar{Q}_n = \exists$, where $i := lv(D[P^{\mathcal{U}} t_1 s_1, \dots, P^{\mathcal{U}} t_m s_m])$. $S'_k(ID_\xi)$ is defined in the same way as the former definition of $S_k(ID_\xi)$ with using the above new notation of lv . It is easily seen that $S'_k(ID_\xi)$ is equivalent to $S_k(ID_\xi)$. In particular $|S'_k(ID_\xi)| = |\omega(\xi, k+1, 0)|_0$ with $k > 2$.

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