

Generalized slip-flow theory and its related Knudsen-layer analysis

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Contents

Preface	vii
1 Parabolic temperature profile and second-order temperature jump of a slightly rarefied gas in an unsteady two-surface problem	1
1.1 Introduction	1
1.2 Problem	2
1.3 Hilbert solution	4
1.3.1 Outline of the analysis	5
1.3.2 Summary	7
1.4 Knudsen-layer correction and the solution for the entire gap	8
1.5 Numerical data	11
1.5.1 The functions A , F , F_d , and related constant γ_2	13
1.5.2 The coefficient $\beta^{(2)}$ of the second-order jump	13
1.5.3 Direct numerical solution of the half-space problem for $\psi^{(2)}$	14
1.6 Conclusion	17
2 Asymptotic theory for the time-dependent behavior of a slightly rarefied gas over a smooth solid boundary	23
2.1 Introduction	23
2.2 Problem and assumption	24
2.3 Hilbert solution	27
2.3.1 The leading (the zeroth) order	28
2.3.2 The first order	28
2.3.3 The second order	29
2.3.4 The third and fourth orders	30
2.4 Knudsen-layer analysis	31
2.4.1 The first order	33
2.4.2 The second order	35

2.5	Summary of the asymptotic analysis	38
2.6	Discussions	41
2.6.1	New features	41
2.6.2	Comments on the initial data	41
2.6.3	How is the gas behavior determined?	42
2.6.3.1	Open domain	42
2.6.3.2	Closed domain	43
2.7	Data of slip and jump coefficients: Application of a symmetric relation . . .	45
2.8	Conclusion	47
2.9	Appendix A: Linearization	48
2.10	Appendix B: Isotropic functions and related constants	50
2.11	Appendix C: Axial symmetry of operators $\mathcal{L}$ and $\mathcal{K}$	50
2.12	Appendix D: Formulas for curved coordinates	52
2.13	Appendix E: Knudsen-layer corrections to the stress tensor and heat-flow vector	55
2.14	Appendix F: Formulas for quantities on the boundary	56
2.14.1	Derivatives of u_{iH0}	56
2.14.2	Derivatives of u_{iH1} and τ_{H1}	57
2.15	Appendix G: Second-order Knudsen-layer problems	58
2.15.1	Fundamental solutions in $\phi_{K2}^{(0)}$	59
2.15.2	Fundamental solutions in $\phi_{K2}^{(1)}$	59
2.15.3	Fundamental solutions in $\phi_{K2}^{(2)}$	60
2.15.4	Fundamental solutions in $\phi_{K2}^{(3)}$	61

3 Second-order Knudsen-layer analysis for the generalized slip-flow theory

I		65
3.1	Introduction	65
3.2	Generalized slip-flow theory: An outline	66
3.2.1	Physical setting	66
3.2.2	Resulting framework	66
3.2.3	Remaining issue	69
3.3	Knudsen-layer problems	70

3.4	Numerical analysis	72
3.4.1	Integral formulation	72
3.4.2	Plan of numerical computation	73
3.4.3	Discretization and some details of numerical method	75
3.5	Numerical results	78
3.5.1	Velocity distribution function	78
3.5.2	Slip/jump coefficient and Knudsen-layer function	79
3.6	Appendix A: Stress and heat flow	81
3.7	Appendix B: Data of computations	83
3.7.1	Grid system	83
3.7.2	Measure of accuracy	85
4	Second-order Knudsen-layer analysis for the generalized slip-flow theory	
	II: Curvature effects	91
4.1	Introduction	91
4.2	Generalized slip-flow theory: Outline and present concern	92
4.3	Knudsen-layer problems	96
4.4	Numerical analysis and singularity of the solution	98
4.4.1	Integral formulation	99
4.4.2	Singular properties of Ψ	100
4.4.3	Influence of the singularity of Ψ on its moments and $C[\Psi]$	102
4.4.4	Discretization and some details of numerical method	103
4.5	Numerical results	105
4.5.1	Velocity distribution functions	105
4.5.2	Slip/jump coefficients and Knudsen-layer functions	108
4.6	Concluding remarks	109
4.7	Appendix A: Stress and heat flow	111
4.8	Appendix B: Sketch of derivation of Eq. (4.16)	112
4.9	Appendix C: Data of computations and measure of accuracy	117
	Conclusion	123

Acknowledgment	125
List of publications	127
Additional information	129

Preface

In ordinary circumstances, gas flows are described by conventional fluid dynamics. In contrast, the behavior of a low pressure gas or a gas in a small system, which is important in aerospace engineering or micro engineering, is not described correctly by conventional fluid dynamics. This is because the mean free path of gas molecules can be comparable to the characteristic length of the system, so that the underlying assumption that the gas is very close to the local equilibrium breaks down due to insufficient intermolecular collisions. The gas in such a system is called a rarefied gas. The ratio of the mean free path to the characteristic length is a fundamental parameter which represents the degree of gas rarefaction and is called the Knudsen number.

In order to describe the behavior of a rarefied gas, we should use molecular gas dynamics, in which the microscopic information is taken into account. The basic physical quantity in molecular gas dynamics is the velocity distribution function of gas molecules, and the fundamental equation governing it is the celebrated Boltzmann equation. However, when we try to solve the Boltzmann equation, we encounter great difficulties since the Boltzmann equation is an integro-differential equation which contains the complex collision integral. In particular, it is a formidable task to solve spatially multidimensional boundary-value problems of the Boltzmann equation, which is of great importance in practical applications.

Fortunately however, when a gas is slightly rarefied, namely the Knudsen number is small, the behavior of the gas can be investigated by modifying conventional fluid dynamics appropriately. This means that we do not need to solve the Boltzmann equation directly for each specific problem that we are interested in. Such an approach is not only of theoretical but also of practical importance, since the Knudsen number is often small for the gases in actual micro devices. As to what modification should be applied to conventional fluid dynamics, there exists a systematic asymptotic theory of the Boltzmann equation established by Sone. According to it, the overall behavior of the gas is described by fluid-dynamic-type equations and their appropriate *slip* and *jump* boundary conditions (the fluid-dynamic-type system). Here, *slip* and *jump* means that there are differences of velocity (the “slip”) and temperature (the “jump”) between the gas and body on its surface. A correction to the

overall solution is required within a thin layer with a few mean-free-path thickness adjacent to the body surface (the Knudsen-layer correction). Here, the values of the slip and jump coefficients appearing in the boundary condition and data of the Knudsen-layer correction, which are required to apply this framework to specific problems, are obtained by solving spatially one-dimensional half-space problems of the linearized Boltzmann equation which determine the structure of the Knudsen layer (the Knudsen-layer problems).

In Sone's asymptotic theory, derived fluid-dynamic-type systems and formulas of the Knudsen-layer correction are different depending on the considered physical situations. Based on the size of the deviation from an equilibrium state at rest and the Reynolds number, situations are classified into ones in which (i) the deviation and the Reynolds number are both very small, (ii) the deviation is small, while the Reynolds number is the order of unity, and (iii) the deviation is the order of unity and the Reynolds number is the order of unity or very large. Results for the situations (i), (ii), and (iii) are summarized respectively as linear theory, weakly nonlinear theory, and nonlinear theory.

In the present thesis, we consider the linear theory and hereafter call it the generalized slip-flow theory. Originally, the generalized slip-flow theory was established for steady problems in the late 1960s and early 1970s, based on the Bhatnagar–Gross–Krook (BGK) model equation. The Stokes set, its appropriate slip and jump boundary conditions, and formulas of the Knudsen-layer correction were derived up to the second order in the Knudsen number expansion. After that, the results were extended to the case of the original Boltzmann equation. This theory has been successfully applied to various fundamental problems in molecular gas dynamics and was revealed to be practical. However, the application has been limited to steady problems. For unsteady problems, analyses of which are as important as those of steady problems, while the fluid-dynamic-type system and formulas of the Knudsen-layer correction have been derived up to the first order by Sone, those at the second order, which draw many researchers' attention recently, are still unknown.

In the present thesis, we develop the generalized slip-flow theory for unsteady problems up to the second order. Simultaneously, we complete required numerical data of slip and jump coefficients and Knudsen-layer correction for the Boltzmann equation, most of which at the second order have not been available even for steady problems. In fact, the complete set of required numerical data has been available only for the BGK model. For the original

Boltzmann equation, a part of the information have been reported in the late 1980s and early 1990s, assuming the hard-sphere molecules. However, difficulties in solving the Knudsen-layer problems, which could be bypassed for the BGK model, have prevented completing the data for a long time.

The present thesis is organized as follows. Chapter 1 is devoted to a study of a new temperature jump occurring in unsteady systems. Chapter 2 is devoted to developing the generalized slip-flow theory for unsteady problems under a general problem setting. Chapters 3 and 4 are devoted to completing necessary data for the original Boltzmann equation by solving the Knudsen-layer problems. A more detailed description of these chapters follows.

In Chapter 1, an unsteady problem of a slightly rarefied monatomic gas between two parallel plates whose temperature grows slowly and linearly in time is considered. This is motivated by the simulation results which were obtained recently by Radtke, Hadjiconstantinou, Takata, and Aoki (RHTA). They solved a problem of a rarefied gas which subjects to constant volumetric heating by a low variance stochastic method and showed that the second-order temperature-jump coefficient observed by RHTA is not covered by the generalized slip-flow theory for steady problems. At first, we show that the problem considered in this chapter is equivalent to the one studied by RHTA. This correspondence between two problems implies that the temperature jump observed by RHTA is a new one that can occur in unsteady systems. A systematic asymptotic analysis of the time-dependent problem for small Knudsen numbers is carried out and the complete fluid-dynamic description, as well as the related half-space problems that determine the structure of the Knudsen layer and the coefficients of temperature jump, are obtained. Finally, a numerical solution is presented for both the BGK model and hard-sphere molecules.

In Chapter 2, we develop the generalized slip-flow theory for unsteady problems under a general problem setting. This is motivated by the possibility that there might be other new slip or jump phenomena which do not appear in steady problems other than one observed in Chapter 1. A time-evolution of a slightly rarefied monatomic gas which is perturbed slowly and slightly from a reference uniform equilibrium state at rest is investigated on the basis of the linearized Boltzmann equation. By a systematic asymptotic analysis, fluid-dynamic-type equations given by a Stokes-type set, their boundary conditions, formulas of the Knudsen-layer correction, and a series of formulated Knudsen-layer problems up to the

second order of the Knudsen number are derived. The differences between the obtained results and the existing theory for steady problems are discussed. In principle, we have to solve the Knudsen-layer problems to obtain numerical data of slip and jump coefficients and Knudsen-layer corrections, which is a difficult matter. Fortunately however, if we focus on obtaining the values of slip and jump coefficients, we can bypass the difficulty by the use of the theory of symmetry relation developed recently by Takata. Hence, numerical values of all the slip and jump coefficients are obtained for a hard-sphere gas not by directly solving the Knudsen-layer problems.

Up to this point, the information of the fluid-dynamic-type system for unsteady problems, namely, fluid-dynamic-type equations, their appropriate slip and jump boundary condition, and numerical values of the slip and jump coefficients for a hard-sphere gas are completed. Based on them, one can investigate the behavior of a slightly rarefied gas in the bulk region. However, the information of the Knudsen-layer structure, which is necessary to understand the behavior of the gas near the boundary, is still incomplete. We attempt to complete it.

In Chapter 3, the second-order Knudsen-layer problems, except for the curvature effect, are solved on the basis of the linearized Boltzmann equation for hard-sphere molecules. These problems might be handled by the finite-difference method as in the studies conducted in 1980s and 1990s. Nonetheless, we take a new approach that makes use of the integral formulation of the Boltzmann equation which is very effective to capture the detailed information on the solution. The primary motivation of this approach is that the remaining Knudsen-layer problems, which are related to the curvature effect and left to Chapter 4, contain a much delicate matter, requiring a new methodology for the numerical analysis. Numerical data of the Knudsen-layer correction, which are necessary in the application to the problems with planar boundary, are prepared up to the second order in the Knudsen number expansion.

In Chapter 4, we tackle the remaining Knudsen-layer problems and the effects of the boundary curvature on the solutions have been clarified in details. Even though the macroscopic quantities are finite and well defined, the velocity distribution function exhibits non-obvious singularities (i.e., diverges indefinitely) for the molecular velocity tangent to the boundary. This feature forced us to separate the present study from the previous one in Chapter 3. Based on the approach devised in Chapter 3, we have developed the numerical

method that handles such a singularity safely, thereby completing the data required in the generalized slip-flow theory.

Chapter 1

Parabolic temperature profile and second-order temperature jump of a slightly rarefied gas in an unsteady two-surface problem

1.1 Introduction

The recent development of small-scale devices has renewed interest in microscale or rarefied gas flows. [1,2,3,4,5,6] In these flows, the Knudsen number, typically defined as the ratio of the molecular mean free path to the characteristic length scale of the physical system, is no longer negligibly small, signaling that the conventional continuum description, namely the Navier–Stokes equation with the no-slip boundary condition (NS system), does not apply. In such cases, a kinetic approach (Boltzmann system) is appropriate and is typically used.

The connection between the Boltzmann and the NS systems has been studied since the days of Hilbert, and a number of useful results have been obtained in the limit of small Knudsen numbers. [7,8,9,10,11,12,13,14,15,16] Specifically, fluid-dynamic-type sets of equations and appropriate slip and jump boundary conditions for describing the *steady* gas behavior in the regime of small Knudsen numbers (the so-called slip flow regime) have been established [11,12,13] since the late 1960s and early 1970s. For rigorous and complete descriptions of the general theory of slip flow, the reader is referred to Refs. [2] and [6]. In the present paper, we shed light on a new feature of the jump of temperature that can occur at the second order of the Knudsen number in *unsteady* problems. The present work is motivated by the simulation results recently obtained by Radtke, Hadjiconstantinou, Takata, and Aoki (RHTA, see Ref. [17]) and is intended to contribute to a growing research topic on the second-order slip (e.g., Refs. [3,18,19,20,21,22]).

Using a low variance stochastic method [23,24,25,26], RHTA carried out numerical simulations of a slightly rarefied gas bounded by two parallel walls subject to constant volumetric heating. The parabolic temperature profile associated with the volumetric heating allowed

them to evaluate the second-order temperature-jump coefficient, which, in the case of the Bhatnagar–Gross–Krook (BGK) [or Boltzmann–Krook–Wendler (BKW)] model [27,28], did not agree with the one predicted (coefficient d_3 in Sec. 3.1.5 of Ref. [6]) by the general theory of slip flow for steady problems (Refs. [11,2,6]).

In order to explain the above observation, in the present paper, we consider a slightly rarefied gas between two parallel plates whose temperature grows slowly and linearly in time which, as we show in Sec. 1.2, can be reduced to the problem studied by RHTA. We subsequently investigate the behavior of the gas by a systematic asymptotic analysis for small Knudsen numbers (Secs. 1.3 and 1.4). Our results show that, as expected, the reduced half-space problem that determines the second-order jump of temperature is new and does not appear in the theory for steady problems. We also perform numerical computations to determine the second-order temperature-jump coefficient and the structure of the related Knudsen layer for the BGK model and for hard-sphere molecules (Sec. 1.5). The solution is compared, not only in the fluid-dynamic region but also in the Knudsen-layer, with the simulation results that RHTA have obtained by the low variance stochastic method.

1.2 Problem

Consider a rarefied monatomic gas between two parallel plates located at $X_1 = \pm D/2$ and kept at the same uniform temperature T_w , where X_i is the Cartesian space coordinate (Fig. 1.1). The temperature T_w is uniform on the plates and grows slowly and linearly in time $\tilde{t}$, i.e., $T_w = T_0(1 + \tilde{\alpha}\tilde{t})$, where T_0 is a reference temperature and $\tilde{\alpha}$ is a small constant. We will investigate the behavior of this gas after a long time has passed from some initial state under the following assumptions: (i) The behavior of the gas is described by the Boltzmann equation (or its model equation such as the BGK model). (ii) The gas molecules are diffusely reflected on the plates. (iii) The change of plate temperature during the characteristic time of heat conduction is small, i.e., $\alpha \equiv \tilde{\alpha}t_0 \ll 1$ with $t_0 \sim \rho_0 R D^2 / \lambda$, and thus the equation and boundary condition can be linearized around the reference equilibrium state at rest with density ρ_0 and temperature T_0 . Here, ρ_0 is the average density of the gas, λ is the thermal conductivity, and R is the specific gas constant, i.e., $R = k/m$ with k denoting the Boltzmann constant and m the mass of a molecule. (iv) The mean free path ℓ_0 of a molecule

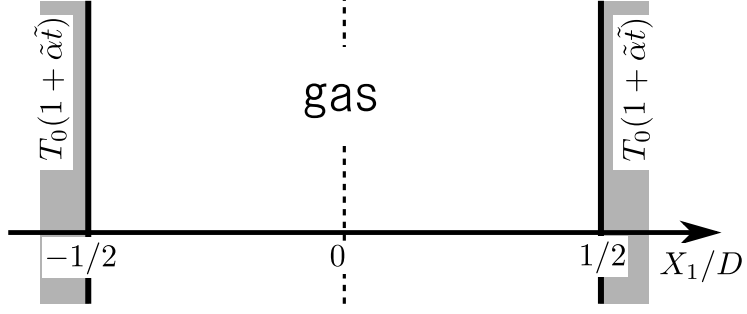


Figure 1.1: Sketch of the problem.

at the reference state is much shorter than the distance between the plates D , or, in other words, the Knudsen number $\text{Kn} = \ell_0/D$ is small. Hereinafter we set the time scale t_0 as $t_0 = D^2/\ell_0(\pi RT_0/2)^{1/2}$.

Let us denote the time $\tilde{t}$ by $t_0 t$, the position X_1 by Dx , the molecular velocity by $(2RT_0)^{1/2}\boldsymbol{\zeta}$, and the velocity distribution function by $\rho_0(2RT_0)^{-3/2}[1 + \alpha t(|\boldsymbol{\zeta}|^2 - \frac{3}{2}) + \phi]E$, where E is the normalized Maxwellian defined by $E(|\boldsymbol{\zeta}|) = \pi^{-3/2} \exp(-|\boldsymbol{\zeta}|^2)$. By assumption (iii), ϕ is a small quantity and its higher order contributions will be neglected. Then, ϕ can be sought as a function of x and $\boldsymbol{\zeta}$ that is even both in ζ_2 and ζ_3 and satisfies the following (dimensionless) steady inhomogeneous linearized Boltzmann equation and the boundary condition:

$$\zeta_1 \partial_x \phi = \frac{1}{\varepsilon} \mathcal{L}[\phi] - \varepsilon \alpha (|\boldsymbol{\zeta}|^2 - \frac{3}{2}), \quad (1.1a)$$

$$\phi = \pm 2\sqrt{\pi} \int_{\zeta_1 \gtrless 0} \zeta_1 \phi E d\boldsymbol{\zeta}, \quad \zeta_1 \leq 0, \quad x = \pm \frac{1}{2}, \quad (1.1b)$$

where $\partial_x = \partial/\partial x$, $\varepsilon = (\sqrt{\pi}/2)\text{Kn}$, $\mathcal{L}$ is the linearized collision operator, and $d\boldsymbol{\zeta} = d\zeta_1 d\zeta_2 d\zeta_3$.

For later convenience, we define the macroscopic quantities and their notation here. The density, flow velocity, temperature, and pressure of the gas are given by $\rho_0(1 + \omega)$, $(2RT_0)^{1/2}(u, 0, 0)$, $T_0(1 + \alpha t + \tau)$, and $p_0(1 + \alpha t + P)$, where $p_0 = \rho_0 RT_0$, while ω , u , τ , and P are defined by

$$\omega = \langle \phi \rangle, \quad u = \langle \zeta_1 \phi \rangle, \quad \tau = \frac{2}{3} \langle (|\boldsymbol{\zeta}|^2 - \frac{3}{2}) \phi \rangle, \quad P = \frac{2}{3} \langle |\boldsymbol{\zeta}|^2 \phi \rangle, \quad (1.2)$$

and the angle brackets denote the following moment:

$$\langle \Phi \rangle = \int \Phi(x, \boldsymbol{\zeta}) E(|\boldsymbol{\zeta}|) d\boldsymbol{\zeta}.$$

Note that τ (more precisely $T_0 \tau$) is a perturbation of the temperature from the plate temperature. By definition, $P = \omega + \tau$.

It should be noted that problem (1.1) is equivalent to the constant volumetric heating problem studied by RHTA.¹ The present formulation shows that the parabolic temperature profile obtained in RHTA via a source term in the governing (Boltzmann) equation can be naturally obtained (no additional source term) by wall temperatures following the time dependence described above. Therefore, as we show below, the temperature jump observed here and in the work by RHTA is not covered by the general theory of slip flow [11,2,6] for steady problems.

The volumetric heating term used in RHTA, $-\varepsilon\alpha(|\zeta|^2 - \frac{3}{2})$, leads to a spatially and temporally constant increase in the gas thermal energy. This heating (or cooling when $\alpha > 0$) is similar to the cooling of the granular gas in that it conserves mass and momentum. However, it is independent of the gas state, in contrast to the granular gas case. It should also be noted that no essential difference between heating ($\alpha < 0$) and cooling ($\alpha > 0$) exists in the case of RHTA. The constant volumetric heating (cooling) implies a controlled energy supply (loss) in the gas, which is balanced by the loss (supply) of energy through the side walls through the heat conduction. In the present time-dependent problem, the temperature profile is determined by the interplay between the gas inertia and the energy supply (loss) due to the change of wall temperature. Our formulation shows that, when measured relative to the time-varying wall temperature, the temperature profile is identical to the heat addition (loss) case of RHTA.

Because of assumption (iv), ε is a small constant. We will investigate the problem (1.1) by a systematic asymptotic analysis for small ε , following Sone's method [11,2,6].

1.3 Hilbert solution

Putting aside the boundary condition, we seek a moderately varying solution ϕ_H (the Hilbert solution) in a power series of ε :

$$\phi_H = \phi_{H0} + \phi_{H1}\varepsilon + \phi_{H2}\varepsilon^2 + \cdots. \quad (1.3)$$

¹This work done in Ref. [17] is closely related to the present work. Incidentally, τ in the present paper is identical to $\hat{T}$ in Ref. [17] when our α is identified with $-(2/3)\epsilon$ there.

Corresponding macroscopic quantities, which we denote by h_H ($h = P, u, \tau, \omega$), are also expanded as

$$h_H = h_{H0} + h_{H1}\varepsilon + h_{H2}\varepsilon^2 + \dots, \quad (1.4)$$

where h_{Hm} ($m = 0, 1, \dots$) is defined by Eq. (1.2) with ϕ being replaced by ϕ_{Hm} . Substitution of Eq. (1.3) into Eq. (1.1a) leads to

$$\mathcal{L}[\phi_{H0}] = 0, \quad (1.5a)$$

$$\mathcal{L}[\phi_{H1}] = \zeta_1 \partial_x \phi_{H0}, \quad (1.5b)$$

$$\mathcal{L}[\phi_{H2}] = \zeta_1 \partial_x \phi_{H1} + \alpha(|\zeta|^2 - \frac{3}{2}), \quad (1.5c)$$

$$\mathcal{L}[\phi_{Hm}] = \zeta_1 \partial_x \phi_{Hm-1}, \quad (m \geq 3). \quad (1.5d)$$

This is a set of linear integral equations and can be solved from the lowest order, provided that the following condition is satisfied:

$$\left\langle \begin{pmatrix} 1 \\ \zeta_1 \\ |\zeta|^2 \end{pmatrix} \times \text{R.H.S. of Eq. (1.5)} \right\rangle = 0, \quad (1.6)$$

because $\mathcal{L}[f] = 0$ has nontrivial solutions $f = 1, \zeta_1, \zeta_2, \zeta_3, |\zeta|^2$ (the so-called collision invariants).

1.3.1 Outline of the analysis

The solvability condition (1.6) for Eq. (1.5a) is automatically satisfied, yielding

$$\phi_{H0} = P_{H0} + 2\zeta_1 u_{H0} + (|\zeta|^2 - \frac{5}{2})\tau_{H0}. \quad (1.7)$$

Let us proceed to Eq. (1.5b). By the substitution of Eq. (1.7), the solvability condition (1.6) is rewritten as

$$\frac{du_{H0}}{dx} = 0, \quad \frac{dP_{H0}}{dx} = 0. \quad (1.8)$$

Thus, Eq. (1.5b) is reduced to

$$\mathcal{L}[\phi_{H1}] = \zeta_1 (|\zeta|^2 - \frac{5}{2}) \frac{d\tau_{H0}}{dx},$$

yielding

$$\phi_{H1} = P_{H1} + 2\zeta_1 u_{H1} + (|\zeta|^2 - \frac{5}{2})\tau_{H1} - \zeta_1 A \frac{d\tau_{H0}}{dx}. \quad (1.9)$$

Here A is a function of $|\zeta|$ defined by

$$\mathcal{L}[\zeta_1 A] = -\zeta_1(|\zeta|^2 - \frac{5}{2}) \quad \text{with } \langle |\zeta|^2 A \rangle = 0.$$

Next, we consider Eq. (1.5c). With the aid of Eq. (1.8), the solvability condition (1.6) can be rewritten as

$$\frac{du_{H1}}{dx} = 0, \quad \frac{dP_{H1}}{dx} = 0, \quad (1.10a)$$

$$-\frac{5}{4}\gamma_2 \frac{d^2\tau_{H0}}{dx^2} + \frac{3}{2}\alpha = 0, \quad (1.10b)$$

where γ_2 is a constant (namely, the dimensionless thermal conductivity) defined by $\gamma_2 = 2I_6(A)$ with

$$I_n(X) = \frac{2}{15} \langle |\zeta|^{n-2} X \rangle.$$

Thus Eq. (1.5c) is reduced to

$$\mathcal{L}[\phi_{H2}] = \zeta_1(|\zeta|^2 - \frac{5}{2}) \frac{d\tau_{H1}}{dx} - \frac{6}{5} \frac{\alpha}{\gamma_2} (\zeta_1^2 - \frac{1}{3}|\zeta|^2)A + \frac{6}{5} \frac{\alpha}{\gamma_2} \left[\frac{5}{6}\gamma_2(|\zeta|^2 - \frac{3}{2}) - \frac{1}{3}|\zeta|^2 A \right],$$

whose solution is

$$\phi_{H2} = P_{H2} + 2\zeta_1 u_{H2} + (|\zeta|^2 - \frac{5}{2})\tau_{H2} - \zeta_1 A \frac{d\tau_{H1}}{dx} - \frac{6}{5} \frac{\alpha}{\gamma_2} (\zeta_1^2 - \frac{1}{3}|\zeta|^2)F - \frac{6}{5} \frac{\alpha}{\gamma_2} F_d. \quad (1.11)$$

Here F and F_d are functions of $|\zeta|$ defined by

$$\mathcal{L}[\zeta_1 \zeta_2 F] = \zeta_1 \zeta_2 A,$$

$$\mathcal{L}[F_d] = -\frac{5}{6}\gamma_2(|\zeta|^2 - \frac{3}{2}) + \frac{1}{3}|\zeta|^2 A \quad \text{with } \langle F_d \rangle = 0 \text{ and } \langle |\zeta|^2 F_d \rangle = 0.$$

We proceed to Eq. (1.5d) with $m = 3$. With the aid of Eqs. (1.8) and (1.10), the solvability condition (1.6) can be rewritten as

$$\frac{du_{H2}}{dx} = 0, \quad \frac{dP_{H2}}{dx} = 0, \quad (1.12a)$$

$$\frac{d^2\tau_{H1}}{dx^2} = 0. \quad (1.12b)$$

Then, Eq. (1.5d) simplifies to

$$\mathcal{L}[\phi_{H3}] = \zeta_1(|\zeta|^2 - \frac{5}{2}) \frac{d\tau_{H2}}{dx},$$

yielding

$$\phi_{H3} = P_{H3} + 2\zeta_1 u_{H3} + (|\zeta|^2 - \frac{5}{2})\tau_{H3} - \zeta_1 A \frac{d\tau_{H2}}{dx}. \quad (1.13)$$

Finally the substitution of this expression reduces the solvability condition (1.6) for Eq. (1.5d) with $m = 4$ to

$$\frac{du_{H3}}{dx} = 0, \quad \frac{dP_{H3}}{dx} = 0, \quad (1.14a)$$

$$\frac{d^2\tau_{H2}}{dx^2} = 0. \quad (1.14b)$$

If we proceed further, we will obtain for *any* $m \geq 4$

$$\phi_{Hm} = P_{Hm} + 2\zeta_1 u_{Hm} + (|\zeta|^2 - \frac{5}{2})\tau_{Hm} - \zeta_1 A \frac{d\tau_{Hm-1}}{dx}, \quad (1.15)$$

and

$$\frac{du_{Hm}}{dx} = 0, \quad \frac{dP_{Hm}}{dx} = 0, \quad \frac{d^2\tau_{Hm-1}}{dx^2} = 0. \quad (1.16)$$

1.3.2 Summary

In Sec. 1.3.1, we have obtained a set of fluid-dynamic-type equations for the macroscopic quantities, namely, Eqs. (1.8) and (1.10b) for $O(\varepsilon^0)$; Eqs. (1.10a) and (1.12b) for $O(\varepsilon^1)$; and Eqs. (1.12a) and (1.14b) for $O(\varepsilon^2)$, and so on.

Since ϕ_{H0} is a linearized Maxwellian, it satisfies the boundary condition (1.1b), if we set

$$u_{H0} = 0, \quad \tau_{H0} = 0 \quad \text{on } x = \pm \frac{1}{2}. \quad (1.17)$$

Note that the reference density ρ_0 is the average density between the plates, so that the average of perturbed density ω , which is related to P and τ by $\omega = P - \tau$, should vanish. Thus, the quantities of $O(\varepsilon^0)$ are obtained from Eqs. (1.8), (1.10b), and (1.17) as

$$u_{H0} = 0, \quad \omega_{H0} = -\frac{3\alpha}{5\gamma_2}(x^2 - \frac{1}{12}), \quad \tau_{H0} = -\frac{3\alpha}{5\gamma_2}(\frac{1}{4} - x^2), \quad P_{H0} = -\frac{\alpha}{10\gamma_2}.$$

The above τ_{H0} shows that ϕ_{H1} is no longer a linearized Maxwellian [see Eq. (1.9)] and does not satisfy the boundary condition (1.1b). The same is true for ϕ_{H2} [see Eq. (1.11)]. Thus, by the analysis in Sec. 1.3.1, we cannot determine the gas behavior completely. What we can see from Eqs. (1.10a) and (1.12b) and from Eqs. (1.12a) and (1.14b) is that

$$\begin{aligned} P_{H1} &= c_1, & u_{H1} &= c_2, & \tau_{H1} &= c_3, \\ P_{H2} &= c_4, & u_{H2} &= c_5, & \tau_{H2} &= c_6, \end{aligned}$$

where c_i ($i = 1, 2, \dots, 6$) are (undetermined) constants. Note that the linear dependence of temperature in x vanishes because the temperature should be symmetric in x with respect

to $x = 0$. In order to find the solution that satisfies the boundary condition, in Sec. 1.4, we will introduce a correction in the microscopic boundary-layer (the Knudsen-layer correction) to the Hilbert solution. The analysis of the correction will determine the constants c_i and the structure of the Knudsen layer. If c_3 and c_6 are nonzero, a jump of temperature at the first- and second-orders of ε exists.

Thanks to the symmetric temperature profile in x , all of τ_{Hm} , P_{Hm} , and u_{Hm} for $m \geq 3$ are found to be constant from Eqs. (1.14a) and (1.16), and thus the corresponding ϕ_{Hm} are reduced to linearized Maxwellians [see Eqs. (1.13) and (1.15)]. Therefore, if we set $u_{Hm} = \tau_{Hm} = 0$, ϕ_{Hm} satisfies the boundary condition (1.1b) at $O(\varepsilon^m)$. Consequently, the Knudsen-layer correction to be discussed in Sec. 1.4 is not required at the third and higher order of ε . P_{Hm} will be determined again by the constraint that the perturbed density averaged in x should vanish.

1.4 Knudsen-layer correction and the solution for the entire gap

Because of the symmetry of the problem $\phi(x, \zeta_1, \cdot) = \phi(-x, -\zeta_1, \cdot)$, it is enough to consider the correction near the left plate at $x = -\frac{1}{2}$. The correction near the right plate is its mirror image.

We will obtain the solution ϕ in the form of $\phi = \phi_H + \phi_K$, where ϕ_K is the Knudsen-layer correction, which is appreciable only in the thin layer adjacent to the left plate [thus, $\partial_x \phi_K = (1/\varepsilon)O(\phi_K)$] and is $O(\varepsilon)$ because the correction is not required at the zeroth order. Corresponding correction to macroscopic quantities will be denoted by h_K , i.e., $h = h_H + h_K$. We introduce the stretched coordinate η which is defined by $\eta = (x + \frac{1}{2})/\varepsilon$ near the left plate. Then, ϕ_K is the solution of the following problem:

$$\zeta_1 \partial_\eta \phi_K = \mathcal{L}[\phi_K], \quad (1.18a)$$

$$\phi_K = -\phi_H - 2\sqrt{\pi} \int_{\zeta_1 < 0} \zeta_1 (\phi_K + \phi_H) E d\boldsymbol{\zeta}, \quad \zeta_1 > 0, \quad \eta = 0, \quad (1.18b)$$

$$\phi_K \rightarrow 0 \quad \text{as } \eta \rightarrow \infty, \quad (1.18c)$$

where $\partial_\eta = \partial/\partial\eta$. It is seen by taking the moment of Eq. (1.18a), i.e., $\langle \text{Eq. (1.18a)} \rangle$, that $u_K \equiv 0$ because of the condition (1.18c). Since $u (= u_H + u_K) = 0$ on the plate, we see that

$u_H = 0$ on the plate and obtain

$$c_2 = c_5 = 0.$$

Since $\phi_K = O(\varepsilon)$, we expand it as $\phi_K = \phi_{K1}\varepsilon + \phi_{K2}\varepsilon^2 + \dots$ and correspondingly h_K as $h_K = h_{K1}\varepsilon + h_{K2}\varepsilon^2 + \dots$, where h_{Km} ($m = 1, 2, \dots$) is given by Eq. (1.2) with ϕ being replaced by ϕ_{Km} . Actually, as mentioned in the last paragraph of Sec. 1.3.2, the expansion terminates at $O(\varepsilon^2)$ in the present problem. Substitution of the expansion of ϕ_K into Eq. (1.18) leads to boundary-value problems for ϕ_{K1} and ϕ_{K2} . To be more specific, if we let

$$\psi^{(1)} = -\left(\frac{3\alpha}{5\gamma_2}\right)^{-1} \phi_{K1}, \quad \psi^{(2)} = \left(\frac{6\alpha}{5\gamma_2}\right)^{-1} \phi_{K2}, \quad (1.19a)$$

$$\beta^{(1)} = -\left(\frac{3\alpha}{5\gamma_2}\right)^{-1} c_3, \quad \beta^{(2)} = \left(\frac{6\alpha}{5\gamma_2}\right)^{-1} c_6, \quad (1.19b)$$

we obtain the following equations and boundary conditions for $\psi^{(i)}$:

$$\zeta_1 \partial_\eta \psi^{(i)} = \mathcal{L}[\psi^{(i)}], \quad (i = 1, 2), \quad (1.20a)$$

$$\psi^{(i)} = I^{(i)} - 2\sqrt{\pi} \int_{\zeta_1 < 0} \zeta_1 \psi^{(i)} E d\zeta, \quad \zeta_1 > 0, \quad \eta = 0, \quad (1.20b)$$

$$\psi^{(i)} \rightarrow 0 \quad \text{as } \eta \rightarrow \infty, \quad (1.20c)$$

where

$$I^{(1)} = -(|\zeta|^2 - 2)\beta^{(1)} + \zeta_1 A,$$

$$I^{(2)} = -(|\zeta|^2 - 2)\beta^{(2)} - \frac{\sqrt{\pi}}{2} (\langle |\zeta| F_d \rangle + \frac{1}{6} \langle |\zeta|^3 F \rangle) + F_d + (\zeta_1^2 - \frac{1}{3} |\zeta|^2) F.$$

The solution of the half-space problem (1.20) exists uniquely and the constant $\beta^{(1)}$ (or $\beta^{(2)}$) is determined uniquely together with the solution $\psi^{(1)}$ (or $\psi^{(2)}$). [10,29,30,31]

The coefficients $-3\alpha/5\gamma_2$ and $6\alpha/5\gamma_2$ in Eq. (1.19) are the first and second derivatives of temperature τ_H at the left plate. Accordingly, $\psi^{(1)}$ (and thus ϕ_{K1}) is seen to be the classical temperature-jump problem, which has been studied by various authors (e.g., Refs. [32,28, 33,34,35,36]). On the other hand, $\psi^{(2)}$ is the solution of the problem of jump caused by the second derivative of temperature with respect to x , which is, however, different from the jump problem caused by the second derivative of temperature predicted by the theory for steady problems.

The fact that the amount of temperature jump caused by the second derivative of temperature is different depending on whether the problem is unsteady or steady may appear to be paradoxical. According to the theory for steady flows (Refs. [2] and [6]), the temperature field satisfies the Laplace equation, which reduces to a vanishing second derivative of temperature ($d^2\tau_H/dx^2 = 0$, where this τ is perturbed temperature from a reference uniform equilibrium state) in one-dimensional problems. In other words, the temperature jump caused by the second derivatives of temperature does not manifest itself in steady one-dimensional problems. In contrast, in the present *unsteady* one-dimensional problem, τ_H satisfies the equation of the form $d^2\tau_H/dx^2 = \text{const}(\neq 0)$ [see Eq. (1.10b)], which is a one-dimensional version of the Poisson equation, not the Laplace equation. As we have seen, the non-zero constant in this equation may be thought of as originating from a volumetric heating source in a steady framework, or, in the unsteady setting, from the time dependence of plate temperature. In other words, the time dependence of the problem introduces a new temperature jump caused by the second derivative of temperature. This difference between steady and unsteady problems is maintained in the general two- and three-dimensional problems, in which the temperature jump associated with the second derivative of temperature also appears in steady problems.

Finally, the constants c_1 and c_4 (namely, P_{H1} and P_{H2}) as well as P_{Hm} for $m \geq 3$ are determined so as to make the perturbed density averaged across the space between the two plates vanish. This procedure yields the relations $\omega_{H1} = 0$, $P_{H1} = c_3$, $\omega_{H2} = -2 \int_0^\infty \omega_{K1} d\eta$, $P_{H2} = c_6 - \omega_{H2}$, $\omega_{H3} = -P_{H3} = -2 \int_0^\infty \omega_{K2} d\eta$, and $\omega_{Hm} = P_{Hm} = 0$ for $m \geq 4$, thus they are completely determined after solving Eq. (1.20).

To summarize, the macroscopic quantities are expressed for the entire gap between the plates by the sum of $h = h_{H0} + (h_{H1} + h_{K1})\varepsilon + (h_{H2} + h_{K2})\varepsilon^2 + h_{H3}\varepsilon^3 + \dots$, where h_{Hm} for $m \geq 3$ vanishes except for ω_{H3} and P_{H3} . We have obtained the following asymptotic solutions of u , τ , ω , and P :

$$u = 0, \tag{1.21a}$$

$$\tau = -\frac{3\alpha}{5\gamma_2} \left\{ \left(\frac{1}{4} - x^2 \right) + [\beta^{(1)} + \Theta^{(1)}(\eta_-) + \Theta^{(1)}(\eta_+)]\varepsilon - 2[\beta^{(2)} + \Theta^{(2)}(\eta_-) + \Theta^{(2)}(\eta_+)]\varepsilon^2 \right\}, \tag{1.21b}$$

$$\omega = -\frac{3\alpha}{5\gamma_2} \left\{ x^2 - \frac{1}{12} + [\Omega^{(1)}(\eta_-) + \Omega^{(1)}(\eta_+)]\varepsilon \right.$$

$$- 2 \left[\int_0^\infty \Omega^{(1)} d\eta + \Omega^{(2)}(\eta_-) + \Omega^{(2)}(\eta_+) \right] \varepsilon^2 + 4 \int_0^\infty \Omega^{(2)} d\eta \varepsilon^3 \Big\}, \quad (1.21c)$$

$$P = \omega + \tau, \quad (1.21d)$$

where

$$\Omega^{(i)} = \langle \psi^{(i)} \rangle, \quad \Theta^{(i)} = \frac{2}{3} \langle (|\zeta|^2 - \frac{3}{2}) \psi^{(i)} \rangle, \quad \eta_+ = \frac{1}{\varepsilon} (-x + \frac{1}{2}), \quad \eta_- = \frac{1}{\varepsilon} (x + \frac{1}{2}).$$

1.5 Numerical data

Because the problem for $\psi^{(1)}$ is classical, reliable data for $\beta^{(1)}$, $\Omega^{(1)}$, and $\Theta^{(1)}$ are available in the literature. For example, $\beta^{(1)}$, $\Omega^{(1)}$, and $\Theta^{(1)}$ correspond to d_1 , Ω_1 , and Θ_1 in Ref. [6], in which data for the BGK (or BKW) model and hard-sphere molecules can be found. $\Omega^{(1)}$ and $\Theta^{(1)}$ are tabulated in Table 1.1 for use in Eq. (1.21) (the data in the table were obtained in Ref. [35]; the accuracy would be slightly better than those in Ref. [6]). The values of $\beta^{(1)}$ and $\int_0^\infty \Omega^{(1)} d\eta$ are

$$\begin{aligned} \beta^{(1)} &= 1.30272 \text{ (BGK), } 2.4001 \text{ (hard sphere),} \\ \int_0^\infty \Omega^{(1)} d\eta &= 0.3230 \text{ (BGK), } 0.5241 \text{ (hard sphere).} \end{aligned}$$

On the other hand, the problem for $\psi^{(2)}$ has not been solved, and thus data for $\beta^{(2)}$, $\Omega^{(2)}$, and $\Theta^{(2)}$ are not available. Data for F_d are also not available. In the present section, we provide these data for the BGK model and hard-sphere molecules to complete our analysis. The specific form of the linearized collision integral for these cases is the following: for the BGK model,

$$\mathcal{L}[\phi] = -\phi + \omega + 2\zeta_1 u + (|\zeta|^2 - \frac{3}{2})\tau,$$

and $\ell_0 = (8RT_0/\pi)^{1/2}/A_c\rho_0$, where A_c is a constant such that $A_c\rho_0$ is the collision frequency of gas molecules at the reference state; for hard-sphere molecules,

$$\begin{aligned} \mathcal{L}[\phi] &= -\nu(|\zeta|)\phi + \int [K_1(\zeta, \xi) - K_2(\zeta, \xi)]\phi(\xi) d\xi, \\ \nu(z) &= \frac{1}{2\sqrt{2}} \left[\exp(-z^2) + \left(2z + \frac{1}{z} \right) \int_0^z \exp(-y^2) dy \right], \\ K_1(\zeta, \xi) &= \frac{1}{\sqrt{2\pi}|\zeta - \xi|} \exp \left(-|\xi|^2 + \frac{|\xi \times \zeta|^2}{|\xi - \zeta|^2} \right), \\ K_2(\zeta, \xi) &= \frac{|\zeta - \xi|}{2\sqrt{2\pi}} \exp(-|\xi|^2), \end{aligned}$$

and $\ell_0 = [\sqrt{2\pi}\sigma^2(\rho_0/m)]^{-1}$, where σ is the diameter of a molecule.

Table 1.1: Knudsen-layer functions $\Omega^{(1)}$ and $\Theta^{(1)}$.

Hard sphere			BGK		
η	$\Omega^{(1)}$	$-\Theta^{(1)}$	η	$\Omega^{(1)}$	$-\Theta^{(1)}$
0.0000	0.5164	0.7378	0.0000	0.3477	0.4492
0.0235	0.4710	0.6766	0.0249	0.3130	0.4101
0.0517	0.4382	0.6314	0.0491	0.2925	0.3860
0.1089	0.3911	0.5655	0.1289	0.2474	0.3314
0.1567	0.3615	0.5234	0.2483	0.2049	0.2779
0.2011	0.3387	0.4906	0.4553	0.1585	0.2175
0.3141	0.2929	0.4241	0.6091	0.1349	0.1860
0.4091	0.2631	0.3805	0.7492	0.1181	0.1633
0.6161	0.2145	0.3085	0.9055	0.1029	0.1425
0.7584	0.1892	0.2707	1.1462	0.0846	0.1173
1.0047	0.1552	0.2198	1.2596	0.0776	0.1076
1.1904	0.1353	0.1899	1.4285	0.0686	0.0950
1.3925	0.1174	0.1632	1.6207	0.0599	0.0830
1.6669	0.0979	0.1342	1.8993	0.0498	0.0688
1.8426	0.0876	0.1190	2.0571	0.0451	0.0621
2.0259	0.0783	0.1053	2.4552	0.0353	0.0485
2.5490	0.0576	0.0754	2.7963	0.0290	0.0396
3.0422	0.0438	0.0559	3.1895	0.0233	0.0316
3.4872	0.0345	0.0431	3.5835	0.0189	0.0255
4.0281	0.0261	0.0318	3.8977	0.0161	0.0215
5.0001	0.0160	0.0188	4.3903	0.0125	0.0167
6.2778	0.0086	0.0097	5.3111	0.0080	0.0106
8.0709	0.0037	0.0040	6.9386	0.0039	0.0050
10.0635	0.0015	0.0015	10.0712	0.0010	0.0013
16.0206	0.0001	0.0001	16.2166	0.0001	0.0001
21.3140	0.0000	0.0000	25.9527	0.0000	0.0000

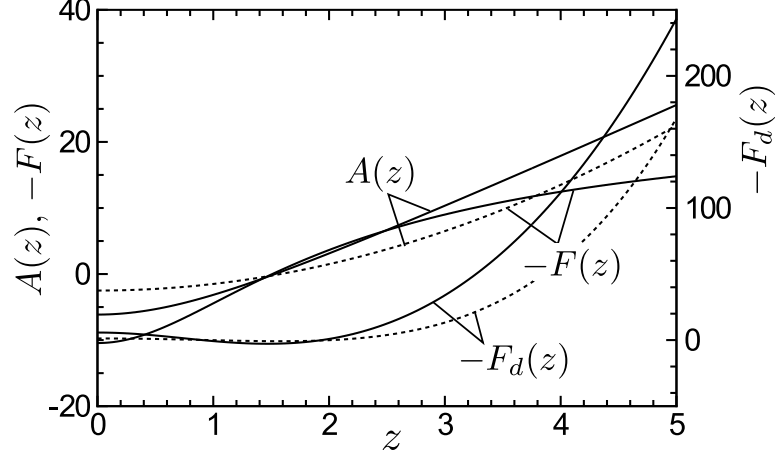


Figure 1.2: Functions A , F , and F_d . Solid lines: hard sphere. Dashed lines: BGK.

1.5.1 The functions A , F , F_d , and related constant γ_2

In the case of the BGK model, the functions A , F , and F_d are immediately obtained from the definition as

$$A(|\zeta|) = |\zeta|^2 - \frac{5}{2}, \quad F(|\zeta|) = -|\zeta|^2 + \frac{5}{2}, \quad F_d(|\zeta|) = -\frac{1}{3}(|\zeta|^4 - 5|\zeta|^2 + \frac{15}{4}),$$

by which we see $\gamma_2 = 1$. In the case of hard-sphere molecules, they are only known numerically (see Fig. 1.2). Note that only F_d has been computed here; the others are available in the literature and are shown here for easy reference. The value of γ_2 for hard-sphere molecules can also be found in the literature: [2,6] $\gamma_2 = 1.922284066$.

1.5.2 The coefficient $\beta^{(2)}$ of the second-order jump

In principle, we have to perform a numerical analysis of the problem (1.20) with $i = 2$ to obtain the value of $\beta^{(2)}$. However, by the use of the symmetry of the linearized kinetic equation, we can relate $\beta^{(2)}$ to quantities occurring in the half-space problem for $\psi^{(1)}$, thereby determining $\beta^{(2)}$ without new computation.

We apply the symmetry relation derived in Ref. [37] to the half-space problems (1.20) for $\psi^{(1)}$ and $\psi^{(2)}$. The basic idea behind the symmetry relation is that a conservative quantity can be constructed from solutions of two independent problems. (This is a basic principle that can be naturally extended to time-dependent problems. [38]) We briefly explain it here. If we multiply Eq. (1.20a) for $\psi^{(1)}$ by $\psi^{(2)}(\eta, -\zeta)$ and take its moment, we find that $\langle \zeta_1 \psi^{(2)}(\eta, -\zeta) \psi^{(1)}(\eta, \zeta) \rangle$, the “flux” of the constructed conservative quantity

$\langle \psi^{(2)}(\eta, -\zeta) \psi^{(1)}(\eta, \zeta) \rangle$, is constant in η and thus vanishes because of condition (1.20c) at infinity. Then, by using the condition (1.20b) at $\eta = 0$, we obtain

$$\frac{5\gamma_2}{4}\beta^{(2)} = \frac{2}{3}I_6(AF) + \frac{5}{2}I_4(AF_d) - \langle \zeta_1[F_d - \frac{1}{3}(|\zeta|^2 - 3\zeta_1^2)F]\psi^{(1)} \rangle|_{\eta=0}.$$

To simplify the last term on the right-hand side further, we consider Eq. (1.20a) for $\psi^{(1)}$ multiplied by $F_d - \frac{1}{3}(|\zeta|^2 - 3\zeta_1^2)F$ and take its moment. Then we obtain

$$\begin{aligned} \frac{d}{d\eta} \langle \zeta_1[F_d - \frac{1}{3}(|\zeta|^2 - 3\zeta_1^2)F]\psi^{(1)} \rangle &= \langle \psi^{(1)} \mathcal{L}[F_d - \frac{1}{3}(|\zeta|^2 - 3\zeta_1^2)F] \rangle \\ &= -\frac{5\gamma_2}{6} \langle (|\zeta|^2 - \frac{3}{2})\psi^{(1)} \rangle + \langle \zeta_1^2 A \psi^{(1)} \rangle, \end{aligned} \quad (1.22)$$

where we have used the self-adjointness of $\mathcal{L}$ and the definitions of F and F_d . We can show that the last term $\langle \zeta_1^2 A \psi^{(1)} \rangle$ vanishes by the result of two integrations with respect to η : one is Eq. (1.20a) for $\psi^{(1)}$ multiplied by $\zeta_1 A$ and the other is that multiplied by $\zeta_1(|\zeta|^2 - \frac{5}{2})$. Thus, from Eq. (1.22), we obtain

$$\langle \zeta_1[F_d - \frac{1}{3}(|\zeta|^2 - 3\zeta_1^2)F]\psi^{(1)} \rangle|_{\eta=0} = \frac{5\gamma_2}{4} \int_0^\infty \Theta^{(1)} d\eta,$$

and finally arrive at the relation

$$\beta^{(2)} = \frac{8}{15\gamma_2} [I_6(AF) + \frac{15}{4}I_4(AF_d)] - \int_0^\infty \Theta^{(1)} d\eta. \quad (1.23)$$

In the case of the BGK model, $I_6(AF) = -9/4$, $I_4(AF_d) = -1/3$, and $\int_0^\infty \Theta^{(1)} d\eta = -0.4391$. In the case of hard-sphere gas, $I_6(AF) = -7.4839567$, $I_4(AF_d) = -1.7512934$, and $\int_0^\infty \Theta^{(1)} d\eta = -0.7184$. Thus we obtain

$$\beta^{(2)} = -1.4276 \text{ (BGK)}, \quad -3.1801 \text{ (hard sphere)}. \quad (1.24)$$

1.5.3 Direct numerical solution of the half-space problem for $\psi^{(2)}$

In order to obtain $\Omega^{(2)}$ and $\Theta^{(2)}$, we carried out a numerical computation of the problem (1.20) for $\psi^{(2)}$ by a finite-difference method both for the BGK model and hard-sphere molecules. For the hard-sphere collision integral, we adopted the method first devised in Ref. [32] and applied later to various fundamental problems, including those for gas mixtures (see Refs. [39,40,35], and the references therein). In the numerical computation, we seek the solution $\psi^{(2)}$ as a function of η , ζ_1 , and $\zeta_\rho (\equiv \sqrt{\zeta_2^2 + \zeta_3^2})$, thanks to the spherical symmetry

Table 1.2: Knudsen-layer functions $\Omega^{(2)}$ and $\Theta^{(2)}$.

Hard sphere			BGK		
η	$-\Omega^{(2)}$	$\Theta^{(2)}$	η	$-\Omega^{(2)}$	$\Theta^{(2)}$
0.0000	1.4907	2.2601	0.0000	0.6746	1.0347
0.0033	1.4743	2.2335	0.0101	0.6546	1.0045
0.1142	1.2440	1.8550	0.1499	0.5309	0.8105
0.4091	0.9400	1.3584	0.5021	0.3797	0.5703
0.9183	0.6507	0.8994	0.9951	0.2644	0.3889
1.4994	0.4547	0.6019	1.4977	0.1924	0.2777
2.0259	0.3375	0.4318	1.9924	0.1446	0.2054
2.4811	0.2641	0.3291	2.4158	0.1149	0.1611
2.8272	0.2204	0.2696	3.0859	0.0814	0.1121
3.1150	0.1901	0.2292	3.7057	0.0601	0.0816
3.6779	0.1432	0.1684	4.0319	0.0515	0.0694
4.1865	0.1114	0.1284	4.3158	0.0451	0.0604
4.6706	0.0881	0.0997	4.7026	0.0378	0.0502
5.0001	0.0752	0.0842	5.1286	0.0312	0.0412
5.6738	0.0546	0.0599	5.9037	0.0222	0.0289
6.4526	0.0379	0.0408	6.5723	0.0167	0.0215
7.0725	0.0285	0.0302	7.1957	0.0129	0.0164
7.5226	0.0231	0.0243	7.6019	0.0109	0.0138
8.0248	0.0184	0.0192	8.1849	0.0086	0.0108
8.5346	0.0146	0.0151	8.8196	0.0066	0.0083
9.0048	0.0118	0.0121	9.5114	0.0051	0.0063
10.0146	0.0075	0.0076	11.9922	0.0020	0.0024
14.5460	0.0010	0.0010	14.0616	0.0009	0.0011
20.0515	0.0001	0.0001	20.4253	0.0001	0.0001
25.0557	0.0000	0.0000	27.7493	0.0000	0.0000

of $\mathcal{L}$. We have used the grid system in Ref. [39] with more grid points (roughly speaking, about 50% increase in ζ_1 and almost double in η and ζ_ρ)² and have adopted a third-order finite-difference scheme as in Ref. [40]. The computation for the BGK model used even more grid points in ζ_1 and ζ_ρ .

Figure 1.3 and Table 1.2 show $\Omega^{(2)}$ and $\Theta^{(2)}$. In the figure, the profiles of $\Omega^{(1)}$ and $\Theta^{(1)}$ are also shown for reference.

The value of $\beta^{(2)}$ was also obtained directly in the computation as

$$\beta^{(2)} = -1.4276 \text{ (BGK)}, -3.1800 \text{ (hard sphere)}. \quad (1.25)$$

²We have applied to η , ζ_1 , and ζ_ρ the grid system for x_1 , ζ_1^α , and ζ_ρ^α generated by Eqs. (52a)–(53c) of Ref. [39] with $\hat{m}^\alpha = 1$. The present numerical data were obtained by the following setting of grid parameters: $\bar{N}_x = 600$, $N_x = 700$, $\bar{Z}_1 = \bar{Z}_\rho = 5$, $\bar{N}_1 = N_1 = 32$, and $\bar{N}_\rho = N_\rho = 36$. By the present choice of $\bar{N}_x$ and N_x the η -space is truncated at $\eta \simeq 35$.

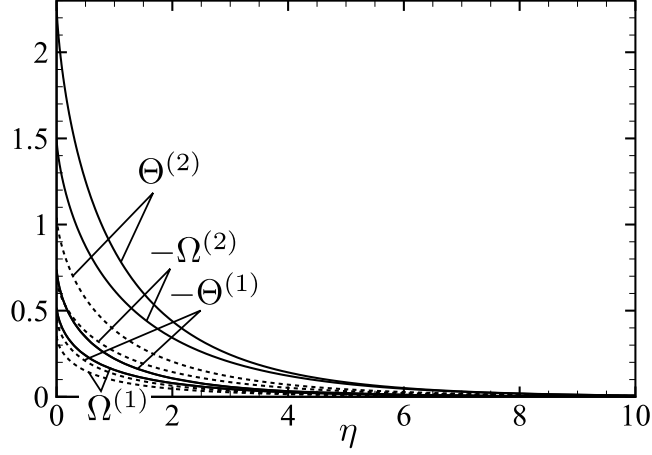


Figure 1.3: Knudsen-layer functions $\Omega^{(2)}$ and $\Theta^{(2)}$. Solid lines: hard sphere. Dashed lines: BGK. The profiles of $\Omega^{(1)}$ and $\Theta^{(1)}$ are also shown for reference.

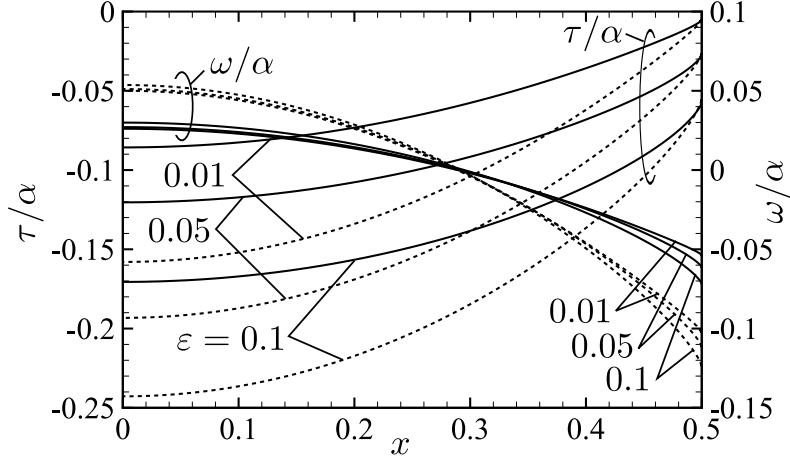


Figure 1.4: Profiles of τ/α and ω/α in the right half of the gap between the plates [see Eq. (1.21)] for $\varepsilon = 0.01, 0.05$, and 0.1 . Solid lines: hard sphere. Dashed lines: BGK.

The values in Eqs. (1.24) and (1.25) agree to four or five digits. This comparison is an indirect measure of computational accuracy.

In Fig. 1.4 we show the profiles of macroscopic quantities τ and ω of the original two-surface problem, i.e., Eqs. (1.21b) and (1.21c), for $\varepsilon = 0.01, 0.05$, and 0.1 . The profiles look highly dependent on the molecular model. This is, however, mainly due to the fact that the relation between the mean free path of a molecule and the thermal conductivity of the gas depends on the molecular model. To account for this, we take the thermal conductivity of the gas as the basic parameter instead of the mean free path, introduce $\varepsilon_* = \gamma_2 \varepsilon$, and plot $\tau\gamma_2/\alpha$ and $\omega\gamma_2/\alpha$ for the same values of ε_* . The resulting profiles become less dependent on the molecular model (Fig. 1.5).

Finally in Fig. 1.6 we compare the profile of τ with that obtained by RHTA (see foot-

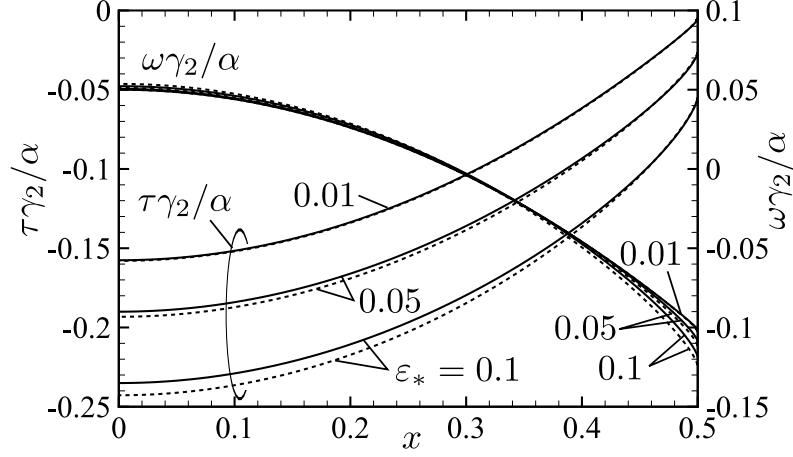


Figure 1.5: Profiles of $\tau\gamma_2/\alpha$ and $\omega\gamma_2/\alpha$ in the right half of the gap between the plates for $\varepsilon_* = 0.01, 0.05$, and 0.1 . Solid lines: hard sphere. Dashed lines: BGK.

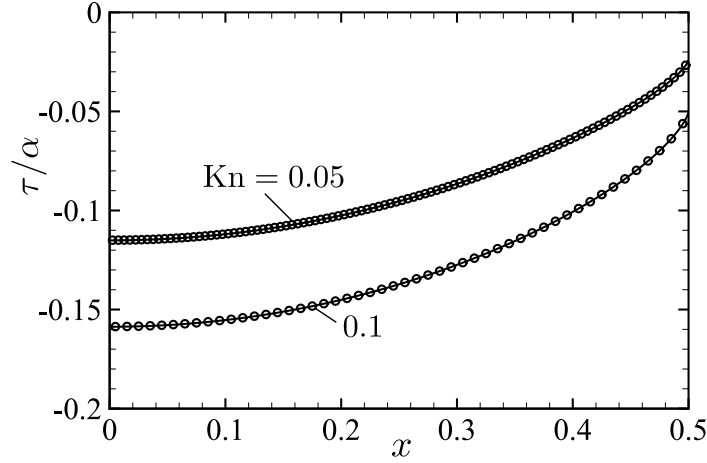


Figure 1.6: Comparisons of the profiles of τ/α with that obtained by Radtke, Hadjiconstantinou, Takata, and Aoki (RHTA) for $\text{Kn} = 0.05$ and 0.1 (hard sphere). Solid lines: Eq. (1.21b). Open circles: RHTA.

note 1) for $\text{Kn} = 0.05$ and 0.1 for hard-sphere molecules. The two profiles agree quite well; the difference is not visible in the figure. This is probably due to the fact that our asymptotic solution (1.21) is valid up to any order of ε .

1.6 Conclusion

We have investigated the behavior of a slightly rarefied monatomic gas between two parallel plates whose temperature grows slowly and linearly in time. To be more precise, we studied the gas behavior that is realized after a long time has passed from some initial state.

We first showed that this problem is equivalent to a boundary-value problem for the

steady linearized Boltzmann equation with a volumetric heating source, recently studied by Radtke, Hadjiconstantinou, Takata, and Aoki who focus on the parabolic temperature profile and related second-order jump. Our formulation gives a natural interpretation to the heating source and shows that the nonzero second derivative of temperature and related second-order jump is induced by a mechanism which is not covered by the general theory of slip flow for steady problems. As a result, naive application of the second-order jump condition for steady problems to unsteady problems can produce incorrect results. Further discussion of the differences between the two cases can be found in the article by RHTA.

We also carried out a systematic asymptotic analysis for small Knudsen numbers. Analysis of the Knudsen-layer correction shows that the second-order jump of temperature is indeed determined by a new half-space problem which does not occur in the general theory of slip flow for steady problems (Refs. [2,6,11,13]). This explains why the jump coefficient observed by RHTA does not agree with any of the jump coefficients in the existing theory. The newly found temperature-jump does not influence the instantaneous heat transfer to the walls, at least up to the second order in the present one-dimensional problem, because its associated Knudsen-layer problem for $\psi^{(2)}$ does not have an inhomogeneous term in Eq. (1.20a) and thus gives no additional contribution to the heat flow. (The heat flux to the wall is *always* equal to the heating/cooling rate integrated over the domain size and is correctly captured even without knowledge of the first-order slip coefficient.) The theoretical treatment presented here can be straightforwardly extended to gas mixtures with qualitatively similar results.

Finally, we carried out numerical computations to determine the coefficient of the new second-order temperature jump and the structure of the related Knudsen layer for the BGK model and hard-sphere molecules. The jump coefficient was also calculated by the use of a symmetry relation. The values obtained by the two different methods agree within our estimates of the numerical accuracy of the computations. Moreover, the present asymptotic solution for the behavior of the gas agrees well with the results that RHTA have obtained by the low variance stochastic method (Refs. [23,24,25,26]).

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Chapter 2

Asymptotic theory for the time-dependent behavior of a slightly rarefied gas over a smooth solid boundary

2.1 Introduction

The connection between the Boltzmann and the fluid-dynamic systems has been studied since the days of Hilbert, and a number of important results have been obtained in the limit of small Knudsen numbers (e.g., Refs. [1,2,3,4,5,6,7,8,9,10]). Specifically, fluid-dynamic-type sets of equations and their appropriate slip and jump boundary conditions for describing the steady gas behavior in the regime of small Knudsen numbers (the so-called slip flow regime) have been established [5,6,7] since the late 1960s and early 1970s. For complete descriptions of the general theory of slip flow for steady problems, the reader is referred to Refs. [11] and [12]. In the present paper, we extend this theory to unsteady problems in a nondeformable domain within a linearized framework. To be more precise, we study the gas behavior on the basis of the linearized Boltzmann equation and a kinetic boundary condition under the diffusion scaling, thereby obtaining a (Stokes-type) set of fluid-dynamic equations and its appropriate (slip and jump) boundary conditions up to the second order of the Knudsen number. The present work is motivated by recent findings in Refs. [13] and [14] on the second-order temperature jump.

In Ref. [13], a slightly rarefied gas, namely a gas for small Knudsen numbers, bounded by two parallel walls subject to constant volumetric heating was studied numerically, using a low variance stochastic method [15]. From the parabolic temperature profile associated with the volumetric heating, Radtke *et al.* evaluated the second-order temperature-jump coefficient, which, in the case of the Bhatnagar–Gross–Krook (BGK) [or Boltzmann–Krook–Wendler (BKW)] model [16,17], did not agree with the one predicted (coefficient d_3 in Sec. 3.1.5 of Ref. [12]) by the general theory of slip flow [5,11,12] for steady problems. Then, Takata *et al.* [14] showed that this problem with volumetric heating corresponds to the

time-evolution problem caused by a change of wall temperature, identified the source of the observed temperature jump by a systematic asymptotic analysis, and concluded that this jump is a new one that is not covered by the theory for steady problems. This result suggests a possibility that there might be other new jump phenomena which do not appear (or degenerate) in steady problems.

In the present paper, following Sone's method [5,11,12] originally developed for steady problems, we carry out a systematic asymptotic analysis of the linearized Boltzmann equation for small Knudsen numbers up to its second order to reach a comprehensive understanding of the general behavior of slightly rarefied gases caused by a slow change of the surroundings. We also provide the numerical data of all the slip and jump coefficients occurring in the theory for a hard-sphere gas, most of which have not been available in the literature.

2.2 Problem and assumption

Consider a monatomic rarefied gas around solid bodies of arbitrary smooth shape. There is no external force. We will investigate the time-dependent behavior of the gas under the assumptions that (i) the behavior of the gas is described by the Boltzmann equation; (ii) the gas molecules are reflected on the surface of bodies by a rule of gas-surface interaction prescribed later in details; (iii) the state of the gas is close to the reference equilibrium state at rest with density ρ_0 and temperature T_0 , and accordingly the equation and initial and boundary conditions can be linearized around that reference state; (iv) the mean free path ℓ_0 (or time) at the reference state is much shorter than the characteristic length L (or time) of the problem; (v) the domain under consideration does not deform in time; and (vi) the time evolution is initiated by a slow change of the surroundings from the reference state.

With a reference time-scale t_0 , which will be specified later in accordance with assumption (vi), we denote by $t_0 t$ the time, by $L \mathbf{x}$ the spatial position, by $(2RT_0)^{1/2} \boldsymbol{\zeta}$ the molecular velocity, and by $\rho_0(2RT_0)^{-3/2}(1 + \phi)E$ the velocity distribution function of molecules, respectively. Here R is the specific gas constant (i.e., $R = k/m$ with k being the Boltzmann constant and m being the mass of a molecule), $E(|\boldsymbol{\zeta}|) = \pi^{-3/2} \exp(-|\boldsymbol{\zeta}|^2)$, and ϕ is a function of t , $\mathbf{x}$, and $\boldsymbol{\zeta}$. Then, because of assumption (iii), ϕ satisfies the dimensionless linearized

Boltzmann equation

$$\text{Sh} \partial_t \phi + \zeta_i \partial_i \phi = \frac{2}{\sqrt{\pi}} \frac{1}{\text{Kn}} \mathcal{L}[\phi], \quad (2.1)$$

where $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, the two constants Sh and Kn are the Strouhal and Knudsen numbers defined by

$$\text{Sh} = \frac{L}{t_0(2RT_0)^{1/2}}, \quad \text{Kn} = \frac{\ell_0}{L},$$

and $\mathcal{L}$ is the linearized collision operator, the explicit form of which is given in Sec. 2.9 for the BGK model and a hard-sphere gas.

The gas molecules are reflected on the surface of bodies by a certain rule of gas-surface interaction (usually assumed as a linear scattering operation on the velocity distribution function) that is determined by the thermodynamic state and motion of the surface, once the gas species and the surface material are specified. If we denote by $\mathcal{K}$ the linear scattering operator that represents the reflection rule on the resting surface at the reference temperature T_0 , the boundary condition for ϕ on the surface is expressed as

$$\phi = g_w + \mathcal{K}[\phi - g_w], \quad \zeta_n > 0, \quad (2.2)$$

where $\zeta_n = \zeta_i n_i$, $\mathbf{n}$ is the normal unit vector to the boundary, pointed to the gas, and g_w is defined in terms of surface velocity $(2RT_0)^{1/2} \mathbf{u}_w$ and temperature $T_0(1 + \tau_w)$ as

$$g_w = 2\zeta_i u_{iw} + (|\zeta|^2 - \frac{5}{2})\tau_w.$$

See Sec. 2.9 for the reduction of the boundary condition for the full (nonlinear) Boltzmann equation to that for the linearized Boltzmann equation (2.2). Hereafter, we generically call g_w (or u_{iw} and τ_w) the boundary data. Note that $u_{iw} n_i = 0$ because of assumption (v). Further, $u_{iw} = 0$ and $\tau_w = 0$ at $t = 0$, because of assumption (vi). The scattering operator $\mathcal{K}$ is often expressed in terms of the reflection kernel $\mathcal{R}$ as

$$\mathcal{K}[\varphi] = \int_{\zeta_n^* < 0} \frac{|\zeta_n^*| E(|\zeta^*|)}{|\zeta_n| E(|\zeta|)} \mathcal{R}(\zeta^*, \zeta) \varphi(t, \mathbf{x}, \zeta^*) d\zeta^*, \quad \zeta_n > 0,$$

where $d\zeta^* = d\zeta_1^* d\zeta_2^* d\zeta_3^*$, and the following fundamental properties are assumed:

1. $\mathcal{R}(\zeta^*, \zeta) \geq 0$ for $\zeta_n^* < 0$, $\zeta_n > 0$.
2. Impermeability: $\int_{\zeta_n > 0} \mathcal{R}(\zeta^*, \zeta) d\zeta = 1$ for $\zeta_n^* < 0$.

3. Let φ be $\varphi = c_0 + c_i \zeta_i + c_4 |\zeta|^2$, where c_0 , c_i , and c_4 are independent of ζ . Among such φ , only $\varphi = c_0$ satisfies the relation $\varphi = \mathcal{K}[\varphi]$ for $\zeta_n > 0$.

4. Detailed balance:

$$|\zeta_n^*| \mathcal{R}(\zeta^*, \zeta) E(|\zeta^*|) = |\zeta_n| \mathcal{R}(-\zeta, -\zeta^*) E(|\zeta|), \quad \text{for } \zeta_n > 0 \text{ and } \zeta_n^* < 0.$$

5. Local isotropy: for any orthogonal transformation matrices l_{ij} (thus, $l_{ik}l_{jk} = \delta_{ij}$, where δ_{ij} is Kronecker's delta) that meet the condition $n_i = l_{ij}n_j$,

$$\mathcal{K}[f(l_{ij}\zeta_j)](\zeta) = \mathcal{K}[f(\zeta)](l_{ij}\zeta_j).$$

By assumption (vi), the initial condition is written as

$$\phi(0, \mathbf{x}, \zeta) = 0. \quad (2.3)$$

We denote the density, flow velocity, and temperature of the gas by $\rho_0(1+\omega)$, $(2RT_0)^{1/2}\mathbf{u}$, and $T_0(1+\tau)$, which are expressed in terms of ϕ as

$$\omega = \langle \phi \rangle, \quad u_i = \langle \zeta_i \phi \rangle, \quad \tau = \frac{2}{3} \langle (|\zeta|^2 - \frac{3}{2}) \phi \rangle, \quad (2.4)$$

where the solid brackets represent the following moment:

$$\langle f \rangle = \int f(t, \mathbf{x}, \zeta) E(|\zeta|) d\zeta,$$

and the range of integration is the whole space of ζ . Further, we denote the pressure, stress tensor, and heat-flow vector by $p_0(1+P)$, $p_0(\delta_{ij} + P_{ij})$, and $p_0(2RT_0)^{1/2}\mathbf{Q}$, respectively. They are defined by

$$P = \frac{2}{3} \langle |\zeta|^2 \phi \rangle, \quad P_{ij} = 2 \langle \zeta_i \zeta_j \phi \rangle, \quad Q_i = \langle \zeta_i (|\zeta|^2 - \frac{5}{2}) \phi \rangle, \quad (2.5)$$

where $p_0 = \rho_0 RT_0$ is the reference pressure. Note that $P = \omega + \tau$ by definition, which is the linearized equation of state.

In the present paper, we are interested in the behavior of the gas caused by a slow motion and/or small temperature variation of the boundary that may change in time in the scale of $t_0 \sim \rho_0 L^2 / \mu_0$ [μ_0 is the reference viscosity; see assumption (vi)] as well as by the corresponding far field deviation from the reference state. This is the time scale that the diffusive nature of the gas is relevant. In view of the relation of the viscosity to the mean

free path, Sh is of $O(\text{Kn})$ and we put $\varepsilon = \text{Sh} = (\sqrt{\pi}/2)\text{Kn}$. Note that ε is small because of assumption (iv). Thus, we rewrite Eq. (2.1) as

$$\varepsilon \partial_t \phi + \zeta_i \partial_i \phi = \frac{1}{\varepsilon} \mathcal{L}[\phi], \quad (2.6)$$

and carry out a systematic asymptotic analysis of the initial- and boundary-value problem, i.e., Eqs. (2.6), (2.2), and (2.3), for small ε . Because we here focus on slow variations (in time) from the reference equilibrium state [see Eq. (2.3)], the present theory will be developed exclusively under the above time scaling (2.6). Further comments on the time scale will be given in Sec. 2.6.2 in the connection with discussions for the time evolution from the initial state that is different from the reference state.

2.3 Hilbert solution

Putting aside the initial and boundary conditions, we seek a solution ϕ_{H} that changes moderately both in $\boldsymbol{x}$ and t (the Hilbert solution) in a power series of ε :

$$\phi_{\text{H}} = \phi_{\text{H}0} + \phi_{\text{H}1}\varepsilon + \phi_{\text{H}2}\varepsilon^2 + \cdots.$$

Corresponding macroscopic quantity h_{H} ($h = P, u_i, \tau, \omega, P_{ij}, Q_i$) is also expanded as

$$h_{\text{H}} = h_{\text{H}0} + h_{\text{H}1}\varepsilon + h_{\text{H}2}\varepsilon^2 + \cdots,$$

where $h_{\text{H}m}$ ($m = 0, 1, \dots$) is defined by Eqs. (2.4) and (2.5) with ϕ being replaced by $\phi_{\text{H}m}$. Substitution of the expansion into Eq. (2.6) leads to

$$\mathcal{L}[\phi_{\text{H}0}] = 0, \quad (2.7a)$$

$$\mathcal{L}[\phi_{\text{H}1}] = \zeta_i \partial_i \phi_{\text{H}0}, \quad (2.7b)$$

$$\mathcal{L}[\phi_{\text{H}2}] = \partial_t \phi_{\text{H}0} + \zeta_i \partial_i \phi_{\text{H}1}, \quad (2.7c)$$

$$\mathcal{L}[\phi_{\text{H}m}] = \partial_t \phi_{\text{H}m-2} + \zeta_i \partial_i \phi_{\text{H}m-1}, \quad (m \geq 3). \quad (2.7d)$$

This set of linear integral equations can be solved from the lowest order, provided that the solvability condition

$$\partial_t \langle \psi_J \phi_{\text{H}m-2} \rangle + \partial_i \langle \zeta_i \psi_J \phi_{\text{H}m-1} \rangle = 0, \quad (J = 0, \dots, 4),$$

or equivalently

$$\partial_t \omega_{\text{H}m-2} + \partial_i u_{i\text{H}m-1} = 0, \quad (2.8a)$$

$$\partial_t u_{i\text{H}m-2} + \frac{1}{2} \partial_j P_{ij\text{H}m-1} = 0, \quad (2.8b)$$

$$\frac{5}{2} \partial_t \tau_{\text{H}m-2} - \partial_t P_{\text{H}m-2} + \partial_i Q_{i\text{H}m-1} = 0, \quad (2.8c)$$

($m = 1, 2, \dots$) is fulfilled, where $\phi_{\text{H}-1} = 0$ and ψ_J is the collision invariant, i.e., $\psi_0 = 1$, $\psi_i = \zeta_i$, and $\psi_4 = |\zeta|^2$. The main objective of this section is to derive a set of fluid-dynamic-type equations that describes the overall behavior of the gas up to $O(\varepsilon^2)$. To this end, we perform calculations up to $O(\varepsilon^4)$. The set of fluid-dynamic-type equations will be obtained as the solvability conditions for the series of integral equations (2.7).

2.3.1 The leading (the zeroth) order

Equation (2.7a) is solved as

$$\phi_{\text{H}0} = \phi_{\text{eH}0}, \quad (2.9)$$

where

$$\phi_{\text{eH}m} = P_{\text{H}m} + 2\zeta_i u_{i\text{H}m} + (|\zeta|^2 - \frac{5}{2})\tau_{\text{H}m}, \quad (m \geq 0).$$

Accordingly, the stress tensor and heat-flow vector at this order are given as

$$P_{ij\text{H}0} = P_{\text{H}0}\delta_{ij}, \quad Q_{i\text{H}0} = 0. \quad (2.10)$$

2.3.2 The first order

Since Eq. (2.7b) is an inhomogeneous linear integral equation for $\phi_{\text{H}1}$, the inhomogeneous term should satisfy the solvability condition (2.8) for $m = 1$, which can be reduced by the use of Eq. (2.10) to

$$\partial_i u_{i\text{H}0} = 0, \quad \partial_i P_{\text{H}0} = 0. \quad (2.11)$$

Under this condition, Eq. (2.7b) can be transformed into

$$\mathcal{L}[\phi_{\text{H}1}] = \zeta_{ij} \overline{\partial_i u_{j\text{H}0}} + \zeta_i (|\zeta|^2 - \frac{5}{2}) \partial_i \tau_{\text{H}0},$$

where $\zeta_{ij} = \zeta_i \zeta_j - (1/3)|\zeta|^2 \delta_{ij}$ and $\overline{f_{ij}} = f_{ij} + f_{ji} - (2/3)f_{kk}\delta_{ij}$ (therefore $\overline{f_{ii}} = 0$), and thus can be solved as

$$\phi_{\text{H}1} = \phi_{\text{eH}1} - \frac{1}{2} \zeta_{ij} B(|\zeta|) \overline{\partial_i u_{j\text{H}0}} - \zeta_i A(|\zeta|) \partial_i \tau_{\text{H}0}, \quad (2.12)$$

where A and B are functions defined in Sec. 2.10. The corresponding stress tensor and heat-flow vector can be obtained by substitution as

$$P_{ijH1} = P_{H1}\delta_{ij} - \gamma_1 \overline{\partial_i u_{jH0}}, \quad Q_{iH1} = -\frac{5}{4}\gamma_2 \partial_i \tau_{H0}, \quad (2.13)$$

where γ_1 and γ_2 are constants (see Sec. 2.10).

2.3.3 The second order

With the aid of Eqs. (2.11) and (2.13), the solvability condition (2.8) for $m = 2$ is reduced to

$$\partial_t \omega_{H0} + \partial_i u_{iH1} = 0, \quad (2.14a)$$

$$\partial_t u_{iH0} + \frac{1}{2} \partial_i P_{H1} - \frac{1}{2} \gamma_1 \Delta u_{iH0} = 0, \quad (2.14b)$$

$$\partial_t \tau_{H0} - \frac{2}{5} \partial_t P_{H0} - \frac{1}{2} \gamma_2 \Delta \tau_{H0} = 0, \quad (2.14c)$$

where Δ is the Laplacian, i.e., $\Delta = \partial_i^2$. It is seen from Eqs. (2.14b) and (2.11) that P_{H1} is a harmonic function.

In order to solve Eq. (2.7c), we first transform its right-hand side by using the solvability conditions (2.11) and (2.14):

$$\begin{aligned} \mathcal{L}[\phi_{H2}] = & \frac{1}{3} \left(\frac{5}{2} \gamma_2 (|\zeta|^2 - \frac{3}{2}) - |\zeta|^2 A(|\zeta|) \right) \Delta \tau_{H0} + \zeta_{ij} \overline{\partial_i u_{jH1}} + \zeta_i (|\zeta|^2 - \frac{5}{2}) \partial_i \tau_{H1} \\ & - \frac{1}{2} \left(\zeta_i \zeta_j \zeta_k B(|\zeta|) - \gamma_1 (\zeta_i \delta_{jk} + \zeta_j \delta_{ik} + \zeta_k \delta_{ij}) \right) \partial_i \overline{\partial_j u_{kH0}} - \frac{1}{2} \zeta_{ij} A(|\zeta|) \overline{\partial_i \partial_j \tau_{H0}}. \end{aligned}$$

This equation can be solved to yield

$$\begin{aligned} \phi_{H2} = & \phi_{eH2} - \frac{1}{2} \zeta_{ij} B(|\zeta|) \overline{\partial_i u_{jH1}} - \zeta_i A(|\zeta|) \partial_i \tau_{H1} \\ & + \frac{1}{2} \left((\zeta_i \delta_{jk} + \zeta_j \delta_{ik} + \zeta_k \delta_{ij}) D_1(|\zeta|) + \zeta_i \zeta_j \zeta_k D_2(|\zeta|) \right) \partial_i \overline{\partial_j u_{kH0}} \\ & - \frac{1}{2} \zeta_{ij} F(|\zeta|) \overline{\partial_i \partial_j \tau_{H0}} - F_d(|\zeta|) \Delta \tau_{H0}, \end{aligned} \quad (2.15)$$

where the functions D_1 , D_2 , F , F_d , and the constant γ_3 that appears soon later are defined in Sec. 2.10. By the substitution of Eq. (2.15), the corresponding stress tensor and heat-flow vector are expressed as

$$P_{ijH2} = P_{H2} \delta_{ij} - \gamma_1 \overline{\partial_i u_{jH1}} + \frac{1}{2} \gamma_3 \overline{\partial_i \partial_j \tau_{H0}}, \quad (2.16a)$$

$$Q_{iH2} = -\frac{5}{4} \gamma_2 \partial_i \tau_{H1} + \frac{1}{2} \gamma_3 \Delta u_{iH0}. \quad (2.16b)$$

2.3.4 The third and fourth orders

By using Eqs. (2.11), (2.14), and (2.16), the solvability condition (2.8) for $m = 3$ is reduced to

$$\partial_t \omega_{H1} + \partial_i u_{iH2} = 0, \quad (2.17a)$$

$$\partial_t u_{iH1} + \frac{1}{2} \partial_i P_{H2}^* - \frac{1}{2} \gamma_1 \Delta u_{iH1} = 0, \quad (2.17b)$$

$$\partial_t \tau_{H1} - \frac{2}{5} \partial_t P_{H1} - \frac{1}{2} \gamma_2 \Delta \tau_{H1} = 0, \quad (2.17c)$$

where

$$P_{H2}^* = P_{H2} - \frac{1}{6} (\gamma_2 \gamma_1 - 4\gamma_3) \Delta \tau_{H0}. \quad (2.17d)$$

In the same way as before, ϕ_{H3} can be solved under the solvability conditions derived so far. However, we do not need its specific form when examining the solvability condition for ϕ_{H4} . Indeed, as is seen from Eq. (2.8), the information of P_{ijH3} and Q_{iH3} is enough to discuss the solvability of Eq. (2.7d) for $m = 4$. This information can be obtained by using the self-adjointness of $\mathcal{L}$ and the definitions of A and B in Sec. 2.10. That is, we make a transformation

$$\begin{aligned} P_{ijH3} - P_{H3} \delta_{ij} &\equiv \langle 2\zeta_{ij} \phi_{H3} \rangle = \langle -\mathcal{L}[\zeta_{ij} B] \phi_{H3} \rangle = -\langle \mathcal{L}[\phi_{H3}] \zeta_{ij} B \rangle, \\ Q_{iH3} &\equiv \langle \zeta_i (|\zeta|^2 - \frac{5}{2}) \phi_{H3} \rangle = \langle -\mathcal{L}[\zeta_i A] \phi_{H3} \rangle = -\langle \mathcal{L}[\phi_{H3}] \zeta_i A \rangle, \end{aligned}$$

and use Eqs. (2.7d) for $m = 3$, (2.12), and (2.15). Then, we eventually obtain

$$\begin{aligned} P_{ijH3} &= P_{H3} \delta_{ij} - \gamma_1 \overline{\partial_i u_{jH2}} + \frac{\gamma_3}{2} \overline{\partial_i \partial_j \tau_{H1}} - \frac{\gamma_{10}}{2} \overline{\partial_i \partial_j P_{H1}} + \frac{1}{2} (\gamma_1 \gamma_{10} - 2\gamma_6) \Delta \overline{\partial_i u_{jH0}}, \\ Q_{iH3} &= -\frac{5}{4} \gamma_2 \partial_i \tau_{H2} + \frac{\gamma_3}{2} \Delta u_{iH1} - \frac{13}{8} \gamma_{11} \partial_i \Delta \tau_{H0}, \end{aligned}$$

where γ_6 , γ_{10} , and γ_{11} are constants defined in Sec. 2.10.

By the substitution of the expressions of P_{ijH3} and Q_{iH3} , solvability condition (2.8) for $m = 4$ is written as

$$\partial_t \omega_{H2} + \partial_i u_{iH3} = 0, \quad (2.18a)$$

$$\partial_t u_{iH2} + \frac{1}{2} \partial_i P_{H3}^* - \frac{1}{2} \gamma_1 \Delta u_{iH2} + \frac{1}{4} (\gamma_1 \gamma_{10} - 2\gamma_6) \Delta^2 u_{iH0} = 0, \quad (2.18b)$$

$$\partial_t \tau_{H2} - \frac{2}{5} \partial_t P_{H2} - \frac{1}{2} \gamma_2 \Delta \tau_{H2} + \frac{1}{10} (\gamma_2 \gamma_3 - \frac{13}{2} \gamma_{11}) \Delta^2 \tau_{H0} = 0, \quad (2.18c)$$

where

$$P_{\text{H3}}^* = P_{\text{H3}} - \frac{1}{6} (\gamma_2 \gamma_1 - 4\gamma_3) \Delta \tau_{\text{H1}} + \frac{1}{5} \gamma_1 \partial_t P_{\text{H1}}. \quad (2.18\text{d})$$

This completes the derivation of the set of fluid-dynamic-type equations that describes the overall behavior of the gas up to $O(\varepsilon^2)$.

2.4 Knudsen-layer analysis

In Sec. 2.3, a set of fluid-dynamic-type equations that describes the overall behavior of the gas up to $O(\varepsilon^2)$ has been derived. However, we did not take into account the initial and boundary conditions there.

At the initial instance, the gas is in the reference equilibrium state and $u_{i\text{w}} = 0$ and $\tau_{\text{w}} = 0$. Thus, the Hilbert solution satisfies the initial and boundary conditions. However, as the time goes on, it becomes non-Maxwellian (except for the leading order) and no longer satisfies the boundary condition. We have to introduce the so-called Knudsen-layer correction in the thin layer with the thickness of a few mean free paths in the vicinity of the boundary (the Knudsen-layer). In the following discussions, for the sake of generality, we expand the boundary data g_{w} (and thus $u_{i\text{w}}$ and τ_{w}) in a power series of ε : $g_{\text{w}} = g_{\text{w}0} + g_{\text{w}1}\varepsilon + \dots$, $u_{i\text{w}} = u_{i\text{w}0} + u_{i\text{w}1}\varepsilon + \dots$, $\tau_{\text{w}} = \tau_{\text{w}0} + \tau_{\text{w}1}\varepsilon + \dots$, and $g_{\text{wm}} = 2\zeta_i u_{i\text{wm}} + (|\zeta|^2 - \frac{5}{2})\tau_{\text{wm}}$ ($m = 0, 1, 2, \dots$).

Since the Hilbert solution is a Maxwellian at the leading order, it can satisfy the boundary condition (2.2) at the leading order if we put

$$u_{i\text{H}0} = u_{i\text{w}0}, \quad \tau_{\text{H}0} = \tau_{\text{w}0}, \quad (2.19)$$

on the boundary [see Eqs. (2.9) and (2.2)]. Thus the Knudsen-layer correction is required from the first order. Let us express ϕ as $\phi = \phi_{\text{H}} + \phi_{\text{K}}$, where ϕ_{K} is appreciable only in the thin layer adjacent to the boundary [thus, $n_i \partial_i \phi_{\text{K}} = (1/\varepsilon)O(\phi_{\text{K}})$] and is $O(\varepsilon)$. Corresponding correction to macroscopic quantities will be denoted by h_{K} , i.e., $h = h_{\text{H}} + h_{\text{K}}$. Then, ϕ_{K} is the solution of the following problem:

$$\begin{aligned} \varepsilon \partial_t \phi_{\text{K}} + \zeta_i \partial_i \phi_{\text{K}} &= \frac{1}{\varepsilon} \mathcal{L}[\phi_{\text{K}}], \\ \phi_{\text{K}} &= -\phi_{\text{H}} + g_{\text{w}} + \mathcal{K}[\phi_{\text{K}} + \phi_{\text{H}} - g_{\text{w}}], \quad \zeta_n > 0, \eta = 0, \\ \phi_{\text{K}} &\rightarrow 0 \quad \text{as } \eta \rightarrow \infty, \end{aligned}$$

$$\phi_K = 0 \quad \text{at } t = 0.$$

We introduce a new coordinate system (η, χ_1, χ_2) , which is stretched in the direction normal to the boundary:

$$x_i = x_{iw}(\chi_1, \chi_2) + \varepsilon \eta n_i(\chi_1, \chi_2),$$

where x_{iw} is a position on the boundary. As shown in Sec. 2.12,

$$\partial_i = \frac{1}{\varepsilon} n_i \partial_\eta + (\partial_i \chi_1)_w \partial_{\chi_1} + (\partial_i \chi_2)_w \partial_{\chi_2} + \varepsilon \eta (\partial_n \partial_i \chi_1)_w \partial_{\chi_1} + \varepsilon \eta (\partial_n \partial_i \chi_2)_w \partial_{\chi_2} + \cdots,$$

where $(\cdot)_w$ denotes the value at $\eta = 0$ and $\partial_n = n_j \partial_j$. Since $\phi_K = O(\varepsilon)$, we expand it in the power series of ε : $\phi_K = \phi_{K1}\varepsilon + \phi_{K2}\varepsilon^2 + \cdots$. Correspondingly, h_K is expanded as $h_K = h_{K1}\varepsilon + h_{K2}\varepsilon^2 + \cdots$, where h_{Km} ($m = 1, 2, \dots$) is given by Eqs. (2.4) and (2.5) with ϕ being replaced by ϕ_{Km} . Substitution of the expansion of ϕ_K leads to the following series of boundary-value problems: for $m = 1, 2, \dots$,

$$\zeta_n \partial_\eta \phi_{Km} = \mathcal{L}[\phi_{Km}] - I_{Km}, \quad (2.20a)$$

$$\phi_{Km} = \mathcal{K}[\phi_{Km}] - \widetilde{\mathcal{K}}[\phi_{Hm} - g_{wm}], \quad \zeta_n > 0, \quad \eta = 0, \quad (2.20b)$$

$$\phi_{Km} \rightarrow 0 \quad \text{as } \eta \rightarrow \infty, \quad (2.20c)$$

where $\widetilde{\mathcal{K}}[f] = f - \mathcal{K}[f]$ ($\widetilde{\mathcal{K}}_1, \widetilde{\mathcal{K}}_2$, etc. that appear later are defined in the same way),

$$I_{K1} = 0, \quad I_{K2} = \zeta_i \mathcal{D}_i[\phi_{K1}],$$

$$I_{K3} = \zeta_i \mathcal{D}_i[\phi_{K2}] + \partial_t \phi_{K1} + \eta \zeta_i ((\partial_n \partial_i \chi_1)_w \partial_{\chi_1} + (\partial_n \partial_i \chi_2)_w \partial_{\chi_2}) \phi_{K1}, \quad \text{etc.},$$

and $\mathcal{D}_i = (\partial_i \chi_1)_w \partial_{\chi_1} + (\partial_i \chi_2)_w \partial_{\chi_2}$ (see Sec. 2.12). Note that the time dependence of ϕ_{Km} occurs only implicitly and the initial condition for ϕ_K has been discarded, because it is automatically satisfied. In fact, the results of the following analysis are compatible with the initial condition, as will be seen below.

We perform the analysis up to $O(\varepsilon^2)$. In the rest of Sec. 2.4 and in Secs. 2.13–2.15, the quantities with subscript H [for instance, $\partial_i \tau_{H0} n_i$, $\overline{\partial_j u_{kH0} n_k} (\delta_{ij} - n_i n_j)$, $\partial_j \tau_{H0} (\delta_{ij} - n_i n_j)$ in Eq. (2.22)] represent their value on the boundary, unless otherwise stated. Hereafter, the comments on this notation will not be repeated.

2.4.1 The first order

By the substitution of Eq. (2.12), the boundary condition (2.20b) ($m = 1$) is reduced to

$$\begin{aligned}\phi_{K1} = & -2\tilde{\mathcal{K}}[(\delta_{ij} - n_i n_j)\zeta_j](u_{iH1} - u_{iw1}) - \tilde{\mathcal{K}}[|\boldsymbol{\zeta}|^2](\tau_{H1} - \tau_{w1}) + \mathcal{K}[\phi_{K1}] \\ & - 2\tilde{\mathcal{K}}[\zeta_n]u_{iH1}n_i + \tilde{\mathcal{K}}[\zeta_n A]\partial_j \tau_{H0}n_j + \tilde{\mathcal{K}}[(\delta_{ij} - n_i n_j)\zeta_i A]\partial_j \tau_{H0} \\ & + \frac{1}{2}\tilde{\mathcal{K}}[\zeta_i \zeta_j B]\overline{\partial_i u_{jH0}}, \quad \zeta_n > 0, \quad \eta = 0,\end{aligned}$$

where we have used that $\mathcal{K}[1] = 1$. The right-hand side can be reduced by the following three steps:

1. Because the body surface is not permeable, $(u_{iH1} + u_{iK1})n_i = 0$ at $\eta = 0$, but $u_{iK1}n_i \equiv 0$ is deduced by the integration of Eq. (2.20a) with the condition (2.20c) for $m = 1$ (the mass conservation). Thus, $u_{iH1}n_i = 0$ at $\eta = 0$, and the fourth term vanishes.

2. By the axial symmetry of $\mathcal{K}$ and $\mathcal{L}$ in Sec. 2.11, the last term can be split as

$$\begin{aligned}\frac{1}{2}\tilde{\mathcal{K}}[\zeta_i \zeta_j B]\overline{\partial_i u_{jH0}} = & (\delta_{ik} - n_i n_k)\zeta_k \tilde{\mathcal{K}}_1[\zeta_n B]\overline{\partial_i u_{jH0}n_j} + \frac{1}{2}\tilde{\mathcal{K}}[\zeta_n^2 B]\overline{\partial_i u_{jH0}n_i n_j} \\ & + \frac{1}{2}\left((\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l)\zeta_k \zeta_l \tilde{\mathcal{K}}_2[B] + (\delta_{ij} - n_i n_j)\tilde{\mathcal{K}}_3[B]\right)\overline{\partial_i u_{jH0}}.\end{aligned}$$

3. Because it is a solid body surface, the boundary moves under the constraint of rigid motion. Therefore, from Eq. (2.19) and the first equation of (2.11), we have

$$\overline{\partial_i u_{jH0}}(\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) = 0, \quad \overline{\partial_i u_{jH0}n_i n_j} = 0,$$

on the boundary ($\eta = 0$), and thus only the first term in the above split survives, namely

$$\frac{1}{2}\tilde{\mathcal{K}}[\zeta_i \zeta_j B]\overline{\partial_i u_{jH0}} = (\delta_{ik} - n_i n_k)\zeta_k \tilde{\mathcal{K}}_1[\zeta_n B]\overline{\partial_i u_{jH0}n_j}.$$

Therefore, the boundary condition (2.20b) ($m = 1$) is finally reduced to

$$\begin{aligned}\phi_{K1} = & \mathcal{K}[\phi_{K1}] - 2\zeta_j \tilde{\mathcal{K}}_1[1](u_{iH1} - u_{iw1})(\delta_{ij} - n_i n_j) - \tilde{\mathcal{K}}[|\boldsymbol{\zeta}|^2](\tau_{H1} - \tau_{w1}) \\ & + \tilde{\mathcal{K}}[\zeta_n A]\partial_i \tau_{H0}n_i + \zeta_i \tilde{\mathcal{K}}_1[A]\partial_j \tau_{H0}(\delta_{ij} - n_i n_j) \\ & + \zeta_i \tilde{\mathcal{K}}_1[\zeta_n B]\overline{\partial_j u_{kH0}n_k}(\delta_{ij} - n_i n_j), \quad \zeta_n > 0, \quad \eta = 0,\end{aligned}\tag{2.21}$$

and thus ϕ_{K1} is expressed in terms of three fundamental solutions as

$$\begin{aligned}\phi_{K1} = & \phi_1^{(0)}(\eta, \zeta_n, |\boldsymbol{\zeta}|)\partial_i \tau_{H0}n_i + \zeta_i \phi_1^{(1)}(\eta, \zeta_n, |\boldsymbol{\zeta}|)\overline{\partial_j u_{kH0}n_k}(\delta_{ij} - n_i n_j) \\ & + \zeta_i \phi_2^{(1)}(\eta, \zeta_n, |\boldsymbol{\zeta}|)\partial_j \tau_{H0}(\delta_{ij} - n_i n_j).\end{aligned}\tag{2.22}$$

Here $\phi_1^{(0)}$, $\phi_1^{(1)}$, and $\phi_2^{(1)}$ are, respectively, the solutions of the following boundary-value problems:

$$\zeta_n \partial_\eta \phi_1^{(0)} = \mathcal{L}[\phi_1^{(0)}], \quad (2.23a)$$

$$\phi_1^{(0)} = \mathcal{K}[\phi_1^{(0)}] - \widetilde{\mathcal{K}}[|\zeta|^2] c_1^{(0)} + \widetilde{\mathcal{K}}[\zeta_n A], \quad \zeta_n > 0, \quad \eta = 0, \quad (2.23b)$$

$$\phi_1^{(0)} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (2.23c)$$

$$\zeta_n \partial_\eta \phi_1^{(1)} = \mathcal{L}_1[\phi_1^{(1)}], \quad (2.24a)$$

$$\phi_1^{(1)} = \mathcal{K}_1[\phi_1^{(1)}] - 2\widetilde{\mathcal{K}}_1[1]b_1^{(1)} + \widetilde{\mathcal{K}}_1[\zeta_n B], \quad \zeta_n > 0, \quad \eta = 0, \quad (2.24b)$$

$$\phi_1^{(1)} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (2.24c)$$

$$\zeta_n \partial_\eta \phi_2^{(1)} = \mathcal{L}_1[\phi_2^{(1)}], \quad (2.25a)$$

$$\phi_2^{(1)} = \mathcal{K}_1[\phi_2^{(1)}] - 2\widetilde{\mathcal{K}}_1[1]b_2^{(1)} + \widetilde{\mathcal{K}}_1[A], \quad \zeta_n > 0, \quad \eta = 0, \quad (2.25b)$$

$$\phi_2^{(1)} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (2.25c)$$

It is known [18,19,20,11] that for each of the problems (2.23)–(2.25) there is a unique solution, and the constants $c_1^{(0)}$, $b_1^{(1)}$, and $b_2^{(1)}$ are determined together with the solutions $\phi_1^{(0)}$, $\phi_1^{(1)}$, and $\phi_2^{(1)}$, respectively.

By the comparison of Eq. (2.21) with Eqs. (2.23b), (2.24b), and (2.25b), the boundary condition for the flow velocity and temperature at $O(\varepsilon)$ are obtained as

$$u_{iH1}n_i = 0, \quad (u_{iH1} - u_{iw1})t_i = b_2^{(1)}\partial_i\tau_{H0}t_i + b_1^{(1)}\overline{\partial_i u_{jH0}n_i t_j}, \quad (2.26)$$

$$\tau_{H1} - \tau_{w1} = c_1^{(0)}\partial_i\tau_{H0}n_i, \quad (2.27)$$

while the Knudsen-layer correction at $O(\varepsilon)$ are obtained from Eq. (2.22) as

$$\omega_{K1} = \Omega_1^{(0)}(\eta)\partial_i\tau_{H0}n_i, \quad \tau_{K1} = \Theta_1^{(0)}(\eta)\partial_i\tau_{H0}n_i, \quad (2.28)$$

$$u_{iK1}n_i = 0, \quad u_{iK1}t_i = Y_1^{(1)}(\eta)\overline{\partial_i u_{jH0}n_i t_j} + Y_2^{(1)}(\eta)\partial_i\tau_{H0}t_i, \quad (2.29)$$

where $\mathbf{t}$ is an arbitrary tangential unit vector to the boundary and

$$\Omega_j^{(i)}(\eta) = \langle \phi_j^{(i)} \rangle, \quad \Theta_j^{(i)}(\eta) = \frac{2}{3} \langle (|\zeta|^2 - \frac{3}{2}) \phi_j^{(i)} \rangle, \quad Y_j^{(i)}(\eta) = \frac{1}{2} \langle (|\zeta|^2 - \zeta_n^2) \phi_j^{(i)} \rangle.$$

As to the Knudsen-layer corrections to the stress tensor and heat-flow vector, see Sec. 2.13. There are many reductions for these quantities, because the following relations (the conservation laws) hold:

$$\langle \zeta_n \phi_1^{(0)} \rangle = 0, \quad \langle \zeta_n^2 \phi_1^{(0)} \rangle = 0, \quad \langle \zeta_n (|\zeta|^2 - \frac{5}{2}) \phi_1^{(0)} \rangle = 0, \quad (2.30a)$$

$$\langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_1^{(1)} \rangle = 0, \quad \langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_2^{(1)} \rangle = 0. \quad (2.30b)$$

The fundamental solutions $\phi_1^{(0)}$, $\phi_1^{(1)}$, and $\phi_2^{(1)}$ are respectively called the solution of the temperature-jump, shear-slip (or viscous-slip), and thermal-slip (or thermal creep) problems in the literature. The above analysis shows that up to $O(\varepsilon)$ the Knudsen-layer correction and the slip and jump conditions are the same as those in the steady case, as discussed in Refs. [11] and [12]. However, this is not the case for higher orders, which will be clarified in Sec. 2.4.2.

2.4.2 The second order

We now consider the second order. The analysis requires long manipulations, and we show below only its outline. Key formulas used in the manipulations are summarized in Sec. 2.14.

We first substitute Eq. (2.22) into the definition of I_{K2} and rewrite it as

$$\begin{aligned} I_{K2} = \zeta_i \mathcal{D}_i [\phi_{K1}] = & \zeta_i \mathcal{D}_i \left[\phi_1^{(0)} \partial_j \tau_{H0} n_j \right] + \zeta_i \mathcal{D}_i \left[(\delta_{jk} - n_j n_k) \zeta_j \phi_2^{(1)} \partial_k \tau_{H0} \right] \\ & + \zeta_i \mathcal{D}_i \left[(\delta_{jk} - n_j n_k) \zeta_j \phi_1^{(1)} \overline{\partial_k u_{IH0} n_l} \right]. \end{aligned}$$

With the aid of the formulas in Sec. 2.12, the above is eventually reduced to

$$\begin{aligned} I_{K2} = & \frac{1}{2} (|\zeta|^2 - \zeta_n^2) \phi_2^{(1)} \partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) - \frac{1}{4} (|\zeta|^2 - \zeta_n^2) \phi_1^{(1)} \partial_i \overline{\partial_j u_{kH0} n_i n_j n_k} \\ & + (|\zeta|^2 - \zeta_n^2) (\partial_{\zeta_n} \phi_1^{(0)} - \phi_2^{(1)}) \bar{\kappa} \partial_i \tau_{H0} n_i + \zeta_i \phi_1^{(0)} \partial_j \partial_k \tau_{H0} n_k (\delta_{ij} - n_i n_j) \\ & + \frac{1}{2} \zeta_i \partial_{\zeta_n} \left((|\zeta|^2 - \zeta_n^2) \phi_1^{(1)} \right) \kappa_{ij} \overline{\partial_j u_{kH0} n_k} \\ & + \zeta_i \left(\frac{1}{2} \partial_{\zeta_n} \left((|\zeta|^2 - \zeta_n^2) \phi_2^{(1)} \right) + \phi_1^{(0)} \right) \kappa_{ij} \partial_j \tau_{H0} \\ & + \frac{1}{2} \zeta_i (|\zeta|^2 - \zeta_n^2) \partial_{\zeta_n} \phi_1^{(1)} \bar{\kappa} \overline{\partial_j u_{kH0} n_k} (\delta_{ij} - n_i n_j) \\ & + \frac{1}{2} \zeta_i (|\zeta|^2 - \zeta_n^2) \partial_{\zeta_n} \phi_2^{(1)} \bar{\kappa} \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) \\ & + \left(\zeta_i \zeta_j - \frac{1}{2} (|\zeta|^2 - \zeta_n^2) \delta_{ij} \right) \left((\partial_{\zeta_n} \phi_1^{(0)} - \phi_2^{(1)}) \kappa_{ij} \partial_k \tau_{H0} n_k \right. \\ & \quad \left. + [\phi_2^{(1)} \partial_k \partial_l \tau_{H0} + \phi_1^{(1)} \partial_k \overline{\partial_l u_{pH0} n_p}] (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) \right) \end{aligned}$$

$$\begin{aligned}
& + \left(\zeta_i \zeta_j \zeta_k - \frac{1}{4} (\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) (|\zeta|^2 - \zeta_n^2) \right) \\
& \times \kappa_{ij} (\partial_{\zeta_n} \phi_2^{(1)} \partial_l \tau_{H0} + \partial_{\zeta_n} \phi_1^{(1)} \overline{\partial_l u_{pH0} n_p}) (\delta_{kl} - n_k n_l),
\end{aligned} \tag{2.31}$$

where κ_{ij} and $\bar{\kappa}$ are the curvature matrix and mean curvature of the boundary (see Sec. 2.12).

We next substitute Eq. (2.15) into Eq. (2.20b) for $m = 2$ to reduce the boundary condition at $\eta = 0$. To this end, we use the axial symmetry of $\mathcal{K}$ and $\mathcal{L}$ (see Sec. 2.11) and the formulas in Sec. 2.14. After a long manipulation, the boundary condition at $\eta = 0$ ($\zeta_n > 0$) is reduced to

$$\begin{aligned}
\phi_{K2} = & -2\zeta_i \widetilde{\mathcal{K}}_1[1] (u_{jH2} - u_{jw2}) (\delta_{ij} - n_i n_j) - \widetilde{\mathcal{K}}[|\zeta|^2] (\tau_{H2} - \tau_{w2}) + \mathcal{K}[\phi_{K2}] \\
& - 2\widetilde{\mathcal{K}}[\zeta_n] u_{iH2} n_i + \widetilde{\mathcal{K}}[\zeta_n A] \partial_i \tau_{H1} n_i - \frac{1}{3} \widetilde{\mathcal{K}}[(|\zeta|^2 - 3\zeta_n^2) B] \partial_i u_{iH1} \\
& - \frac{1}{4} \widetilde{\mathcal{K}}[(|\zeta|^2 - 3\zeta_n^2) b_1^{(1)} B + 2\zeta_n D_1 - 2\zeta_n (|\zeta|^2 - 2\zeta_n^2) D_2] \partial_i \overline{\partial_j u_{kH0} n_i n_j n_k} \\
& + \frac{1}{2} \widetilde{\mathcal{K}}[(|\zeta|^2 - 3\zeta_n^2) (b_2^{(1)} B + F)] \partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) \\
& - b_2^{(1)} \widetilde{\mathcal{K}}[(|\zeta|^2 - 3\zeta_n^2) B] \bar{\kappa} \partial_i \tau_{H0} n_i - \widetilde{\mathcal{K}}[\frac{1}{3} (|\zeta|^2 - 3\zeta_n^2) F - F_d] \Delta \tau_{H0} \\
& + \zeta_i \widetilde{\mathcal{K}}_1[\zeta_n B] \overline{\partial_j u_{kH1} n_k} (\delta_{ij} - n_i n_j) + \zeta_i \widetilde{\mathcal{K}}_1[A] \partial_j \tau_{H1} (\delta_{ij} - n_i n_j) \\
& - \zeta_i \widetilde{\mathcal{K}}_1[D_1 + \zeta_n^2 D_2] \partial_j \overline{\partial_k u_{lH0}} (\delta_{ik} - n_i n_k) n_j n_l \\
& + 2\zeta_i \widetilde{\mathcal{K}}_1[\zeta_n F] \partial_j \partial_k \tau_{H0} n_k (\delta_{ij} - n_i n_j) \\
& - \zeta_i \widetilde{\mathcal{K}}_1[D_1 + \frac{1}{2} (|\zeta|^2 - 3\zeta_n^2) D_2] \kappa_{ij} \overline{\partial_j u_{kH0} n_k} \\
& - \zeta_i \widetilde{\mathcal{K}}_1[2D_1 + \frac{1}{2} (|\zeta|^2 - \zeta_n^2) D_2] \bar{\kappa} \overline{\partial_j u_{kH0} n_k} (\delta_{ij} - n_i n_j) \\
& + \left(\zeta_i \zeta_j - \frac{1}{2} (|\zeta|^2 - \zeta_n^2) \delta_{ij} \right) (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) \\
& \times \left(\widetilde{\mathcal{K}}_2[b_2^{(1)} B + F] \partial_k \partial_l \tau_{H0} - \frac{1}{2} \widetilde{\mathcal{K}}_2[\zeta_n D_2] \partial_q \overline{\partial_k u_{lH0} n_q} \right. \\
& \quad \left. + \widetilde{\mathcal{K}}_2[b_1^{(1)} B - \zeta_n D_2] \partial_k \overline{\partial_q u_{lH0} n_q} \right) \\
& - b_2^{(1)} \left(\zeta_i \zeta_j - \frac{1}{2} (|\zeta|^2 - \zeta_n^2) \delta_{ij} \right) \widetilde{\mathcal{K}}_2[B] \kappa_{ij} \partial_k \tau_{H0} n_k \\
& - \left(\zeta_i \zeta_j \zeta_k - \frac{1}{4} (\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) (|\zeta|^2 - \zeta_n^2) \right) \\
& \times \widetilde{\mathcal{K}}_4[D_2] \kappa_{ij} \overline{\partial_l u_{pH0} n_l} (\delta_{kp} - n_k n_p).
\end{aligned} \tag{2.32}$$

It should be noted that, with the aid of Eq. (2.31), $u_{iH2} n_i$ on the boundary ($\eta = 0$) is obtained by the integration of $\langle \text{Eq. (2.20a)} \rangle$ from $\eta = 0$ to ∞ under the condition (2.20c)

for $m = 2$ (the mass conservation) as

$$u_{iH2}n_i = \int_0^\infty Y_2^{(1)}(z)dz (2\bar{\kappa}\partial_i\tau_{H0}n_i - \partial_i\partial_j\tau_{H0}(\delta_{ij} - n_in_j)) \\ + \frac{1}{2} \int_0^\infty Y_1^{(1)}(z)dz \partial_i\overline{\partial_j u_{kH0}n_in_jn_k}.$$

By substituting the above into Eq. (2.32) and rearranging the terms, it is seen, together with the form of Eq. (2.31), that we can express the second-order Knudsen-layer correction as

$$\begin{aligned} \phi_{K2} = & \phi_1^{(0)}\partial_i\tau_{H1}n_i + \phi_2^{(0)}\partial_i\partial_j\tau_{H0}(\delta_{ij} - n_in_j) + \phi_3^{(0)}\partial_i\overline{\partial_j u_{kH0}n_in_jn_k} \\ & + \phi_4^{(0)}\bar{\kappa}\partial_i\tau_{H0}n_i + \phi_5^{(0)}\partial_i u_{iH1} + \phi_6^{(0)}\Delta\tau_{H0} \\ & + \zeta_i(\delta_{ij} - n_in_j)(\phi_1^{(1)}\overline{\partial_j u_{kH1}n_k} + \phi_2^{(1)}\partial_j\tau_{H1} + \phi_3^{(1)}\partial_j\partial_k\tau_{H0}n_k \\ & + \phi_4^{(1)}\partial_l\overline{\partial_j u_{kH0}n_kn_l} + \phi_5^{(1)}\bar{\kappa}\overline{\partial_j u_{kH0}n_k} + \phi_8^{(1)}\bar{\kappa}\partial_j\tau_{H0}) \\ & + \zeta_i\kappa_{ij}(\phi_6^{(1)}\overline{\partial_j u_{kH0}n_k} + \phi_7^{(1)}\partial_j\tau_{H0}) \\ & + (\zeta_i\zeta_j - \frac{1}{2}(|\zeta|^2 - \zeta_n^2)\delta_{ij})\phi_1^{(2)}\kappa_{ij}\partial_k\tau_{H0}n_k \\ & + (\zeta_i\zeta_j - \frac{1}{2}(|\zeta|^2 - \zeta_n^2)\delta_{ij})(\delta_{jl} - n_jn_l)(\delta_{ik} - n_in_k) \\ & \quad \times (\phi_2^{(2)}\partial_k\partial_l\tau_{H0} + \phi_3^{(2)}\partial_k\overline{\partial_l u_{pH0}n_p} + \phi_4^{(2)}\partial_p\overline{\partial_k u_{lH0}n_p}) \\ & + (\zeta_i\zeta_j\zeta_k - \frac{1}{4}(\zeta_i\delta_{jk} + \zeta_j\delta_{ki} + \zeta_k\delta_{ij})(|\zeta|^2 - \zeta_n^2)) \\ & \quad \times \kappa_{ij}(\delta_{kl} - n_kn_l)(\phi_1^{(3)}\partial_l\tau_{H0} + \phi_2^{(3)}\overline{\partial_l u_{pH0}n_p}), \end{aligned} \quad (2.33)$$

where $\phi_j^{(i)}$ ($i = 0, 1, 2, 3$; $j = 1, 2, \dots$) on the right-hand side are fundamental solutions that depend only on η , ζ_n , and $|\zeta|$ (see Sec. 2.15). In view of their boundary-value problems in Sec. 2.15, the second-order slip and jump conditions are obtained as

$$\begin{aligned} \tau_{H2} = & \tau_{w2} + c_1^{(0)}\partial_i\tau_{H1}n_i + c_2^{(0)}\partial_i\partial_j\tau_{H0}(\delta_{ij} - n_in_j) + c_3^{(0)}\partial_i\overline{\partial_j u_{kH0}n_in_jn_k} \\ & + c_4^{(0)}\bar{\kappa}\partial_i\tau_{H0}n_i + c_5^{(0)}\partial_i u_{iH1} + c_6^{(0)}\Delta\tau_{H0}, \end{aligned} \quad (2.34)$$

$$\begin{aligned} u_{iH2}n_i = & \int_0^\infty Y_2^{(1)}(z)dz (2\bar{\kappa}\partial_i\tau_{H0}n_i - \partial_i\partial_j\tau_{H0}(\delta_{ij} - n_in_j)) \\ & + \frac{1}{2} \int_0^\infty Y_1^{(1)}(z)dz \partial_i\overline{\partial_j u_{kH0}n_in_jn_k}, \end{aligned} \quad (2.35)$$

$$\begin{aligned} u_{iH2}t_i = & u_{iw2}t_i + b_1^{(1)}\overline{\partial_i u_{jH1}n_it_j} + b_2^{(1)}\partial_i\tau_{H1}t_i + b_3^{(1)}\partial_i\partial_j\tau_{H0}n_it_j \\ & + b_4^{(1)}\partial_i\overline{\partial_j u_{kH0}n_in_jt_k} + b_5^{(1)}\bar{\kappa}\overline{\partial_i u_{jH0}n_it_j} \\ & + b_6^{(1)}\kappa_{ij}\overline{\partial_j u_{kH0}n_kt_i} + b_7^{(1)}\kappa_{ij}\partial_i\tau_{H0}t_j + b_8^{(1)}\bar{\kappa}\partial_i\tau_{H0}t_i, \end{aligned} \quad (2.36)$$

while the Knudsen-layer corrections are obtained as

$$\begin{aligned}\omega_{K2} = & \Omega_1^{(0)}(\eta) \partial_i \tau_{H1} n_i + \Omega_2^{(0)}(\eta) \partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) + \Omega_3^{(0)}(\eta) \partial_i \overline{\partial_j u_{kH0} n_i n_j n_k} \\ & + \Omega_4^{(0)}(\eta) \bar{\kappa} \partial_i \tau_{H0} n_i + \Omega_5^{(0)}(\eta) \partial_i u_{iH1} + \Omega_6^{(0)}(\eta) \Delta \tau_{H0},\end{aligned}\quad (2.37)$$

$$\begin{aligned}\tau_{K2} = & \Theta_1^{(0)}(\eta) \partial_i \tau_{H1} n_i + \Theta_2^{(0)}(\eta) \partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) + \Theta_3^{(0)}(\eta) \partial_i \overline{\partial_j u_{kH0} n_i n_j n_k} \\ & + \Theta_4^{(0)}(\eta) \bar{\kappa} \partial_i \tau_{H0} n_i + \Theta_5^{(0)}(\eta) \partial_i u_{iH1} + \Theta_6^{(0)}(\eta) \Delta \tau_{H0},\end{aligned}\quad (2.38)$$

$$\begin{aligned}u_{iK2} n_i = & - \int_{\eta}^{\infty} Y_2^{(1)}(z) dz \left(2\bar{\kappa} \partial_i \tau_{H0} n_i - \partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) \right) \\ & - \frac{1}{2} \int_{\eta}^{\infty} Y_1^{(1)}(z) dz \partial_i \overline{\partial_j u_{kH0} n_i n_j n_k},\end{aligned}\quad (2.39)$$

$$\begin{aligned}u_{iK2} t_i = & Y_1^{(1)}(\eta) \overline{\partial_i u_{jH1} n_i t_j} + Y_2^{(1)}(\eta) \partial_i \tau_{H1} t_i + Y_3^{(1)}(\eta) \partial_i \partial_j \tau_{H0} n_i t_j \\ & + Y_4^{(1)}(\eta) \partial_i \overline{\partial_j u_{kH0} n_i n_j t_k} + Y_5^{(1)}(\eta) \bar{\kappa} \overline{\partial_i u_{jH0} n_i t_j} \\ & + Y_6^{(1)}(\eta) \kappa_{ij} \overline{\partial_j u_{kH0} n_k t_i} + Y_7^{(1)}(\eta) \kappa_{ij} \partial_i \tau_{H0} t_j + Y_8^{(1)}(\eta) \bar{\kappa} \partial_i \tau_{H0} t_i.\end{aligned}\quad (2.40)$$

Here some reductions have been made in deriving the expressions for $u_{iK2} n_i$ by using the mass conservation law [see Eq. (2.30) and Sec. 2.15]. This completes the analysis of the Knudsen-layer correction and the slip and jump conditions for the set of fluid-dynamic-type equations, up to $O(\varepsilon^2)$.

2.5 Summary of the asymptotic analysis

In summary, in Secs. 2.3 and 2.4, we have derived a set of fluid-dynamic-type equations and its appropriate boundary conditions that describe the overall behavior of the gas up to $O(\varepsilon^2)$. We repeat the main results below because of their importance.

Leading order

$$\partial_i P_{H0} = 0, \quad (2.41)$$

$$\partial_i u_{iH0} = 0, \quad (2.42a)$$

$$\partial_t u_{iH0} + \frac{1}{2} \partial_i P_{H1} - \frac{1}{2} \gamma_1 \Delta u_{iH0} = 0, \quad (2.42b)$$

$$\partial_t \tau_{H0} - \frac{2}{5} \partial_t P_{H0} - \frac{1}{2} \gamma_2 \Delta \tau_{H0} = 0. \quad (2.42c)$$

$$\text{b.c.} \quad u_{i\text{H}0} = u_{i\text{w}0}, \quad (2.43\text{a})$$

$$\tau_{\text{H}0} = \tau_{\text{w}0}. \quad (2.43\text{b})$$

First order

$$\partial_t \omega_{\text{H}0} + \partial_i u_{i\text{H}1} = 0, \quad (2.44\text{a})$$

$$\partial_t u_{i\text{H}1} + \frac{1}{2} \partial_i P_{\text{H}2}^* - \frac{1}{2} \gamma_1 \Delta u_{i\text{H}1} = 0, \quad (2.44\text{b})$$

$$\partial_t \tau_{\text{H}1} - \frac{2}{5} \partial_t P_{\text{H}1} - \frac{1}{2} \gamma_2 \Delta \tau_{\text{H}1} = 0, \quad (2.44\text{c})$$

where

$$P_{\text{H}2}^* = P_{\text{H}2} - \frac{1}{6} (\gamma_2 \gamma_1 - 4\gamma_3) \Delta \tau_{\text{H}0}. \quad (2.44\text{d})$$

$$\text{b.c.} \quad u_{i\text{H}1} n_i = 0, \quad (2.45\text{a})$$

$$u_{i\text{H}1} t_i = u_{i\text{w}1} t_i + b_2^{(1)} \partial_i \tau_{\text{H}0} t_i + b_1^{(1)} \overline{\partial_i u_{j\text{H}0} n_i t_j}, \quad (2.45\text{b})$$

$$\tau_{\text{H}1} = \tau_{\text{w}1} + c_1^{(0)} \partial_i \tau_{\text{H}0} n_i. \quad (2.45\text{c})$$

Second order

$$\partial_t \omega_{\text{H}1} + \partial_i u_{i\text{H}2} = 0, \quad (2.46\text{a})$$

$$\partial_t u_{i\text{H}2} + \frac{1}{2} \partial_i P_{\text{H}3}^* - \frac{1}{2} \gamma_1 \Delta u_{i\text{H}2} + \frac{1}{4} (\gamma_1 \gamma_{10} - 2\gamma_6) \Delta^2 u_{i\text{H}0} = 0, \quad (2.46\text{b})$$

$$\partial_t \tau_{\text{H}2} - \frac{2}{5} \partial_t P_{\text{H}2} - \frac{1}{2} \gamma_2 \Delta \tau_{\text{H}2} + \frac{1}{10} (\gamma_2 \gamma_3 - \frac{13}{2} \gamma_{11}) \Delta^2 \tau_{\text{H}0} = 0, \quad (2.46\text{c})$$

where

$$P_{\text{H}3}^* = P_{\text{H}3} - \frac{1}{6} (\gamma_2 \gamma_1 - 4\gamma_3) \Delta \tau_{\text{H}1} + \frac{1}{5} \gamma_1 \partial_t P_{\text{H}1}. \quad (2.46\text{d})$$

$$\begin{aligned} \text{b.c.} \quad u_{i\text{H}2} n_i &= \int_0^\infty Y_2^{(1)}(z) dz (2\bar{\kappa} \partial_i \tau_{\text{H}0} n_i - \Delta \tau_{\text{H}0} + \partial_i \partial_j \tau_{\text{H}0} n_i n_j) \\ &\quad + \frac{1}{2} \int_0^\infty Y_1^{(1)}(z) dz \partial_i \overline{\partial_j u_{k\text{H}0} n_i n_j n_k}, \end{aligned} \quad (2.47\text{a})$$

$$\begin{aligned} u_{i\text{H}2} t_i &= u_{i\text{w}2} t_i + b_1^{(1)} \overline{\partial_i u_{j\text{H}1} n_i t_j} + b_2^{(1)} \partial_i \tau_{\text{H}1} t_i + b_3^{(1)} \partial_i \partial_j \tau_{\text{H}0} n_i t_j \\ &\quad + b_4^{(1)} \partial_i \overline{\partial_j u_{k\text{H}0} n_i n_j t_k} + b_5^{(1)} \bar{\kappa} \overline{\partial_j u_{k\text{H}0} n_j t_k} \end{aligned}$$

Table 2.1: Correspondence of the notation to Refs. [11] and [12].

present notation	notation in Refs. [11] and [12]	present notation	notation in Refs. [11] and [12]
$(c_1^{(0)}, \Omega_1^{(0)}, \Theta_1^{(0)})$	$(d_1, \Omega_1, \Theta_1)$	$(b_3^{(1)}, Y_3^{(1)})$	$d_1(K_1, \frac{1}{2}Y_1) - (a_4, Y_{a4})$
$(c_2^{(0)}, \Omega_2^{(0)}, \Theta_2^{(0)})$	$-(d_3, \Omega_3, \Theta_3)$	$(b_4^{(1)}, Y_4^{(1)})$	$-(a_1, Y_{a1})$
$(c_3^{(0)}, \Omega_3^{(0)}, \Theta_3^{(0)})$	$(d_4, \Omega_4, \Theta_4)$	$(b_5^{(1)}, Y_5^{(1)})$	$-(a_2, Y_{a2})$
$(c_4^{(0)}, \Omega_4^{(0)}, \Theta_4^{(0)})$	$(d_5, \Omega_5, \Theta_5)$	$(b_6^{(1)}, Y_6^{(1)})$	$-(a_3, Y_{a3})$
$(b_1^{(1)}, Y_1^{(1)}, H_1^{(1)})$	$(-k_0, -Y_0, H_A)$	$(b_7^{(1)}, Y_7^{(1)})$	$d_1(K_1, \frac{1}{2}Y_1) - (a_6, Y_{a6})$
$(b_2^{(1)}, Y_2^{(1)}, H_2^{(1)})$	$(-K_1, -\frac{1}{2}Y_1, H_B)$	$(b_8^{(1)}, Y_8^{(1)})$	$-(a_5, Y_{a5})$

$$+ b_6^{(1)} \kappa_{ij} \overline{\partial_j u_{kH0}} n_k t_i + b_7^{(1)} \kappa_{ij} \partial_i \tau_{H0} t_j + b_8^{(1)} \bar{\kappa} \partial_i \tau_{H0} t_i, \quad (2.47b)$$

$$\begin{aligned} \tau_{H2} = \tau_{w2} + c_1^{(0)} \partial_i \tau_{H1} n_i - c_2^{(0)} \partial_i \partial_j \tau_{H0} n_i n_j + c_3^{(0)} \partial_i \overline{\partial_j u_{kH0}} n_i n_j n_k \\ + c_4^{(0)} \bar{\kappa} \partial_i \tau_{H0} n_i + (c_6^{(0)} + c_2^{(0)}) \Delta \tau_{H0} + c_5^{(0)} \partial_i u_{iH1}. \end{aligned} \quad (2.47c)$$

It should be noted that the above system is closed because it is supplemented by the equation of state:

$$P_{Hm} = \omega_{Hm} + \tau_{Hm} \quad (m = 0, 1, 2, \dots). \quad (2.48)$$

In Sec. 2.3, we also obtained the expressions of the stress tensor P_{ijHm} and heat-flow vector Q_{iHm} for $m \leq 3$ and the velocity distribution function ϕ_{Hm} for $m \leq 2$ of the Hilbert solution. Moreover, in Sec. 2.4, we obtained the expressions of the Knudsen-layer corrections not only to the density ω_{Km} , flow velocity u_{iKm} , and temperature τ_{Km} but also to the stress tensor P_{ijKm} , heat-flow vector Q_{iKm} (see Sec. 2.13), and velocity distribution function ϕ_{Km} for $m \leq 2$. Their specific forms are not repeated here.

In Table 2.1, we show the correspondence of our notation to that in Refs. [11] and [12], from which the data of slip and jump coefficients and the table of Knudsen-layer functions for the BGK model (also a partial data for a hard-sphere gas) are available. Note that the functions corresponding to $H_j^{(1)}$ ($j = 3, 4, \dots, 8$) and $S_j^{(2)}$ ($j = 1, 2, 3, 4$) listed in Sec. 2.13, as well as the data of $c_6^{(0)}$, $\Omega_6^{(0)}$, and $\Theta_6^{(0)}$, are not found in Refs. [11] and [12] (see Sec. 2.6.1). In the case of the diffuse reflection condition, $(c_5^{(0)}, \Omega_5^{(0)}, \Theta_5^{(0)})$ is equivalent to $(4/3)(d_6, \Omega_6, \Theta_6)$ in Refs. [11] and [12] for the complete condensation condition.

The constants $\gamma_1 \sim \gamma_{11}$ are all unity in the case of the BGK model, while their values for a hard-sphere gas are shown in Sec. 2.10. The values of $b_1^{(1)} \sim b_8^{(1)}$, $c_1^{(0)} \sim c_6^{(0)}$, $\int_0^\infty Y_1^{(1)}(z) dz$, and $\int_0^\infty Y_2^{(1)}(z) dz$ for a hard-sphere gas will be provided in Table 2.3 in Sec. 2.7.

2.6 Discussions

2.6.1 New features

In the fluid-dynamic-type equations, the effect of compressibility manifests itself in the continuity [Eqs. (2.44a) and (2.46a)] and energy [Eqs. (2.42c), (2.44c), and (2.46c)] equations as in the case of the steady weakly nonlinear theory. This resemblance has already been discussed by Sone [12]. In the momentum equation (2.44b) at the first order, the structure of the equation is unchanged from that of the leading order, thanks to using the modified pressure. However, in the momentum and energy equations, (2.46b) and (2.46c), at the second order, new terms appear as the effect of unsteadiness, namely the double Laplacian terms of flow velocity and temperature.

In the Knudsen-layer corrections and the slip and jump conditions, the difference from the steady case appears at the second order as the terms of $\partial_k u_{kH1}$ and $\Delta\tau_{H0}$ in the density, temperature (ω_{K2} , τ_{K2} , and τ_{H2}), and diagonal components of the stress tensor $P_{ijK2}t_it_j$ (Sec. 2.13). As to the flow velocity, there is no change from the steady case, provided that $\partial_i\partial_j\tau_{H0}n_in_j$ in the steady theory is reinterpreted as $\partial_i\partial_j\tau_{H0}n_in_j - \Delta\tau_{H0}$. The form of the heat-flow vector also remains unchanged by the same reinterpretation. It should be noted that when the physical system reaches the steady state, both $\partial_k u_{kH1}$ and $\Delta\tau_{H0}$ vanish, and we recover the results of the existing theory for the steady case [11,12].

The appearance of $\partial_k u_{kH1}$ term is a new feature of time-dependent case. However, under the diffuse reflection boundary condition, the corresponding Knudsen-layer problem for $\phi_5^{(0)}$ is equivalent to the steady weak evaporation/condensation problem in a half space under the complete condensation boundary condition. In this sense, the jump coefficient and Knudsen-layer correction related to $\phi_5^{(0)}$ are classical. On the other hand, the appearance of $\Delta\tau_{H0}$ term has an impact that the related Knudsen-layer problem for $\phi_6^{(0)}$ has not occurred in the theory for steady flows. It is this problem that causes the second-order temperature jump associated with the parabolic temperature profile in Refs. [13] and [14].

2.6.2 Comments on the initial data

We have discussed a time evolution from the reference equilibrium state by a slow change of the boundary data. This condition can be relaxed to a time evolution from a solution of

the steady linearized Boltzmann equation for non-zero boundary data.

On the other hand, if we consider a time evolution from the reference state by a sudden change of the boundary data (followed by a slow change), the situation becomes a little more complicated. In the case, the system experiences more rapid change than the present time scale, and we have to consider first the initial-layer problem and then the acoustic scale of change. The time evolution in the scale of the present paper comes after these two initial stages. To be more specific, for $0 < t \lesssim \varepsilon^2$, the gas remains the initial reference equilibrium state except for the thin layer adjacent to the boundary. In this layer, the state of the gas evolves to form the Knudsen layer and pulsive disturbance. The dynamics in this first stage is purely kinetic. Then, in the next stage, i.e., $\varepsilon^2 \lesssim t \lesssim \varepsilon$, the clear separation into the Knudsen layer, fluid-dynamic boundary layer with the thickness of $\sqrt{\varepsilon}$, and propagating pulse occurs. The dynamics of this second stage is acoustic as a whole, and the pulse travels through the domain without perturbing the initial uniform state after its passage. In this way, the initial state for the evolution after $t \sim \varepsilon$ is prepared. Then, for $t \gtrsim \varepsilon$, the system obeys the set of equations summarized in Sec. 2.5. This last stage may be considered as the process that the fluid-dynamic boundary layer which has emerged in the second stage ($\varepsilon^2 \lesssim t \lesssim \varepsilon$) extends over the entire domain.

2.6.3 How is the gas behavior determined?

In this small section, we discuss the solution process of the system summarized in Sec. 2.5. The case of open domain and that of closed domain will be treated separately. For simplicity, we do not consider the fluid–body coupled problem. Hence the boundary data are given and are not expanded in the power series of ε .

2.6.3.1 Open domain

In this case, the pressure in a far field is given. Let us denote $P(t, \mathbf{x})$ in the far field by $P_0(t) + \varepsilon P_1(t, \mathbf{x})$. Because of Eq. (2.41), $P_{H0}(t, \mathbf{x}) = P_0(t)$. Thus, the term $\partial_t P_{H0}$ is known, and the energy equation (2.42c) is solved for τ_{H0} under the non-jump condition (2.43b) with the given initial data. Next, u_{iH0} and P_{H1} are determined from the continuity and momentum equations (2.42a) and (2.42b) under the non-slip condition (2.43a) with the given initial data [note that P_{H1} is a harmonic function, which is seen from the divergence of Eq. (2.42b) by

taking account of Eq. (2.42a)]. In this way, the gas behavior at the leading order has been determined, together with the pressure at the first order.

The first order is treated in the same way. Because the pressure field P_{H1} has already been determined, the term $\partial_t P_{H1}$ in Eq. (2.44c) is known. Thus, the energy equation (2.44c) is solved for τ_{H1} under the jump condition (2.45c) with the given initial data. Since $\omega_{H0} = P_{H0} - \tau_{H0}$, the term $\partial_t \omega_{H0}$ in Eq. (2.44a) is now known. Thus u_{iH1} and P_{H2}^* are determined from the continuity and momentum equations (2.44a) and (2.44b) with the slip and jump conditions (2.45a) and (2.45b) with the given initial data. The difference from the leading order case is the effect of compressibility, namely $\partial_t \omega_{H0}$, in Eq. (2.44a) and the modified pressure P_{H2}^* in place of P_{H2} in Eq. (2.44b) (note that P_{H2}^* is a solution of the Poisson equation, not the Laplace equation, in contrast to P_{H1}). Since the difference between P_{H2} and P_{H2}^* is given [see Eq. (2.44d)], the first order quantities and the second order pressure P_{H2} have been determined.

Finally we proceed to the second order. Since both τ_{H0} and P_{H2} are known, the energy equation (2.46c) is solved for τ_{H2} under the jump condition (2.47c) with the given initial data. Since $\omega_{H1} = P_{H1} - \tau_{H1}$ is known, u_{iH2} and P_{H3}^* are determined from the continuity and momentum equations (2.46a) and (2.46b) with the slip and jump conditions (2.47a) and (2.47b) with the given initial data. Then, P_{H3} is determined from Eq. (2.46d). In this way, the second order quantities and the third order pressure P_{H3} are determined. This completes the solution procedure up to $O(\varepsilon^2)$.

2.6.3.2 Closed domain

In this case, the total mass of the gas is given. We take the mean density as the reference ρ_0 , hence $\int_D \omega d\mathbf{x} = 0$, where D is a domain of the gas. By taking account of the volume fraction of the Knudsen-layer, we can rewrite this as

$$\int_D \omega_{H0} d\mathbf{x} + \varepsilon \int_D \omega_{H1} d\mathbf{x} + \varepsilon^2 \int_D (\omega_{H2} + \omega_{K1}) d\mathbf{x} + \cdots = 0. \quad (2.49)$$

Thus, $\int_D P_{H0} d\mathbf{x} = \int_D \tau_{H0} d\mathbf{x}$. Because of Eq. (2.41), P_{H0} is independent of $\mathbf{x}$ and thus $P_{H0}(t) = \int_D \tau_{H0} d\mathbf{x} / V_D$, where V_D is the volume of the domain D . Then, the energy equation (2.42c) is rewritten as

$$\partial_t \tau_{H0} - \frac{2}{5} \frac{1}{V_D} \frac{d}{dt} \int_D \tau_{H0} d\mathbf{x} - \frac{1}{2} \gamma_2 \Delta \tau_{H0} = 0,$$

which is solved under the non-jump condition (2.43b) and the given initial data. Once τ_{H0} is determined, P_{H0} is determined by $P_{H0}(t) = \int_D \tau_{H0} d\mathbf{x} / V_D$. Next the continuity and momentum equations (2.42a) and (2.42b) are solved to determine u_{iH0} and P_{H1} under the non-slip condition (2.43a) with the given initial data. Here it should be noted that if we split P_{H1} into two parts: $P_{H1} = P_{H1}^\dagger + \int_D P_{H1} d\mathbf{x} / V_D$, only $P_{H1}^\dagger$ is determined and the other part $\int_D P_{H1} d\mathbf{x} / V_D$ is left unknown. In this sense, in contrast to the open domain case, only the leading order quantities are determined at this stage.

At the first order, because of Eq. (2.49), $\int_D P_{H1} d\mathbf{x} = \int_D \tau_{H1} d\mathbf{x}$, and the energy equation (2.44c) is reduced to

$$\partial_t \tau_{H1} - \frac{2}{5} \frac{1}{V_D} \frac{d}{dt} \int_D \tau_{H1} d\mathbf{x} - \frac{1}{2} \gamma_2 \Delta \tau_{H1} = \frac{2}{5} \partial_t P_{H1}^\dagger.$$

Since the right-hand side is known, this equation is solved for τ_{H1} under the jump condition (2.45c) with the given initial data. Then P_{H1} is also determined by the relation $P_{H1} = P_{H1}^\dagger + \int_D \tau_{H1} d\mathbf{x} / V_D$. Next, the continuity and momentum equations (2.44a) and (2.44b) are solved for u_{iH1} and P_{H2}^* under the slip and jump conditions (2.45a) and (2.45b) with the given initial data. Here again, if we split P_{H2} into two parts: $P_{H2} = P_{H2}^\dagger + \int_D P_{H2} d\mathbf{x} / V_D$ and introduce $P_{H2}^{*\dagger} = P_{H2}^\dagger - (1/6)(\gamma_2 \gamma_1 - 4\gamma_3) \Delta \tau_{H0}$, only $P_{H2}^{*\dagger}$ (and thus $P_{H2}^\dagger$) is determined. In this way, the first order quantities are determined at this stage, with the part $\int_D P_{H2} d\mathbf{x} / V_D$ left unknown.

Finally let us proceed to the second order. Because of Eq. (2.49), $\int_D P_{H2} d\mathbf{x} = \int_D (\tau_{H2} - \omega_{K1}) d\mathbf{x}$ and the energy equation (2.46c) is reduced to

$$\begin{aligned} \partial_t \tau_{H2} - \frac{2}{5} \frac{1}{V_D} \frac{d}{dt} \int_D \tau_{H2} d\mathbf{x} - \frac{1}{2} \gamma_2 \Delta \tau_{H2} \\ = \frac{2}{5} \partial_t \left(P_{H2}^\dagger - \frac{1}{V_D} \int_D \omega_{K1} d\mathbf{x} \right) - \frac{1}{10} (\gamma_2 \gamma_3 - \frac{13}{2} \gamma_{11}) \Delta^2 \tau_{H0}. \end{aligned}$$

Since the right-hand side is known, this equation is solved for τ_{H2} under the jump condition (2.47c) with the given initial data. Once τ_{H2} is obtained, P_{H2} is determined by the relation $P_{H2} = P_{H2}^\dagger + \int_D (\tau_{H2} - \omega_{K1}) d\mathbf{x} / V_D$. Then, u_{iH2} and P_{H3}^* are determined by solving the continuity and momentum equations (2.46a) and (2.46b) under the slip and jump conditions (2.47a) and (2.47b) with the given initial data (here again, P_{H3}^* is determined only partially at this stage). This ends the solution process up to $O(\varepsilon^2)$.

2.7 Data of slip and jump coefficients: Application of a symmetric relation

As discussed in Ref. [21], by the use of the symmetry of the linearized kinetic equation, we can find cross relations among the slip and jump coefficients. Some relations among the first-order slip and jump problems, such as Eq. (2.50) that appears later, have been discussed by several authors (see, for instance, Refs. [23,24,21]). We refer to the pioneering works by Waldmann, Kuřčer, and their collaborators in this direction of research, which mainly discuss the entropy production and related Onsager reciprocity relation for rarefied gas flows and are compiled in Ref. [23]. Here, we use a general direct approach (the symmetric relation) that has been developed recently in Ref. [21], in order to obtain all the data of the slip and jump coefficients up to the second order.

The concerned half-space problems are generically written as

$$\begin{aligned}\zeta_n \partial_\eta \phi^\alpha &= \mathcal{L}(\phi^\alpha) + I^\alpha, \\ \phi^\alpha &= \tilde{\mathcal{K}}[-2\zeta_j(\delta_{ij} - n_i n_j) b_i^\alpha - |\boldsymbol{\zeta}|^2 c^\alpha + g^\alpha] + \mathcal{K}[\phi^\alpha], \quad \zeta_n > 0, \quad \eta = 0, \\ \phi^\alpha &\rightarrow 0 \quad \text{as } \eta \rightarrow \infty,\end{aligned}$$

where the superscript α (and β below) is attached to distinguish the problems. Applying the symmetric relation in Ref. [21] to the pairs of solutions for this type of problems, we obtain

$$\begin{aligned}&\langle \zeta_n(2\zeta_j(\delta_{ij} - n_i n_j) b_i^\beta - |\boldsymbol{\zeta}|^2 c^\beta)(\phi^\alpha - g^\alpha) \rangle_w + \langle \zeta_n g^{\beta-}(\phi^\alpha - \frac{1}{2}g^\alpha) \rangle_w - \int_0^\infty \langle I^{\beta-} \phi^\alpha \rangle d\eta \\ &= \langle \zeta_n(2\zeta_j(\delta_{ij} - n_i n_j) b_i^\alpha - |\boldsymbol{\zeta}|^2 c^\alpha)(\phi^\beta - g^\beta) \rangle_w + \langle \zeta_n g^{\alpha-}(\phi^\beta - \frac{1}{2}g^\beta) \rangle_w - \int_0^\infty \langle I^{\alpha-} \phi^\beta \rangle d\eta,\end{aligned}$$

where subscript w denotes the value at $\eta = 0$ and $f^-(\cdot, \cdot, \boldsymbol{\zeta}) = f(\cdot, \cdot, -\boldsymbol{\zeta})$. Applying this relation to problem pairs listed in Table 2.2, we obtain the following identities:

$$b_2^{(1)} = \frac{1}{\gamma_1} \left(\frac{1}{2} \gamma_3 - \int_0^\infty H_1^{(1)} d\eta \right), \quad (2.50)$$

$$c_5^{(0)} = \frac{8}{15\gamma_2} (\gamma_3 + \int_0^\infty \Pi_1^{(0)} d\eta), \quad (2.51)$$

$$c_6^{(0)} = \frac{8}{15\gamma_2} \left(I_6(AF) + \frac{15}{4} I_4(AF_d) \right) - \int_0^\infty \Theta_1^{(0)} d\eta, \quad (2.52)$$

$$b_4^{(1)} = -\frac{\gamma_6}{\gamma_1} - \int_0^\infty Y_1^{(1)} d\eta, \quad (2.53)$$

Table 2.2: Symmetric relations and derived identities. Here $\phi_{2'}^{(0)} = \phi_2^{(0)} + \frac{3}{2}b_2^{(1)}\phi_5^{(0)}$, $\phi_{3'}^{(0)} = \phi_3^{(0)} - \frac{3}{4}b_1^{(1)}\phi_5^{(0)}$, $\phi_{4'}^{(0)} = \frac{1}{2}\phi_4^{(0)} + \phi_2^{(0)}$, $\phi_{5'}^{(1)} = \phi_5^{(1)} - \phi_6^{(1)} - \phi_4^{(1)}$, $\phi_{7'}^{(1)} = \phi_7^{(1)} - \phi_3^{(1)}$, $\phi_{8'}^{(1)} = \phi_8^{(1)} - \phi_7^{(1)} + \phi_3^{(1)}$, and $\zeta_t = \zeta_i t_i$.

symmetric relation	derived identity	symmetric relation	derived identity
$\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_2^{(1)}$	(2.50)	$\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_3^{(1)}$	(2.56)
$\phi_1^{(0)}$ vs $\phi_5^{(0)}$	(2.51)	$\phi_1^{(0)}$ vs $\phi_{3'}^{(0)}$ and $\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_3^{(1)}$	(2.57)
$\phi_1^{(0)}$ vs $\phi_6^{(0)}$	(2.52)	$\phi_1^{(0)}$ vs $\phi_{2'}^{(0)}$	(2.58)
$\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_4^{(1)}$	(2.53)	$\phi_1^{(0)}$ vs $\phi_{4'}^{(0)}$	(2.59)
$\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_6^{(1)}$	(2.54)	$\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_{7'}^{(1)}$	(2.60)
$\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_{5'}^{(1)}$	(2.55)	$\zeta_t \phi_1^{(1)}$ vs $\zeta_t \phi_{8'}^{(1)}$	(2.61)

$$\begin{aligned}
b_6^{(1)} = & \frac{1}{4\gamma_1} \left(\int_0^\infty \langle |\zeta|^2 (|\zeta|^2 - \zeta_n^2) B \phi_1^{(1)} \rangle d\eta - 2(I_6(BD_1) - \frac{1}{7}I_8(BD_2)) \right. \\
& + 2\langle \zeta_n (|\zeta|^2 - \zeta_n^2) (D_1 + \frac{1}{2}(|\zeta|^2 - 3\zeta_n^2) D_2) \phi_1^{(1)} \rangle_w \\
& \left. - \int_0^\infty \langle (|\zeta|^2 - \zeta_n^2)^2 \phi_1^{(1)-} \partial_{\zeta_n} \phi_1^{(1)} \rangle d\eta \right), \tag{2.54}
\end{aligned}$$

$$b_5^{(1)} = b_4^{(1)} + b_6^{(1)}, \tag{2.55}$$

$$\begin{aligned}
b_3^{(1)} = & -\frac{15}{8} \frac{\gamma_2}{\gamma_1} c_5^{(0)} b_1^{(1)} - \frac{1}{\gamma_1} \int_0^\infty \eta (\Pi_1^{(0)} + 2H_1^{(1)}) d\eta \\
& - \frac{1}{2\gamma_1} \int_0^\infty \langle (|\zeta|^2 - \zeta_n^2) (\zeta_n B + \phi_1^{(1)-}) \phi_1^{(0)} \rangle d\eta, \tag{2.56}
\end{aligned}$$

$$c_3^{(0)} = -\frac{2}{5} \frac{\gamma_1}{\gamma_2} (b_3^{(1)} + c_1^{(0)} b_2^{(1)}), \tag{2.57}$$

$$\begin{aligned}
c_2^{(0)} = & -\frac{3}{2} c_5^{(0)} b_2^{(1)} - \frac{4}{5\gamma_2} \left(\int_0^\infty (2\eta + c_1^{(0)}) H_2^{(1)} d\eta + I_6(AF) \right. \\
& \left. + \frac{1}{2} \int_0^\infty \langle (|\zeta|^2 - \zeta_n^2) \phi_2^{(1)} - |\zeta|^2 A \phi_1^{(0)-} \rangle d\eta \right), \tag{2.58}
\end{aligned}$$

$$\begin{aligned}
c_4^{(0)} = & -2c_2^{(0)} + \frac{8}{5\gamma_2} \left(\int_0^\infty \langle |\zeta|^2 A \phi_1^{(0)} \rangle d\eta - I_6(AF) \right. \\
& \left. - \frac{1}{2} \int_0^\infty \langle (|\zeta|^2 - \zeta_n^2) \phi_1^{(0)-} \partial_{\zeta_n} \phi_1^{(0)} \rangle d\eta \right), \tag{2.59}
\end{aligned}$$

$$\begin{aligned}
b_7^{(1)} = & b_3^{(1)} + b_1^{(1)} \frac{\gamma_3}{\gamma_1} + \frac{2}{\gamma_1} \int_0^\infty \eta H_1^{(1)} d\eta + \frac{1}{4\gamma_1} \int_0^\infty \langle |\zeta|^2 (|\zeta|^2 - \zeta_n^2) B \phi_2^{(1)} \rangle d\eta \\
& - \frac{1}{4\gamma_1} \int_0^\infty \langle (|\zeta|^2 - \zeta_n^2)^2 \phi_1^{(1)-} \partial_{\zeta_n} \phi_2^{(1)} \rangle d\eta, \tag{2.60}
\end{aligned}$$

$$b_8^{(1)} = -b_3^{(1)} + b_7^{(1)} - \frac{\gamma_3}{\gamma_1} b_1^{(1)} - \frac{2}{\gamma_1} \int_0^\infty \eta H_1^{(1)} d\eta. \tag{2.61}$$

The identities (2.51)–(2.61) show that, only from the solutions of the classical first-order Knudsen-layer problems $\phi_1^{(0)}$, $\phi_1^{(1)}$, and $\phi_2^{(1)}$ (and thus $c_1^{(0)}$, $b_1^{(1)}$, and $b_2^{(1)}$), we can determine

Table 2.3: Data of slip and jump coefficients for the BGK model and hard-sphere (HS) gas under the diffuse reflection condition. The data with * are obtained by the symmetric relations (2.51)–(2.61). The data with † are numerically confirmed to meet the symmetric relations.

	BGK	HS		BGK	HS
$b_1^{(1)}$	1.01619	1.2540	$c_1^{(0)}$	1.30272	2.4001
$b_2^{(1)}$	0.38316†	0.6465†	$c_2^{(0)}$	0†	−0.4992*
$b_3^{(1)}$	−0.77837†	−1.5846*	$c_3^{(0)}$	0.11169†	0.0087*[22]
$b_4^{(1)}$	−0.76632†	−0.9039*	$c_4^{(0)}$	1.82181†	4.6181*
$b_5^{(1)}$	−0.50000†	−0.6601*	$c_5^{(0)}$	0.44045†	0.4596*[21]
$b_6^{(1)}$	0.26632†	0.2438*	$c_6^{(0)}$	−1.4276†[14]	−3.1800†[14]
$b_7^{(1)}$	0.26729†	0.4472*	$\int_0^\infty Y_1^{(1)} dz$	−0.23368	−0.2137
$b_8^{(1)}$	−0.26693†	−0.2336*	$\int_0^\infty Y_2^{(1)} dz$	−0.26693	−0.4782

all the values of the remaining slip and jump coefficients, namely $b_3^{(1)} \sim b_8^{(1)}$ and $c_2^{(0)} \sim c_6^{(0)}$, thereby completing the fluid-dynamic system summarized in Sec. 2.5. We list in Table 2.3 the numerical values of slip and jump coefficients which are available in the literature and newly obtained by the symmetric relations (2.51)–(2.61). For the sake of accuracy in using (2.51)–(2.61), $\phi_1^{(0)}$, $\phi_1^{(1)}$, and $\phi_2^{(1)}$ have been recomputed in the present work, and the value of $b_2^{(1)}$ is improved at the last digit from that in Refs. [11] and [12].

2.8 Conclusion

We have studied a time-dependent problem of a slightly rarefied monoatomic gas which is perturbed slowly and slightly from a reference equilibrium state at rest on the basis of the linearized Boltzmann equation. By a systematic asymptotic analysis, we obtain a set of fluid-dynamic-type equations and its appropriate boundary conditions that describe the gas behavior up to the second order of the Knudsen number (or ε^2). The theory covers general intermolecular potential and gas–surface interaction laws. The results show that the compressibility of the gas appears from the leading order in the energy equation and from the first order in the continuity equation. Although the momentum equation is a Stokes-type equation with a modified pressure, it contains a double Laplacian of the leading order flow velocity as a source term at the second order. The energy equation shows a similar feature at the second order. On the other hand, the slip and jump conditions as well as the Knudsen-layer corrections are the same as those for time-independent problems up to

the first order. The difference from the time-independent case appears only at the second order in the jump condition and associated Knudsen-layer correction as a divergence of the first-order flow velocity and a Laplacian of the leading-order temperature.

We have also obtained the numerical data of all the slip and jump coefficients for a hard-sphere gas, most of which have not been available in the literature, by the use of a symmetric relation for linearized kinetic equations.

2.9 Appendix A: Linearization

Linearization of the nonlinear Boltzmann equation and its kinetic boundary condition can be done straightforwardly by neglecting the higher orders of the perturbation from the reference equilibrium state at rest. Because the linearization of the Boltzmann equation is rather common, we show here only the explicit form of $\mathcal{L}$ for the BGK model and a hard-sphere gas: for the BGK model,

$$\mathcal{L}[\phi] = -\phi + \omega + 2\zeta_i u_i + (|\zeta|^2 - \frac{3}{2})\tau,$$

and the reference mean free path ℓ_0 is given by $\ell_0 = (8RT_0/\pi)^{1/2}/A_c\rho_0$, where A_c is a constant such that $A_c\rho_0$ is the collision frequency of gas molecules at the reference state; for a hard-sphere gas,

$$\begin{aligned}\mathcal{L}[\phi] &= -\nu(|\zeta|)\phi + \int [k_1(\zeta, \xi) - k_2(\zeta, \xi)]\phi(\xi)d\xi, \\ \nu(z) &= \frac{1}{2\sqrt{2}} \left[\exp(-z^2) + \left(2z + \frac{1}{z}\right) \int_0^z \exp(-y^2)dy \right], \\ k_1(\zeta, \xi) &= \frac{1}{\sqrt{2\pi}|\zeta - \xi|} \exp\left(-|\xi|^2 + \frac{|\xi \times \zeta|^2}{|\xi - \zeta|^2}\right), \\ k_2(\zeta, \xi) &= \frac{|\zeta - \xi|}{2\sqrt{2\pi}} \exp(-|\xi|^2),\end{aligned}$$

and $\ell_0 = [\sqrt{2\pi}\sigma^2(\rho_0/m)]^{-1}$, where σ and m are the diameter and the mass of a molecule. The above mean free path is related to the viscosity μ_0 at the reference equilibrium state as $\mu_0 = (\sqrt{\pi}/4)\gamma_1\rho_0(2RT_0)^{1/2}\ell_0$, where γ_1 is a constant that depends on the molecular model (see Sec. 2.10).

On the other hand, the process of getting the linearized boundary condition (2.2) is less common. We will demonstrate it here. Usually, the kinetic boundary condition is assumed

to be linear (integral) operation on the full velocity distribution function and is written in the dimensionless form as Ref. [12]

$$f(\zeta) = \int_{\zeta_n^* < 0} K_B(\zeta^*, \zeta; t, \mathbf{x}) f(\zeta^*) d\zeta^*, \quad \zeta_n > 0, \quad (2.62)$$

where $f = (1 + \phi)E$ and the assumption $u_{iw}n_i = 0$ has been used. The kernel K_B may change, in general, depending on the material and temperature and velocity of the boundary, thus having the arguments t and $\mathbf{x}$. Because the boundary condition (2.62) should be satisfied with the (dimensionless) Maxwellian M_w which is characterized by the velocity and temperature of the boundary, the following relation holds:

$$M_w(\zeta) = \int_{\zeta_n^* < 0} K_B(\zeta^*, \zeta; t, \mathbf{x}) M_w(\zeta^*) d\zeta^*, \quad \zeta_n > 0. \quad (2.63)$$

In the linearized framework, we neglect the higher order terms of the perturbation from the reference equilibrium state. In the case, the Maxwellian M_w is written as $M_w = [1 + g_w + c(t, \mathbf{x})]E$. If we split the kernel K_B into the kernel K_0 at the reference equilibrium state at rest and the remainder, the remainder is the perturbation whose higher order terms can be neglected. Physically, the remainder represents the dependence of the kernel K_B on the variations of surface temperature and velocity, which are assumed to be small. Then, subtracting Eq. (2.63) from Eq. (2.62) yields, after the linearization,

$$(\phi - g_w - c)E(|\zeta|) = \int_{\zeta_n^* < 0} K_0(\zeta^*, \zeta; \mathbf{x}) (\phi - g_w - c)E(|\zeta^*|) d\zeta^*, \quad \zeta_n > 0. \quad (2.64)$$

Because Eq. (2.63) is reduced to

$$E(|\zeta|) = \int_{\zeta_n^* < 0} K_0(\zeta^*, \zeta; \mathbf{x}) E(|\zeta^*|) d\zeta^*, \quad \zeta_n > 0,$$

at the reference equilibrium state, the terms containing c in Eq. (2.64) cancel out each other, and the expression (2.2) is obtained.

In the above derivation, the dependence of K_0 on $\mathbf{x}$ is retained, which implies, for instance, that the accommodation coefficients may change along the boundary due to the change of material. Consideration of this dependence would be interesting, but here in the present paper, we assume K_0 is independent of $\mathbf{x}$ along the surface. Thus, the reflection kernel $\mathcal{R}$ introduced in Sec. 2.2 is also independent of $\mathbf{x}$. Incidentally, as is clear from the above derivation, $K_0(\zeta^*, \zeta) = (|\zeta_n^*|/|\zeta_n|)\mathcal{R}(\zeta^*, \zeta)$. In the case of the diffuse reflection condition, $K_0[= (|\zeta_n^*|/|\zeta_n|)\mathcal{R}] = 2\sqrt{\pi}|\zeta_n^*|E(|\zeta|)$.

2.10 Appendix B: Isotropic functions and related constants

We give below the definitions of the isotropic functions A , B , D_1 , D_2 , F , and F_d and related constants γ_i ($i = 1, 2, 3, 6, 10, 11$) that have been introduced in the course of analysis.

$$\begin{aligned}\mathcal{L}[\zeta_i A(|\zeta|)] &= -\zeta_i(|\zeta|^2 - \frac{5}{2}), \quad \mathcal{L}[\zeta_{ij} B(|\zeta|)] = -2\zeta_{ij}, \quad \mathcal{L}[\zeta_{ij} F(|\zeta|)] = \zeta_{ij} A(|\zeta|), \\ \mathcal{L}[(\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) D_1(|\zeta|) + \zeta_i \zeta_j \zeta_k D_2(|\zeta|)] &= \gamma_1(\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) - \zeta_i \zeta_j \zeta_k B(|\zeta|), \\ \mathcal{L}[F_d(|\zeta|)] &= -\frac{5}{6}\gamma_2(|\zeta|^2 - \frac{3}{2}) + \frac{1}{3}|\zeta|^2 A(|\zeta|),\end{aligned}$$

where A , D_1 , D_2 , and F_d respectively satisfy the subsidiary conditions:

$$\langle |\zeta|^2 A(|\zeta|) \rangle = 0, \quad \langle 5|\zeta|^2 D_1(|\zeta|) + |\zeta|^4 D_2(|\zeta|) \rangle = 0, \quad \langle F_d(|\zeta|) \rangle = 0, \quad \langle |\zeta|^2 F_d(|\zeta|) \rangle = 0.$$

The constants γ_i are defined by

$$\begin{aligned}\gamma_1 &= I_6(B), \quad \gamma_2 = 2I_6(A), \quad \gamma_3 = I_6(AB) = 5I_6(D_1) + I_8(D_2) = -2I_6(F), \\ \gamma_6 &= \frac{1}{2}I_6(BD_1) + \frac{3}{14}I_8(BD_2), \quad \gamma_{10} = \frac{1}{2}I_6(B^2), \\ \gamma_{11} &= -\frac{4}{39}\left(\frac{1}{2}\gamma_2(15I_4(A^2) + \gamma_3) + 4I_6(AF) + 15I_4(AF_d)\right),\end{aligned}$$

where $I_n(X) = (8/15\sqrt{\pi}) \int_0^\infty z^n X(z) \exp(-z^2) dz$. For the BGK model, all γ_i are unity and

$$A = -F = |\zeta|^2 - \frac{5}{2}, \quad B = 2, \quad D_1 = -1, \quad D_2 = 2, \quad F_d = -\frac{1}{3}(|\zeta|^4 - 5|\zeta|^2 + \frac{15}{4}).$$

For a hard-sphere gas, γ_i are

$$\begin{aligned}\gamma_1 &= 1.270042427, \quad \gamma_2 = 1.922284066, \quad \gamma_3 = 1.947906335, \quad \gamma_6 = 1.419423836, \\ \gamma_{10} &= 1.63607346, \quad \gamma_{11} = 2.7931173.\end{aligned}$$

2.11 Appendix C: Axial symmetry of operators $\mathcal{L}$ and $\mathcal{K}$

For any function $f(\zeta_n, |\zeta|)$, where ζ_n is the component of ζ in the direction normal to the boundary, the scattering and collision operators $\mathcal{K}$ and $\mathcal{L}$ satisfy the following axial symmetry:

$$J[(\delta_{ij} - n_i n_j) \zeta_j f] = (\delta_{ij} - n_i n_j) \zeta_j J_1[f], \quad (2.65)$$

$$\begin{aligned}
& J[(\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l)\zeta_k \zeta_l f] \\
& = (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l)\zeta_k \zeta_l J_2[f] + J_3[f](\delta_{ij} - n_i n_j),
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
& J[(\delta_{il} - n_i n_l)(\delta_{jp} - n_j n_p)(\delta_{kq} - n_k n_q)\zeta_l \zeta_p \zeta_q f] \\
& = (\delta_{il} - n_i n_l)(\delta_{jp} - n_j n_p)(\delta_{kq} - n_k n_q)\zeta_l \zeta_p \zeta_q J_4[f] \\
& \quad + \zeta_l J_5[f](\delta_{il} - n_i n_l)(\delta_{jk} - n_j n_k) \\
& \quad + \zeta_l J_5[f](\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) \\
& \quad + \zeta_l J_5[f](\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l),
\end{aligned} \tag{2.67}$$

where $J = \mathcal{K}$ and $\mathcal{L}$, and $J_1[f], J_2[f], \dots, J_5[f]$ are functions of ζ_n and $|\zeta|$.

Sketch of proof: Equation (2.65) is a direct consequence of Appendix B.4.2 of Ref. [11] for one suffix case. In order to obtain Eqs. (2.66) and (2.67), we apply the results in Appendix B.4.2 of Ref. [11] for two or three suffixes. Then the manipulation $\delta_{kl}(\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) = \delta_{ij} - n_i n_j$ reduces the terms of J_3 and J_5 into the desired form. $\square$

Further, if we multiply Eq. (2.66) by δ_{ij} , we have $J[(|\zeta|^2 - \zeta_n^2)f] = (|\zeta|^2 - \zeta_n^2)J_2[f] + 2J_3[f]$. Thus, subtracting this multiplied by $(\delta_{ij} - n_i n_j)/2$ from Eq. (2.66) leads to

$$\begin{aligned}
& J\left[\left((\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l)\zeta_k \zeta_l - \frac{1}{2}(|\zeta|^2 - \zeta_n^2)(\delta_{ij} - n_i n_j)\right)f\right] \\
& = \left((\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l)\zeta_k \zeta_l - \frac{1}{2}(|\zeta|^2 - \zeta_n^2)(\delta_{ij} - n_i n_j)\right)J_2[f].
\end{aligned} \tag{2.68}$$

If we multiply Eq. (2.67) by δ_{jk} , we have

$$J[(\delta_{il} - n_i n_l)\zeta_l(|\zeta|^2 - \zeta_n^2)f] = (\delta_{il} - n_i n_l)\zeta_l\left((|\zeta|^2 - \zeta_n^2)J_4[f] + 4J_5[f]\right),$$

and thus obtain

$$\begin{aligned}
& J[(\delta_{il} - n_i n_l)(\delta_{jk} - n_j n_k)\zeta_l(|\zeta|^2 - \zeta_n^2)f] \\
& \quad + J[(\delta_{jl} - n_j n_l)(\delta_{ik} - n_i n_k)\zeta_l(|\zeta|^2 - \zeta_n^2)f] \\
& \quad + J[(\delta_{kl} - n_k n_l)(\delta_{ij} - n_i n_j)\zeta_l(|\zeta|^2 - \zeta_n^2)f] \\
& = (\delta_{il} - n_i n_l)(\delta_{jk} - n_j n_k)\zeta_l\left((|\zeta|^2 - \zeta_n^2)J_4[f] + 4J_5[f]\right) \\
& \quad + (\delta_{jl} - n_j n_l)(\delta_{ik} - n_i n_k)\zeta_l\left((|\zeta|^2 - \zeta_n^2)J_4[f] + 4J_5[f]\right) \\
& \quad + (\delta_{kl} - n_k n_l)(\delta_{ij} - n_i n_j)\zeta_l\left((|\zeta|^2 - \zeta_n^2)J_4[f] + 4J_5[f]\right).
\end{aligned}$$

Subtracting this equation divided by 4 from Eq. (2.67) leads to

$$\begin{aligned}
& J[(\delta_{il} - n_i n_l)(\delta_{jp} - n_j n_p)(\delta_{kq} - n_k n_q) \zeta_l \zeta_p \zeta_q f \\
& - \frac{1}{4} \left((\delta_{il} - n_i n_l)(\delta_{jk} - n_j n_k) + (\delta_{jl} - n_j n_l)(\delta_{ik} - n_i n_k) \right. \\
& \left. + (\delta_{kl} - n_k n_l)(\delta_{ij} - n_i n_j) \right) \zeta_l (|\boldsymbol{\zeta}|^2 - \zeta_n^2) f] \\
& = (\delta_{il} - n_i n_l)(\delta_{jp} - n_j n_p)(\delta_{kq} - n_k n_q) \zeta_l \zeta_p \zeta_q J_4[f] \\
& - \frac{1}{4} \left((\delta_{il} - n_i n_l)(\delta_{jk} - n_j n_k) + (\delta_{jl} - n_j n_l)(\delta_{ik} - n_i n_k) \right. \\
& \left. + (\delta_{kl} - n_k n_l)(\delta_{ij} - n_i n_j) \right) \zeta_l (|\boldsymbol{\zeta}|^2 - \zeta_n^2) J_4[f]. \tag{2.69}
\end{aligned}$$

Formulas (2.68) and (2.69) will be used in Secs. 2.15.3 and 2.15.4, respectively.

2.12 Appendix D: Formulas for curved coordinates

Consider the curved coordinates (η, χ_1, χ_2)

$$x_i = x_{iw}(\chi_1, \chi_2) + \varepsilon \eta n_i(\chi_1, \chi_2), \tag{2.70}$$

where x_{iw} represents the surface of body and n_i its normal unit vector, pointed to the gas. Tangential unit vectors $\ell_i^{(1)}$ and $\ell_i^{(2)}$ along the coordinates χ_1 and χ_2 are given by

$$\ell_i^{(1)} = \frac{\partial_{\chi_1} x_{iw}}{|\partial_{\chi_1} \mathbf{x}_w|}, \quad \ell_i^{(2)} = \frac{\partial_{\chi_2} x_{iw}}{|\partial_{\chi_2} \mathbf{x}_w|}.$$

Note that $\boldsymbol{\ell}^{(1)}$ and $\boldsymbol{\ell}^{(2)}$ are not necessarily orthogonal to each other, though they are orthogonal to $\mathbf{n}$ since $n_i dx_{iw} = 0$ by definition.

Take the derivative of Eq. (2.70) with respect to x_j :

$$\begin{aligned}
\delta_{ij} &= (\partial_j \chi_1) \partial_{\chi_1} x_{iw} + (\partial_j \chi_2) \partial_{\chi_2} x_{iw} + \varepsilon (\partial_j \eta) n_i + \varepsilon \eta (\partial_j \chi_1) \partial_{\chi_1} n_i + \varepsilon \eta (\partial_j \chi_2) \partial_{\chi_2} n_i \\
&= (\partial_j \chi_1) |\partial_{\chi_1} \mathbf{x}_w| \ell_i^{(1)} + (\partial_j \chi_2) |\partial_{\chi_2} \mathbf{x}_w| \ell_i^{(2)} \\
&\quad + \varepsilon (\partial_j \eta) n_i + \varepsilon \eta (\partial_j \chi_1) \partial_{\chi_1} n_i + \varepsilon \eta (\partial_j \chi_2) \partial_{\chi_2} n_i.
\end{aligned}$$

Then the inner product with the unit vectors $\mathbf{n}$, $\boldsymbol{\ell}^{(1)}$, and $\boldsymbol{\ell}^{(2)}$ gives the relations that

$$\begin{aligned}
n_j &= \varepsilon \partial_j \eta, \\
\ell_j^{(1)} &= (\partial_j \chi_1)_w |\partial_{\chi_1} \mathbf{x}_w| + (\partial_j \chi_2)_w |\partial_{\chi_2} \mathbf{x}_w| \ell_i^{(1)} \ell_i^{(2)}, \\
\ell_j^{(2)} &= (\partial_j \chi_1)_w |\partial_{\chi_1} \mathbf{x}_w| \ell_i^{(1)} \ell_i^{(2)} + (\partial_j \chi_2)_w |\partial_{\chi_2} \mathbf{x}_w|,
\end{aligned}$$

where $(\cdots)_w$ denotes the value evaluated on the surface ($\eta = 0$). The above relations are rewritten as

$$\partial_j \eta = (1/\varepsilon) n_j, \quad (2.71)$$

$$(\partial_j \chi_1)_w = \frac{1}{|\partial_{\chi_1} \mathbf{x}_w|} \frac{\ell_j^{(1)} - (\ell_i^{(1)} \ell_i^{(2)}) \ell_j^{(2)}}{1 - (\ell_i^{(1)} \ell_i^{(2)})^2}, \quad (2.72)$$

$$(\partial_j \chi_2)_w = \frac{1}{|\partial_{\chi_2} \mathbf{x}_w|} \frac{\ell_j^{(2)} - (\ell_i^{(1)} \ell_i^{(2)}) \ell_j^{(1)}}{1 - (\ell_i^{(1)} \ell_i^{(2)})^2}, \quad (2.73)$$

and thus ∂_i is written in the new coordinate system as follows:

$$\begin{aligned} \partial_i &= \partial_i \eta \partial_\eta + \partial_i \chi_1 \partial_{\chi_1} + \partial_i \chi_2 \partial_{\chi_2} \\ &= (1/\varepsilon) n_i \partial_\eta + (\partial_i \chi_1)_w \partial_{\chi_1} + (\partial_i \chi_2)_w \partial_{\chi_2} + \varepsilon \eta (\partial_n \partial_i \chi_1)_w \partial_{\chi_1} + \varepsilon \eta (\partial_n \partial_i \chi_2)_w \partial_{\chi_2} + \cdots, \end{aligned}$$

where ∂_n denotes the spatial derivative in the normal direction (in unstretched original coordinate system).

Next we introduce the curvature matrix of the surface defined by

$$\kappa_{ij} \equiv (\partial_i n_j)_w = (\partial_i \chi_1)_w \partial_{\chi_1} n_j + (\partial_i \chi_2)_w \partial_{\chi_2} n_j.$$

Note that $(\partial_i \chi_1)_w$ and $(\partial_i \chi_2)_w$ are tangential to the surface because of Eqs. (2.72) and (2.73). Since $\mathbf{n}$ is a unit vector, $n_j \partial_{\chi_1} n_j = n_j \partial_{\chi_2} n_j = 0$, so that $\partial_{\chi_1} n_j$ and $\partial_{\chi_2} n_j$ are tangential to the surface and can be expressed by a linear combination of $\ell^{(1)}$ and $\ell^{(2)}$:

$$\partial_{\chi_1} n_j = \frac{\partial_{\chi_1} n_i}{1 - (\ell_k^{(1)} \ell_k^{(2)})^2} \left(\ell_i^{(1)} [\ell_j^{(1)} - (\ell_k^{(1)} \ell_k^{(2)}) \ell_j^{(2)}] + \ell_i^{(2)} [\ell_j^{(2)} - (\ell_k^{(1)} \ell_k^{(2)}) \ell_j^{(1)}] \right), \quad (2.74)$$

$$\partial_{\chi_2} n_j = \frac{\partial_{\chi_2} n_i}{1 - (\ell_k^{(1)} \ell_k^{(2)})^2} \left(\ell_i^{(1)} [\ell_j^{(1)} - (\ell_k^{(1)} \ell_k^{(2)}) \ell_j^{(2)}] + \ell_i^{(2)} [\ell_j^{(2)} - (\ell_k^{(1)} \ell_k^{(2)}) \ell_j^{(1)}] \right). \quad (2.75)$$

As is obvious from Eqs. (2.72)–(2.75), the curvature matrix is symmetric and its operation on the normal unit results in zero:

$$\kappa_{ij} = \kappa_{ji}, \quad \kappa_{ij} n_j = \kappa_{ij} n_i = 0.$$

Now suppose that χ_1 and χ_2 are orthogonal to each other and denote by $\mathbf{t}$ and $\mathbf{s}$ the tangential unit vectors $\ell^{(1)}$ and $\ell^{(2)}$ in this case. Then, Eqs. (2.72)–(2.75) are reduced to

$$(\partial_j \chi_1)_w = \frac{t_j}{|\partial_{\chi_1} \mathbf{x}_w|}, \quad (\partial_j \chi_2)_w = \frac{s_j}{|\partial_{\chi_2} \mathbf{x}_w|}, \quad \partial_{\chi_\alpha} n_j = (t_i \partial_{\chi_\alpha} n_i) t_j + (s_i \partial_{\chi_\alpha} n_i) s_j, \quad (\alpha = 1, 2).$$

Thus the curvature matrix is expressed as

$$\kappa_{ij} = \kappa_t t_i t_j + \kappa_s s_i s_j + \frac{s_k \partial_{\chi_1} n_k}{|\partial_{\chi_1} \mathbf{x}_w|} t_i s_j + \frac{t_k \partial_{\chi_2} n_k}{|\partial_{\chi_2} \mathbf{x}_w|} s_i t_j,$$

where κ_t and κ_s are the curvature in the directions of $\mathbf{t}$ and $\mathbf{s}$ respectively:

$$\kappa_t = (\partial_i \chi_1)_w (\partial_{\chi_1} n_i), \quad \kappa_s = (\partial_i \chi_2)_w (\partial_{\chi_2} n_i),$$

and the last two terms represent the effect of geodesic torsion. In particular, when χ_1 and χ_2 are taken in the principal directions of the surface, $s_k \partial_{\chi_1} n_k = 0$ and $t_k \partial_{\chi_2} n_k = 0$, so that the above equation is written as

$$\kappa_{ij} = \kappa_1 \ell_i \ell_j + \kappa_2 m_i m_j, \quad (2.76)$$

where $\boldsymbol{\ell}$ and $\mathbf{m}$ are the tangential unit vectors in the principal directions. The above κ_1 and κ_2 are called the principal curvature. Note that

$$\kappa_{ii} = \kappa_t + \kappa_s = \kappa_1 + \kappa_2 \equiv 2\bar{\kappa}. \quad (2.77)$$

Thus, the sum of the curvatures in two orthogonal directions are universal, and their average $\bar{\kappa} = (1/2)(\kappa_1 + \kappa_2)$ is called the mean curvature.

With the above preparation, we present some useful formulas related to the differential operator $\mathcal{D}_i$:

$$\mathcal{D}_i = (\delta_{ij} - n_i n_j) \partial_j|_{\mathbf{x}=\mathbf{x}_w} = (\partial_i \chi_1)_w \partial_{\chi_1} + (\partial_i \chi_2)_w \partial_{\chi_2}, \quad (2.78)$$

which are listed below:

$$\mathcal{D}_i[n_j] = (\partial_i \chi_1)_w \partial_{\chi_1} n_j + (\partial_i \chi_2)_w \partial_{\chi_2} n_j = \kappa_{ij}, \quad (2.79)$$

$$\mathcal{D}_i[f(\mathbf{x} = \mathbf{x}_w)] = (\delta_{ij} - n_i n_j) \partial_j|_{\mathbf{x}=\mathbf{x}_w} f(\mathbf{x} = \mathbf{x}_w) = (\delta_{ij} - n_i n_j) (\partial_j f)_w, \quad (2.80)$$

$$\mathcal{D}_i[f(\eta, \zeta_n, |\boldsymbol{\zeta}|)] = \mathcal{D}_i[\zeta_n] \partial_{\zeta_n} f = \kappa_{ij} \zeta_j \partial_{\zeta_n} f, \quad (2.81)$$

$$\mathcal{D}_i[\delta_{jk} - n_j n_k] = -\kappa_{ij} n_k - \kappa_{ik} n_j. \quad (2.82)$$

These can be proved in order from top to bottom.

2.13 Appendix E: Knudsen-layer corrections to the stress tensor and heat-flow vector

We summarize the Knudsen-layer corrections to the stress tensor and heat-flow vector up to $O(\varepsilon^2)$:

$$\begin{aligned}
P_{ijK1} &= \Pi_1^{(0)}(\eta) \partial_k \tau_{H0} n_k (\delta_{ij} - n_i n_j), \\
Q_{iK1} n_i &= 0, \quad Q_{iK1} t_i = H_1^{(1)}(\eta) \overline{\partial_i u_{jH0}} n_i t_j + H_2^{(1)}(\eta) \partial_i \tau_{H0} t_i, \\
P_{ijK2} n_i t_j &= \int_{\eta}^{\infty} \Pi_1^{(0)}(z) dz (\partial_i \partial_j \tau_{H0} n_i t_j + \kappa_{ij} \partial_i \tau_{H0} t_j), \\
P_{ijK2} n_i n_j &= -2 \int_{\eta}^{\infty} \Pi_1^{(0)}(z) dz \bar{\kappa} \partial_i \tau_{H0} n_i, \\
P_{ijK2} t_i t_j &= \Pi_1^{(0)}(\eta) \partial_i \tau_{H1} n_i + (\Pi_2^{(0)}(\eta) - S_2^{(2)}(\eta)) \partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) \\
&\quad + (\Pi_3^{(0)}(\eta) + \frac{1}{2} S_3^{(2)}(\eta) + S_4^{(2)}(\eta)) \partial_i \overline{\partial_j u_{kH0}} n_i n_j n_k \\
&\quad + (\Pi_4^{(0)}(\eta) + \int_{\eta}^{\infty} \Pi_1^{(0)}(z) dz - 2S_1^{(2)}(\eta)) \bar{\kappa} \partial_i \tau_{H0} n_i + \Pi_5^{(0)}(\eta) \partial_i u_{iH1} \\
&\quad + \Pi_6^{(0)}(\eta) \Delta \tau_{H0} + 2S_1^{(2)}(\eta) \kappa_{ij} t_i t_j \partial_k \tau_{H0} n_k + 2S_2^{(2)}(\eta) \partial_i \partial_j \tau_{H0} t_i t_j \\
&\quad + 2S_3^{(2)}(\eta) \partial_i \overline{\partial_j u_{kH0}} n_k t_i t_j + 2S_4^{(2)}(\eta) \partial_i \overline{\partial_j u_{kH0}} n_i t_j t_k, \\
P_{ijK2} t_i s_j &= 2S_1^{(2)}(\eta) \kappa_{ij} t_i s_j \partial_k \tau_{H0} n_k + 2S_2^{(2)}(\eta) \partial_i \partial_j \tau_{H0} t_i s_j \\
&\quad + S_3^{(2)}(\eta) \partial_i \overline{\partial_j u_{kH0}} (t_i s_j + t_j s_i) n_k + 2S_4^{(2)}(\eta) \partial_i \overline{\partial_j u_{kH0}} n_i t_j s_k, \\
Q_{iK2} n_i &= -\frac{1}{2} \int_{\eta}^{\infty} H_1^{(1)}(z) dz \partial_i \overline{\partial_j u_{kH0}} n_i n_j n_k \\
&\quad + \int_{\eta}^{\infty} H_2^{(1)}(z) dz (\partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) - 2\bar{\kappa} \partial_i \tau_{H0} n_i), \\
Q_{iK2} t_i &= H_1^{(1)}(\eta) \overline{\partial_i u_{jH1}} n_i t_j + H_2^{(1)}(\eta) \partial_i \tau_{H1} t_i + H_3^{(1)}(\eta) \partial_i \partial_j \tau_{H0} n_i t_j \\
&\quad + H_4^{(1)}(\eta) \partial_i \overline{\partial_j u_{kH0}} n_i n_j t_k + H_5^{(1)}(\eta) \bar{\kappa} \overline{\partial_i u_{jH0}} n_i t_j + H_6^{(1)}(\eta) \kappa_{ij} \overline{\partial_j u_{kH0}} n_k t_i \\
&\quad + H_7^{(1)}(\eta) \kappa_{ij} \partial_i \tau_{H0} t_j + H_8^{(1)}(\eta) \bar{\kappa} \partial_i \tau_{H0} t_i,
\end{aligned}$$

where $\mathbf{t}$ and $\mathbf{s}$ are tangential unit vectors orthogonal to each other and

$$\begin{aligned}
\Pi_j^{(i)}(\eta) &= \frac{3}{2} (\Omega_j^{(i)}(\eta) + \Theta_j^{(i)}(\eta)), \quad S_j^{(i)}(\eta) = \frac{1}{4} \langle (|\boldsymbol{\zeta}|^2 - \zeta_n^2)^2 \phi_j^{(i)} \rangle, \\
H_j^{(i)}(\eta) &= \frac{1}{2} \langle (|\boldsymbol{\zeta}|^2 - \zeta_n^2) (|\boldsymbol{\zeta}|^2 - \frac{5}{2}) \phi_j^{(i)} \rangle.
\end{aligned}$$

Here some reductions have been made in deriving the expressions for $P_{ijK2}n_it_j$, $P_{ijK2}n_in_j$, and $Q_{iK2}n_i$ by the use of the conservation laws of momentum and energy [see Eq. (2.30) and Sec. 2.15].

2.14 Appendix F: Formulas for quantities on the boundary

2.14.1 Derivatives of u_{iH0}

$$\partial_i \overline{\partial_j u_{iH0} n_j} = \frac{1}{2} \overline{\partial_k \partial_j u_{iH0} n_j n_i n_k}, \quad (2.83)$$

$$\partial_i \overline{\partial_j u_{iH0}} (\delta_{jk} - n_j n_k) = (\partial_l \overline{\partial_j u_{iH0} n_i n_l} + 2\bar{\kappa} n_i \overline{\partial_j u_{iH0}}) (\delta_{jk} - n_j n_k) + \kappa_{ki} n_j \overline{\partial_j u_{iH0}}, \quad (2.84)$$

$$\partial_l \overline{\partial_j u_{iH0}} (\delta_{il} - n_i n_l) n_j n_k = -2\kappa_{ij} n_k \overline{\partial_j u_{iH0}}, \quad (2.85)$$

$$\begin{aligned} \partial_l \overline{\partial_p u_{qH0}} (\delta_{il} - n_i n_l) (\delta_{jp} - n_j n_p) (\delta_{kq} - n_k n_q) \\ = (\kappa_{ij} n_l (\delta_{kp} - n_k n_p) + \kappa_{ik} n_l (\delta_{jp} - n_j n_p)) \overline{\partial_l u_{pH0}}. \end{aligned} \quad (2.86)$$

Sketch of proof: To prove Eq. (2.83), decompose u_{iH0} as $u_{iH0}(\mathbf{x}) = U_i(\mathbf{x}) + v_i(\mathbf{x})$, where $U_i(\mathbf{x})$ is the rigid motion that coincides with that of the body on its surface and $v_i(\mathbf{x})$ is the remainder. Note that $\overline{\partial_j U_i} = 0$ and $\partial_i v_i = 0$ everywhere and that $\overline{\partial_j v_i n_i n_j} = 0$, $\overline{\partial_j v_i} (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) = 0$, and $v_i = 0$ on the surface. Then

$$\begin{aligned} \partial_i \overline{\partial_j u_{iH0} n_j} &= \partial_i \overline{\partial_j v_i n_j} = \Delta v_j n_j = \partial_i \partial_k v_j n_j n_i n_k + \partial_i \partial_k v_j n_j (\delta_{ik} - n_i n_k) \\ &= \frac{1}{2} \overline{\partial_i \partial_k v_j n_i n_j n_k} + \partial_i \partial_k v_j n_j (\delta_{il} - n_i n_l) (\delta_{lk} - n_l n_k) \\ &= \frac{1}{2} \overline{\partial_i \partial_k u_{jH0} n_i n_j n_k} + \mathcal{D}_l [\partial_k v_j n_j (\delta_{lk} - n_l n_k)] - \mathcal{D}_l [n_j (\delta_{lk} - n_l n_k)] \partial_k v_j. \end{aligned}$$

The last two terms are further reduced to $-(1/2)\kappa_{ij} \overline{\partial_i v_j} + \bar{\kappa} n_i n_j \overline{\partial_i v_j}$ by using the formulas in Sec. 2.12, which vanishes because of the properties of v_i mentioned above.

In order to prove Eq. (2.84), we first make the following transformation

$$\begin{aligned} \partial_i \overline{\partial_j u_{iH0}} (\delta_{jk} - n_j n_k) &= \partial_i \overline{\partial_j v_i} (\delta_{jk} - n_j n_k) \\ &= \partial_l \overline{\partial_j v_i} (\delta_{jk} - n_j n_k) n_i n_l + \partial_l \overline{\partial_j v_i} (\delta_{jk} - n_j n_k) (\delta_{il} - n_i n_l) \\ &= \partial_l \overline{\partial_j u_{iH0}} (\delta_{jk} - n_j n_k) n_i n_l + (\partial_l \partial_j v_i + \partial_l \partial_i v_j) (\delta_{jk} - n_j n_k) (\delta_{il} - n_i n_l) \\ &= \partial_l \overline{\partial_j u_{iH0}} (\delta_{jk} - n_j n_k) n_i n_l + \mathcal{D}_i [\partial_j v_i (\delta_{jk} - n_j n_k)] \end{aligned}$$

$$- \mathcal{D}_i[(\delta_{jk} - n_j n_k)] \partial_j v_i + \mathcal{D}_i[\partial_i v_j (\delta_{jk} - n_j n_k)] - \mathcal{D}_i[(\delta_{jk} - n_j n_k)] \partial_i v_j,$$

and apply the formulas in Sec. 2.12. Then the last four terms are eventually reduced to $\kappa_{ki} n_j \overline{\partial_j v_i} + 2\bar{\kappa} n_i \overline{\partial_i v_j} (\delta_{jk} - n_j n_k)$, which is identical to $\kappa_{ki} n_j \overline{\partial_j u_{iH0}} + 2\bar{\kappa} n_i \overline{\partial_j u_{iH0}} (\delta_{jk} - n_j n_k)$.

Equations (2.85) and (2.86) are easily obtained by the use of the formulas in Sec. 2.12. $\square$

2.14.2 Derivatives of u_{iH1} and τ_{H1}

$$\begin{aligned} \overline{\partial_i u_{jH1}} n_i n_j &= b_1^{(1)} \overline{\partial_i \partial_k u_{jH0}} n_i n_j n_k + 2b_2^{(1)} \partial_i \partial_j \tau_{H0} n_i n_j + 4b_2^{(1)} \bar{\kappa} \partial_j \tau_{H0} n_j \\ &\quad - 2b_2^{(1)} \Delta \tau_{H0} + \frac{4}{3} \partial_i u_{iH1}, \end{aligned} \quad (2.87)$$

$$\begin{aligned} \overline{\partial_k u_{lH1}} (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) \\ &= (b_1^{(1)} \partial_k \overline{\partial_m u_{lH0}} n_m + b_1^{(1)} \partial_l \overline{\partial_m u_{kH0}} n_m + 2b_2^{(1)} \partial_k \partial_l \tau_{H0}) (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) \\ &\quad - 2b_2^{(1)} \kappa_{ij} \partial_k \tau_{H0} n_k - \frac{2}{3} \partial_m u_{mH1} (\delta_{ij} - n_i n_j), \end{aligned} \quad (2.88)$$

$$\partial_j \tau_{H1} (\delta_{ij} - n_i n_j) = \partial_j \tau_{w1} (\delta_{ij} - n_i n_j) + c_1^{(0)} \kappa_{ik} \partial_k \tau_{H0} + c_1^{(0)} \partial_j \partial_k \tau_{H0} n_k (\delta_{ij} - n_i n_j). \quad (2.89)$$

Sketch of Proof: To obtain Eqs. (2.87)–(2.89), we make use of the first-order slip and jump conditions (2.45a)–(2.45c) in addition to the formulas in Sec. 2.12.

Actually, to prove Eq. (2.87), using $\overline{\partial_i u_{iH1}} = 0$, we first transform its left-hand side as

$$\begin{aligned} \overline{\partial_i u_{jH1}} n_i n_j &= -\overline{\partial_i u_{jH1}} (\delta_{ij} - n_i n_j) = -2\partial_i u_{jH1} (\delta_{ij} - n_i n_j) + \frac{2}{3} \delta_{ij} \partial_l u_{lH1} (\delta_{ij} - n_i n_j) \\ &= -2\partial_i u_{jH1} (\delta_{ik} - n_i n_k) (\delta_{jk} - n_j n_k) + \frac{4}{3} \partial_k u_{kH1} \\ &= -2\mathcal{D}_k[u_{jH1} (\delta_{jk} - n_j n_k)] + \frac{4}{3} \partial_k u_{kH1}. \end{aligned}$$

Then, substitute Eq. (2.45b) and apply the formulas in Sec. 2.12. The desired equation is obtained after a long but straightforward manipulation.

Equation (2.88) can be proved in the same way. We first transform the left-hand side as

$$\begin{aligned} \overline{\partial_k u_{lH1}} (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) \\ &= (\partial_k u_{lH1} + \partial_l u_{kH1}) (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) - \frac{2}{3} \delta_{kl} \partial_p u_{pH1} (\delta_{ik} - n_i n_k) (\delta_{jl} - n_j n_l) \\ &= \mathcal{D}_i[u_{lH1} (\delta_{jl} - n_j n_l)] - \mathcal{D}_i[(\delta_{jl} - n_j n_l)] u_{lH1} + \mathcal{D}_j[u_{kH1} (\delta_{ik} - n_i n_k)] \\ &\quad - \mathcal{D}_j[(\delta_{ik} - n_i n_k)] u_{kH1} - \frac{2}{3} \partial_p u_{pH1} (\delta_{ij} - n_i n_j). \end{aligned}$$

Then, again, apply the formulas in Sec. 2.12, after substituting Eq. (2.45b). The desired equation is obtained after a long but straightforward manipulation.

Finally, Eq. (2.89) can be easily obtained by first substituting Eq. (2.45c) and then applying the formulas in Sec. 2.12. $\square$

2.15 Appendix G: Second-order Knudsen-layer problems

In view of the parity with respect to the tangential component of molecular velocity to the boundary, the second order Knudsen-layer correction ϕ_{K2} can be split into four parts:

$$\phi_{K2} = \phi_{K2}^{(0)} + \phi_{K2}^{(1)} + \phi_{K2}^{(2)} + \phi_{K2}^{(3)},$$

each part of which is expressed as follows:

$$\begin{aligned} \phi_{K2}^{(0)} &= \phi_1^{(0)} \partial_i \tau_{H1} n_i + \phi_2^{(0)} \partial_i \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) + \phi_3^{(0)} \partial_i \overline{\partial_j u_{kH0} n_i n_j n_k} + \phi_4^{(0)} \overline{\kappa} \partial_i \tau_{H0} n_i \\ &\quad + \phi_5^{(0)} \partial_i u_{iH1} + \phi_6^{(0)} \Delta \tau_{H0}, \\ \phi_{K2}^{(1)} &= \zeta_i \left(\phi_1^{(1)} \overline{\partial_j u_{kH1} n_k} (\delta_{ij} - n_i n_j) + \phi_2^{(1)} \partial_j \tau_{H1} (\delta_{ij} - n_i n_j) + \phi_3^{(1)} \partial_j \partial_k \tau_{H0} n_k (\delta_{ij} - n_i n_j) \right. \\ &\quad + \phi_4^{(1)} \partial_j \overline{\partial_k u_{lH0}} (\delta_{ik} - n_i n_k) n_j n_l + \phi_5^{(1)} \overline{\kappa} \partial_j u_{kH0} n_k (\delta_{ij} - n_i n_j) + \phi_6^{(1)} \kappa_{ij} \overline{\partial_j u_{kH0} n_k} \\ &\quad \left. + \phi_7^{(1)} \kappa_{ij} \partial_j \tau_{H0} + \phi_8^{(1)} \overline{\kappa} \partial_j \tau_{H0} (\delta_{ij} - n_i n_j) \right), \\ \phi_{K2}^{(2)} &= \left(\zeta_i \zeta_j - \frac{1}{2} (|\zeta|^2 - \zeta_n^2) \delta_{ij} \right) \left(\phi_1^{(2)} \kappa_{ij} \partial_k \tau_{H0} n_k + (\phi_2^{(2)} \partial_k \partial_l \tau_{H0} \right. \\ &\quad \left. + \phi_3^{(2)} \partial_k \overline{\partial_l u_{pH0} n_p} + \phi_4^{(2)} \partial_p \overline{\partial_l u_{kH0} n_p}) (\delta_{jl} - n_j n_l) (\delta_{ik} - n_i n_k) \right), \\ \phi_{K2}^{(3)} &= \left(\zeta_i \zeta_j \zeta_k - \frac{1}{4} (\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) (|\zeta|^2 - \zeta_n^2) \right) \\ &\quad \times \kappa_{ij} \left(\phi_1^{(3)} \partial_l \tau_{H0} + \phi_2^{(3)} \overline{\partial_l u_{pH0} n_p} \right) (\delta_{kl} - n_k n_l). \end{aligned}$$

Here $\phi_j^{(i)}$ ($i = 0, 1, 2, 3$ and $j = 1, 2, \dots$) are functions of η , ζ_n , and $|\zeta|$ only, which are the solution of the half-space problems listed in the subsequent subsections.

2.15.1 Fundamental solutions in $\phi_{K2}^{(0)}$

Element $\phi_1^{(0)}$ is the solution of the temperature jump (2.23). The other elements $\phi_j^{(0)}$ ($j = 2, 3, \dots, 6$) are respectively the solution of the following problem:

$$\zeta_n \partial_\eta \phi_j^{(0)} = \mathcal{L}[\phi_j^{(0)}] - I_j^{(0)}, \quad (2.90a)$$

$$\phi_j^{(0)} = -\tilde{\mathcal{K}}[|\zeta|^2] c_j^{(0)} + \mathcal{K}[\phi_j^{(0)}] + \tilde{\mathcal{K}}[g_j^{(0)}], \quad \zeta_n > 0, \eta = 0, \quad (2.90b)$$

$$\phi_j^{(0)} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty, \quad (2.90c)$$

where

$$I_2^{(0)} = \frac{1}{2}(|\zeta|^2 - \zeta_n^2) \phi_2^{(1)}, \quad g_2^{(0)} = 2 \int_0^\infty Y_2^{(1)}(z) dz \zeta_n + \frac{1}{2}(|\zeta|^2 - 3\zeta_n^2)(b_2^{(1)}B + F),$$

$$I_3^{(0)} = -\frac{1}{4}(|\zeta|^2 - \zeta_n^2) \phi_1^{(1)},$$

$$g_3^{(0)} = -\int_0^\infty Y_1^{(1)}(z) dz \zeta_n - \frac{1}{4}b_1^{(1)}(|\zeta|^2 - 3\zeta_n^2)B - \frac{1}{2}\zeta_n(D_1 - (|\zeta|^2 - 2\zeta_n^2)D_2),$$

$$I_4^{(0)} = (|\zeta|^2 - \zeta_n^2)(\partial_{\zeta_n} \phi_1^{(0)} - \phi_2^{(1)}), \quad g_4^{(0)} = -4 \int_0^\infty Y_2^{(1)}(z) dz \zeta_n - b_2^{(1)}(|\zeta|^2 - 3\zeta_n^2)B,$$

$$I_5^{(0)} = 0, \quad g_5^{(0)} = -\frac{1}{3}(|\zeta|^2 - 3\zeta_n^2)B, \quad I_6^{(0)} = 0, \quad g_6^{(0)} = -\frac{1}{3}(|\zeta|^2 - 3\zeta_n^2)F + F_d,$$

and $c_j^{(0)}$ is a constant to be determined together with the solution $\phi_j^{(0)}$. It should be noted that, by taking the $\langle \dots \rangle$ moment of Eq. (2.90a) multiplied by 1, ζ_n , and $|\zeta|^2$ (conservation laws) and taking into account Eqs. (2.30a) and (2.30b), we have the relations that

$$\begin{aligned} \langle \zeta_n \phi_2^{(0)} \rangle &= \int_\eta^\infty Y_2^{(1)}(z) dz, \quad \langle \zeta_n^2 \phi_2^{(0)} \rangle = 0, \quad \langle \zeta_n (|\zeta|^2 - \frac{5}{2}) \phi_2^{(0)} \rangle = \int_\eta^\infty H_2^{(1)}(z) dz, \\ \langle \zeta_n \phi_3^{(0)} \rangle &= -\frac{1}{2} \int_\eta^\infty Y_1^{(1)}(z) dz, \quad \langle \zeta_n^2 \phi_3^{(0)} \rangle = 0, \quad \langle \zeta_n (|\zeta|^2 - \frac{5}{2}) \phi_3^{(0)} \rangle = -\frac{1}{2} \int_\eta^\infty H_1^{(1)}(z) dz, \\ \langle \zeta_n \phi_4^{(0)} \rangle &= -2 \int_\eta^\infty Y_2^{(1)}(z) dz, \quad \langle \zeta_n^2 \phi_4^{(0)} \rangle = -\int_\eta^\infty \Pi_1^{(0)}(z) dz, \\ \langle \zeta_n (|\zeta|^2 - \frac{5}{2}) \phi_4^{(0)} \rangle &= -2 \int_\eta^\infty H_2^{(1)}(z) dz, \\ \langle \zeta_n \phi_5^{(0)} \rangle &= 0, \quad \langle \zeta_n^2 \phi_5^{(0)} \rangle = 0, \quad \langle \zeta_n (|\zeta|^2 - \frac{5}{2}) \phi_5^{(0)} \rangle = 0, \\ \langle \zeta_n \phi_6^{(0)} \rangle &= 0, \quad \langle \zeta_n^2 \phi_6^{(0)} \rangle = 0, \quad \langle \zeta_n (|\zeta|^2 - \frac{5}{2}) \phi_6^{(0)} \rangle = 0. \end{aligned}$$

2.15.2 Fundamental solutions in $\phi_{K2}^{(1)}$

Elements $\phi_1^{(1)}$ and $\phi_2^{(1)}$ are the solution of the shear-slip (2.24) and thermal-slip (2.25), respectively. Because of Eq. (2.65), the other elements $\phi_j^{(1)}$ ($j = 3, 4, \dots, 8$) are respectively

the solution of the following problem:

$$\zeta_n \partial_\eta \phi_j^{(1)} = \mathcal{L}_1[\phi_j^{(1)}] - I_j^{(1)}, \quad (2.91a)$$

$$\phi_j^{(1)} = -2\widetilde{\mathcal{K}}_1[1] b_j^{(1)} + \mathcal{K}_1[\phi_j^{(1)}] + \widetilde{\mathcal{K}}_1[g_j^{(1)}], \quad \zeta_n > 0, \eta = 0, \quad (2.91b)$$

$$\phi_j^{(1)} \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty, \quad (2.91c)$$

where

$$\begin{aligned} I_3^{(1)} &= \phi_1^{(0)}, \quad g_3^{(1)} = 2\zeta_n F, \quad I_4^{(1)} = 0, \quad g_4^{(1)} = -(D_1 + \zeta_n^2 D_2), \\ I_5^{(1)} &= \frac{1}{2}(|\zeta|^2 - \zeta_n^2) \partial_{\zeta_n} \phi_1^{(1)}, \quad g_5^{(1)} = -2D_1 - \frac{1}{2}(|\zeta|^2 - \zeta_n^2) D_2, \\ I_6^{(1)} &= \frac{1}{2} \partial_{\zeta_n} \left((|\zeta|^2 - \zeta_n^2) \phi_1^{(1)} \right), \quad g_6^{(1)} = -D_1 - \frac{1}{2}(|\zeta|^2 - 3\zeta_n^2) D_2, \\ I_7^{(1)} &= \frac{1}{2} \partial_{\zeta_n} \left((|\zeta|^2 - \zeta_n^2) \phi_2^{(1)} \right) + \phi_1^{(0)}, \quad g_7^{(1)} = 0, \quad I_8^{(1)} = \frac{1}{2}(|\zeta|^2 - \zeta_n^2) \partial_{\zeta_n} \phi_2^{(1)}, \quad g_8^{(1)} = 0, \end{aligned}$$

and $b_j^{(1)}$ is a constant to be determined together with the solution $\phi_j^{(1)}$. Again, by taking the $\langle \cdots \rangle$ moment of Eq. (2.91a) multiplied by $(|\zeta|^2 - \zeta_n^2)$ (momentum conservation) and taking into account Eqs. (2.30a) and (2.30b), we have the relations that

$$\begin{aligned} \langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_3^{(1)} \rangle &= \int_\eta^\infty \Pi_1^{(0)}(z) dz, \quad \langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_4^{(1)} \rangle = 0, \quad \langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_5^{(1)} \rangle = 0, \\ \langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_6^{(1)} \rangle &= 0, \quad \langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_7^{(1)} \rangle = \int_\eta^\infty \Pi_1^{(0)}(z) dz, \quad \langle \zeta_n (|\zeta|^2 - \zeta_n^2) \phi_8^{(1)} \rangle = 0. \end{aligned}$$

2.15.3 Fundamental solutions in $\phi_{K2}^{(2)}$

Thanks to Eq. (2.68), the problem for $\phi_j^{(2)}$ ($j = 1, 2, 3, 4$) is reduced to

$$\begin{aligned} \zeta_n \partial_\eta \phi_j^{(2)} &= \mathcal{L}_2[\phi_j^{(2)}] - I_j^{(2)}, \\ \phi_j^{(2)} &= \mathcal{K}_2[\phi_j^{(2)}] + \widetilde{\mathcal{K}}_2[g_j^{(2)}], \quad \zeta_n > 0, \eta = 0, \\ \phi_j^{(2)} &\rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty, \end{aligned}$$

where

$$\begin{aligned} I_1^{(2)} &= \partial_{\zeta_n} \phi_1^{(0)} - \phi_2^{(1)}, \quad g_1^{(2)} = -b_2^{(1)} B, \quad I_2^{(2)} = \phi_2^{(1)}, \quad g_2^{(2)} = b_2^{(1)} B + F, \\ I_3^{(2)} &= \phi_1^{(1)}, \quad g_3^{(2)} = b_1^{(1)} B - \zeta_n D_2, \quad I_4^{(2)} = 0, \quad g_4^{(2)} = -\frac{1}{2} \zeta_n D_2. \end{aligned}$$

2.15.4 Fundamental solutions in $\phi_{K2}^{(3)}$

Thanks to Eq. (2.69), the problem for $\phi_j^{(3)}$ ($j = 1, 2$) is reduced to

$$\begin{aligned}\zeta_n \partial_\eta \phi_j^{(3)} &= \mathcal{L}_4[\phi_j^{(3)}] - I_j^{(3)}, \\ \phi_j^{(3)} &= \mathcal{K}_4[\phi_j^{(3)}] + \widetilde{\mathcal{K}}_4[g_j^{(3)}], \quad \zeta_n > 0, \quad \eta = 0, \\ \phi_j^{(3)} &\rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty,\end{aligned}$$

where

$$I_1^{(3)} = \partial_{\zeta_n} \phi_2^{(1)}, \quad g_1^{(3)} = 0, \quad I_2^{(3)} = \partial_{\zeta_n} \phi_1^{(1)}, \quad g_2^{(3)} = -D_2.$$

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Chapter 3

Second-order Knudsen-layer analysis for the generalized slip-flow theory I

3.1 Introduction

Study on the connection between the kinetic theory and the fluid-dynamics has a long history [1,2,3] and a number of important results have been obtained for a small or a vanishing limit of the Knudsen number, e.g., Refs. [4,5,6,7,8,9]. Mathematical studies are developing to include the argument of boundary condition for the fluid-dynamical equation [10,11,12]. It seems, however, that they are mainly concerned with the vanishing limit of the Knudsen number and are not intended to describe the gas rarefaction effect itself. In the meantime, a systematic asymptotic theory established in the late 1960s and early 1970s [4,5] and developed further since then [6] provides not only the fluid-dynamic description in the bulk region but also the slip/jump boundary condition and the non-fluid-like correction in a thin layer adjacent to the boundary (the Knudsen layer) for small Knudsen numbers. We call the linear case of this theory the generalized slip-flow theory in the present paper. The generalized slip-flow theory contains rich information on the gas rarefaction effect, giving a fluid-dynamical interpretation to various phenomena occurring for small Knudsen numbers. It also motivates recent studies on time-dependent problems in the framework of the theory [13,14,15].

The theory has been applied to various fundamental problems and is revealed to be practical. However, such applications have been limited mostly to the studies based on the Bhatnagar–Gross–Krook (BGK) [or Boltzmann–Krook–Wendler (BKW)] model equation [16,17]. This is due to lack of numerical data at the second order of the Knudsen number for the original Boltzmann equation. Some difficulties in numerical analyses, which could be bypassed for the BGK model, have prevented the preparation of the desired data. Here, the difficulties do not mean a mere numerical cost but rather mean possible singularities inherent in the elemental (or component) half-space problems to be solved.

The present paper reports the first part of our recent attempts to prepare the missing

data that are necessary in applying the theory to specific problems on the basis of the original Boltzmann equation. Assuming the hard-sphere molecules and the diffuse reflection boundary condition, we provide the data for the generalized slip-flow theory, up to the second order of the Knudsen number expansion, except for the curvature effects.

3.2 Generalized slip-flow theory: An outline

3.2.1 Physical setting

We consider a gas around smooth solid bodies which do not change in time their shape and position under the following assumptions: (i) There is no external force and the behavior of the gas is described by the Boltzmann equation for monatomic molecules; (ii) The gas molecules are reflected locally isotropically [6] on the surface of the solid bodies (no net flow across their surface); (iii) The deviation from the reference equilibrium state at rest with density ρ_0 and temperature T_0 is so small that the equation and the initial and the boundary condition can be linearized around that equilibrium state; (iv) The mean free path ℓ_0 of a molecule at the reference equilibrium state is much smaller than the characteristic length L of the physical system (i.e., the Knudsen number $\text{Kn} = \ell_0/L \ll 1$); (v) The time evolution is initiated by a slow change of the surroundings from the reference equilibrium state.

The time scale of the change t_0 in (v) is the same order as that of the viscous and the thermal diffusion, and we set it as $t_0 = (2/\sqrt{\pi})(L^2/\ell_0\sqrt{2RT_0})$, where R is the specific gas constant (the Boltzmann constant k divided by the mass of a molecule m). In the actual computations for the Knudsen-layer analysis, we assume the hard-sphere molecules in (i) and the diffuse reflection boundary condition in (ii). For hard-sphere molecules, $\ell_0 = [\sqrt{2}\pi d_m^2(\rho_0/m)]^{-1}$, where d_m is the diameter of a molecule.

3.2.2 Resulting framework

The generalized slip-flow theory consists of first considering the overall behavior of the gas that changes in the scale of the characteristic length (and time) of the system and then introducing the correction in the vicinity of the boundary. The first part is conducted by the Hilbert (or Grad–Hilbert) expansion to yield a set of fluid-dynamic equations. The solution of this set is called the Hilbert part (or solution). The second part is conducted by the

expansion after rescaling (actually stretching) the spatial coordinate in the direction normal to the boundary. This yields a set of slip/jump boundary condition and the associated correction to the fluid-dynamic solution near the boundary. The correction is called the Knudsen-layer correction, which is the primary concern in the present paper.

Let us denote by Lx_i the space coordinates, by $t_0 t$ the time, by $\rho_0(1+\omega)$ the density of the gas, by $(2RT_0)^{1/2}u_i$ the flow velocity, by $T_0(1+\tau)$ the temperature, by $p_0(1+P)$ the pressure with $p_0 = \rho_0 RT_0$, and by $(2RT_0)^{1/2}u_{iw}$ and $T_0(1+\tau_w)$ the velocity and the temperature of the body surface. Since solid bodies change neither the shape nor the position, $u_{iw}n_i = 0$, where n_i is the unit vector normal to the surface, pointed to the gas. We denote the Hilbert part and the Knudsen-layer correction of the macroscopic quantity h ($h = \omega, u_i, \tau, P$) by h_H and h_K , respectively: $h = h_H + h_K$. Since the Knudsen number is small, we use $\varepsilon = (\sqrt{\pi}/2)\text{Kn}$ as a small parameter. The Hilbert part and the Knudsen-layer correction are obtained by a power series expansion in ε , which will be denoted as $h_H = h_{H0} + h_{H1}\varepsilon + h_{H2}\varepsilon^2 + \dots$ and $h_K = h_{K0} + h_{K1}\varepsilon + h_{K2}\varepsilon^2 + \dots$. Actually, the expansion of h_K starts from $O(\varepsilon)$, because no correction to the Hilbert solution is required at $O(1)$.

The resulting set of fluid-dynamic equations, their slip/jump boundary conditions, and the Knudsen-layer corrections up to the second order of the expansion in ε are summarized as follows: [13]

Fluid-dynamic equations

$$\frac{\partial P_{H0}}{\partial x_i} = 0, \quad (3.1a)$$

$$\frac{\partial u_{iHm}}{\partial x_i} + \frac{\partial \omega_{Hm-1}}{\partial t} = 0, \quad (3.1b)$$

$$\frac{\partial u_{iHm}}{\partial t} + \frac{1}{2} \frac{\partial P_{Hm+1}^*}{\partial x_i} - \frac{1}{2} \gamma_1 \frac{\partial^2 u_{iHm}}{\partial x_j^2} + \frac{1}{4} (\gamma_1 \gamma_{10} - 2\gamma_6) \frac{\partial^4 u_{iHm-2}}{\partial x_j^2 \partial x_k^2} = 0, \quad (3.1c)$$

$$\frac{\partial \tau_{Hm}}{\partial t} - \frac{2}{5} \frac{\partial P_{Hm}}{\partial t} - \frac{1}{2} \gamma_2 \frac{\partial^2 \tau_{Hm}}{\partial x_j^2} + \frac{1}{10} \left(\gamma_2 \gamma_3 - \frac{13}{2} \gamma_{11} \right) \frac{\partial^4 \tau_{Hm-2}}{\partial x_j^2 \partial x_k^2} = 0, \quad (3.1d)$$

$$P_{Hm+1}^* = P_{Hm+1} - \frac{1}{6} (\gamma_2 \gamma_1 - 4\gamma_3) \frac{\partial^2 \tau_{Hm-1}}{\partial x_j^2} + \frac{1}{5} \gamma_1 \frac{\partial P_{Hm-1}}{\partial t}, \quad (3.1e)$$

$$P_{Hm} = \omega_{Hm} + \tau_{Hm}, \quad (3.1f)$$

where $m = 0, 1, 2$ and the quantities h_{H-1} and h_{H-2} ($h = \omega, u_i, \tau, P$) should be read as zero. The γ 's occurring in the equations are positive constants corresponding to the transport coefficients at the reference state; γ 's are all unity for the BGK model, while they are

respectively $\gamma_1 = 1.270042427$, $\gamma_2 = 1.922284066$, $\gamma_3 = 1.947906335$, $\gamma_6 = 1.419423836$, $\gamma_{10} = 1.63607346$, and $\gamma_{11} = 2.7931173$ for hard-sphere molecules.

Slip/jump boundary condition and the Knudsen-layer correction

$$\begin{aligned} \begin{bmatrix} (u_{iHm} - u_{iwm})t_i \\ u_{iKm}t_i \end{bmatrix} &= \frac{\overline{\partial u_{iHm-1}}}{\partial x_j} n_i t_j \begin{bmatrix} b_1^{(1)} \\ Y_1^{(1)}(\eta) \end{bmatrix} + \frac{\partial \tau_{Hm-1}}{\partial x_i} t_i \begin{bmatrix} b_2^{(1)} \\ Y_2^{(1)}(\eta) \end{bmatrix} \\ &+ \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} n_i t_j \begin{bmatrix} b_3^{(1)} \\ Y_3^{(1)}(\eta) \end{bmatrix} + \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j t_k \begin{bmatrix} b_4^{(1)} \\ Y_4^{(1)}(\eta) \end{bmatrix} \\ &+ \bar{\kappa} \frac{\overline{\partial u_{iHm-2}}}{\partial x_j} n_i t_j \begin{bmatrix} b_5^{(1)} \\ Y_5^{(1)}(\eta) \end{bmatrix} + \kappa_{ij} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_k t_i \begin{bmatrix} b_6^{(1)} \\ Y_6^{(1)}(\eta) \end{bmatrix} \\ &+ \kappa_{ij} \frac{\partial \tau_{Hm-2}}{\partial x_i} t_j \begin{bmatrix} b_7^{(1)} \\ Y_7^{(1)}(\eta) \end{bmatrix} + \bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} t_i \begin{bmatrix} b_8^{(1)} \\ Y_8^{(1)}(\eta) \end{bmatrix}, \end{aligned} \quad (3.2a)$$

$$\begin{aligned} \begin{bmatrix} u_{iHm} n_i \\ u_{iKm} n_i \end{bmatrix} &= \frac{1}{2} \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j n_k \begin{bmatrix} \int_0^\infty Y_1^{(1)}(z) dz \\ - \int_\eta^\infty Y_1^{(1)}(z) dz \end{bmatrix} \\ &+ \left[2\bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} n_i - \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} (\delta_{ij} - n_i n_j) \right] \begin{bmatrix} \int_0^\infty Y_2^{(1)}(z) dz \\ - \int_\eta^\infty Y_2^{(1)}(z) dz \end{bmatrix}, \end{aligned} \quad (3.2b)$$

$$\begin{aligned} \begin{bmatrix} \tau_{Hm} - \tau_{wm} \\ \omega_{Km} \\ \tau_{Km} \end{bmatrix} &= \frac{\partial \tau_{Hm-1}}{\partial x_i} n_i \begin{bmatrix} c_1^{(0)} \\ \Omega_1^{(0)}(\eta) \\ \Theta_1^{(0)}(\eta) \end{bmatrix} + \frac{\partial u_{iHm-1}}{\partial x_i} \begin{bmatrix} c_5^{(0)} \\ \Omega_5^{(0)}(\eta) \\ \Theta_5^{(0)}(\eta) \end{bmatrix} \\ &+ \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} (\delta_{ij} - n_i n_j) \begin{bmatrix} c_2^{(0)} \\ \Omega_2^{(0)}(\eta) \\ \Theta_2^{(0)}(\eta) \end{bmatrix} + \frac{\partial^2 \tau_{Hm-2}}{\partial x_j^2} \begin{bmatrix} c_6^{(0)} \\ \Omega_6^{(0)}(\eta) \\ \Theta_6^{(0)}(\eta) \end{bmatrix} \\ &+ \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j n_k \begin{bmatrix} c_3^{(0)} \\ \Omega_3^{(0)}(\eta) \\ \Theta_3^{(0)}(\eta) \end{bmatrix} + \bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} n_i \begin{bmatrix} c_4^{(0)} \\ \Omega_4^{(0)}(\eta) \\ \Theta_4^{(0)}(\eta) \end{bmatrix}, \end{aligned} \quad (3.2c)$$

$$P_{Km} = \omega_{Km} + \tau_{Km}, \quad (3.2d)$$

where $m = 0, 1, 2$ and $\overline{f_{ij}} = f_{ij} + f_{ji} - (2/3)f_{kk}\delta_{ij}$ (δ_{ij} is the Kronecker delta). In (3.2), the quantities with the subscript H or w denote their values at the (dimensionless) surface position x_{iw} , and t_i (or n_i) is a unit vector tangential (or normal) to the surface at x_{iw} . The quantities with the subscript K depend on η as well as x_{iw} , where η is the stretched spatial coordinate normal to the surface at x_{iw} such that the position x_i in the Knudsen layer is expressed by $x_i = x_{iw} + \varepsilon \eta n_i$. The surface velocity and temperature are also expanded in a power series of ε : $h_w = h_{w0} + h_{w1}\varepsilon + h_{w2}\varepsilon^2 + \dots$ ($h = u_i, \tau$). The effect of surface curvature

occurs through the terms with κ 's defined by

$$\bar{\kappa} = \frac{1}{2}(\kappa_1 + \kappa_2), \quad \kappa_{ij} = \kappa_1 \ell_i \ell_j + \kappa_2 m_i m_j. \quad (3.3)$$

Here κ_1/L and κ_2/L are the principal curvatures of the boundary, with κ_1 and κ_2 being taken negative when the corresponding center of curvature lies on the gas side; ℓ_i and m_i are the direction cosines of the principal directions corresponding to κ_1 and κ_2 respectively.

Many terms degenerate from Eqs. (3.1) and (3.2) when $m = 0, 1$. For instance, the boundary condition at $O(1)$, namely for $m = 0$, is none other than the non-slip/non-jump condition; the slip/jump may occur at the first or higher order of ε . The reader is referred to Refs. [6] and [13] for the discussions on the main features of the above system. The system can be solved from the lowest order to determine the time-dependent behavior of the gas under the considered situation, provided that the data of the slip/jump coefficients $c_1^{(0)} \sim c_6^{(0)}$, $b_1^{(1)} \sim b_8^{(1)}$, $\int_0^\infty Y_1^{(1)}(z)dz$, $\int_0^\infty Y_2^{(1)}(z)dz$ and the elemental (or component) Knudsen-layer functions $\Omega_1^{(0)}(\eta) \sim \Omega_6^{(0)}(\eta)$, $\Theta_1^{(0)}(\eta) \sim \Theta_6^{(0)}(\eta)$, $Y_1^{(1)}(\eta) \sim Y_8^{(1)}(\eta)$ are available. They are obtained by solving elemental (or component) half-space problems of the linearized Boltzmann equation, which may be homogeneous or inhomogeneous depending on that component.

3.2.3 Remaining issue

Since the first publication of the generalized slip-flow theory [4,5], the complete set of the slip/jump coefficients and the associated Knudsen-layer corrections has been available only for the BGK model. For the original Boltzmann equation, difficulties in directly solving the component problems have prevented preparing the corresponding data. Even for hard-sphere molecules under the diffuse reflection condition, the component problems have been tackled only for the problems of temperature jump ($c_1^{(0)}, \Omega_1^{(0)}, \Theta_1^{(0)}$), shear slip ($b_1^{(1)}, Y_1^{(1)}$), thermal slip ($b_2^{(1)}, Y_2^{(1)}$), thermal-stress slip ($b_3^{(1)}, Y_3^{(1)}$), and thermal inertia due to time evolution ($c_6^{(0)}, \Omega_6^{(0)}, \Theta_6^{(0)}$) [18,19,20,21].

Recently, we have succeeded in Ref. [13] to prepare the complete set of slip/jump coefficients up to the second order of ε by making use of the theory of symmetry relation [22]. This method is, however, indirect, and the information on the Knudsen-layer structure at the second order is still far from complete. In the present work, we numerically solve the component problems at the second order of ε . We shall focus on the component problems

related to $(c_2^{(0)}, \Omega_2^{(0)}, \Theta_2^{(0)})$, $(c_3^{(0)}, \Omega_3^{(0)}, \Theta_3^{(0)})$, $(c_5^{(0)}, \Omega_5^{(0)}, \Theta_5^{(0)})$, and $(b_4^{(1)}, Y_4^{(1)})$ only. These problems might be handled by the finite-difference method as in Ref. [20]. Nevertheless, we take a new approach that makes use of the integral formulation of the Boltzmann equation, the formulation that has been effective in the study of singularities [23,24]. The primary motivation of this approach is that the remaining component problems, which are related to the curvature effect and left to a separate paper, contain a much delicate matter, requiring a new methodology for the numerical analysis.

3.3 Knudsen-layer problems

Let us denote by $(2RT_0)^{1/2}\zeta$ the molecular velocity and introduce the notation $\zeta = |\zeta|$ and $\mu = \zeta_i n_i / \zeta$, where $-1 \leq \mu \leq 1$ and $0 \leq \zeta < \infty$. Then, the component problems for the Knudsen layer admit a similarity solution of three arguments (η, μ, ζ) and are reduced to the following two types of boundary-value problems by assuming the hard-sphere molecules and the diffuse reflection boundary condition:

$$\mu\zeta \frac{\partial \phi_\alpha}{\partial \eta} = -\nu(\zeta)\phi_\alpha + \mathcal{C}[\phi_\alpha] - \mathcal{I}_\alpha(\eta, \mu, \zeta), \quad (3.4a)$$

$$\phi_\alpha = -\sigma_\alpha^{(0)} - c_\alpha^{(0)}\zeta^2 + g_\alpha(\mu, \zeta), \quad (\mu\zeta > 0, \eta = 0), \quad (3.4b)$$

$$\phi_\alpha \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \quad (3.4c)$$

and

$$\mu\zeta \frac{\partial \psi_\beta}{\partial \eta} = -\nu(\zeta)\psi_\beta + \mathcal{C}^S[\psi_\beta] - \mathcal{I}_\beta^S(\eta, \mu, \zeta), \quad (3.5a)$$

$$\psi_\beta = -2b_\beta^{(1)} + g_\beta^S(\mu, \zeta), \quad (\mu\zeta > 0, \eta = 0), \quad (3.5b)$$

$$\psi_\beta \rightarrow 0, \quad \text{as } \eta \rightarrow \infty. \quad (3.5c)$$

Here

$$\nu(\zeta) = \frac{1}{2\sqrt{2}} \left[\exp(-\zeta^2) + \left(2\zeta + \frac{1}{\zeta} \right) \int_0^\zeta \exp(-\xi^2) d\xi \right];$$

$\mathcal{C}$ is an integral operator acting on a function of ζ to be defined soon later; and g_α , $\mathcal{I}_\alpha$, g_β^S , and $\mathcal{I}_\beta^S$ are given functions. $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta^S$ are supposed to decay fast in η . The solution ϕ_α (or ψ_β) ($\alpha = 1, \dots, 6$; $\beta = 1, \dots, 8$) is a function of η , μ , and ζ . It is determined together with the constants $\sigma_\alpha^{(0)}$ and $c_\alpha^{(0)}$ [or $b_\beta^{(1)}$] for every given $(\mathcal{I}_\alpha, g_\alpha)$ [or $(\mathcal{I}_\beta^S, g_\beta^S)$] [25]. The operator $\mathcal{C}$

is defined by

$$\begin{aligned}\mathcal{C}[\phi](\zeta) &= \int [k_1(\zeta, \xi) - k_2(\zeta, \xi)]\phi(\xi)d\xi, \\ k_1(\zeta, \xi) &= \frac{1}{\sqrt{2\pi}|\zeta - \xi|} \exp\left(-|\xi|^2 + \frac{|\xi \times \zeta|^2}{|\xi - \zeta|^2}\right), \\ k_2(\zeta, \xi) &= \frac{|\zeta - \xi|}{2\sqrt{2\pi}} \exp(-|\xi|^2).\end{aligned}$$

Thanks to its spherical and axial symmetry, $\mathcal{C}[\phi_\alpha]$ becomes a function of μ and ζ (and η) for the function $\phi_\alpha(\eta, \mu, \zeta)$. On the other hand, $\mathcal{C}^S$ is defined for functions of μ and ζ (and η) through $\mathcal{C}$, making use of its axial symmetry:

$$\zeta_i t_i \mathcal{C}^S[\psi_\beta] = \mathcal{C}[\zeta_i t_i \psi_\beta].$$

Once the solutions are obtained, $\Omega_\alpha^{(0)}(\eta)$, $\Theta_\alpha^{(0)}(\eta)$, and $Y_\beta^{(1)}(\eta)$ are obtained as their moment:

$$\Omega_\alpha^{(0)}(\eta) = \langle \phi_\alpha \rangle, \quad \Theta_\alpha^{(0)}(\eta) = \frac{2}{3} \langle (\zeta^2 - \frac{3}{2}) \phi_\alpha \rangle, \quad (3.6)$$

$$Y_\beta^{(1)}(\eta) = \frac{1}{2} \langle \zeta^2 (1 - \mu^2) \psi_\beta \rangle, \quad (3.7)$$

where $\langle f \rangle = \int f(\xi) E(|\xi|) d\xi$ with $E(z) = \pi^{-3/2} \exp(-z^2)$.

Our present concern is the case $\alpha = 2, 3, 5$ and $\beta = 4$, for which $(\mathcal{I}_\alpha, g_\alpha)$ and $(\mathcal{I}_\beta^S, g_\beta^S)$ are given as

$$\mathcal{I}_2 = \frac{1}{2} \zeta^2 (1 - \mu^2) \psi_2(\eta, \mu, \zeta), \quad (3.8a)$$

$$g_2 = 2\mu\zeta \int_0^\infty Y_2^{(1)}(z) dz + \frac{1}{2} \zeta^2 (1 - 3\mu^2) [b_2^{(1)} B(\zeta) + F(\zeta)], \quad (3.8b)$$

$$\mathcal{I}_3 = -\frac{1}{4} \zeta^2 (1 - \mu^2) \psi_1(\eta, \mu, \zeta), \quad (3.8c)$$

$$\begin{aligned}g_3 &= -\mu\zeta \int_0^\infty Y_1^{(1)}(z) dz - \frac{1}{4} b_1^{(1)} \zeta^2 (1 - 3\mu^2) B(\zeta) \\ &\quad - \frac{1}{2} \mu\zeta [D_1(\zeta) - \zeta^2 (1 - 2\mu^2) D_2(\zeta)],\end{aligned} \quad (3.8d)$$

$$\mathcal{I}_5 = 0, \quad g_5 = -\frac{1}{3} \zeta^2 (1 - 3\mu^2) B(\zeta), \quad (3.8e)$$

$$\mathcal{I}_4^S = 0, \quad g_4^S = -[D_1(\zeta) + \mu^2 \zeta^2 D_2(\zeta)]. \quad (3.8f)$$

Here ψ_1 and ψ_2 are respectively the solution of the problem (3.5) with

$$\mathcal{I}_1^S = 0, \quad g_1^S = \mu\zeta B(\zeta), \quad \mathcal{I}_2^S = 0, \quad g_2^S = A(\zeta), \quad (3.9)$$

(the so-called shear-slip and thermal-slip problems [26,6]), and $(b_1^{(1)}, Y_1^{(1)})$ and $(b_2^{(1)}, Y_2^{(1)})$ are their associated slip coefficient and Knudsen-layer function, all of which have already been obtained in Ref. [19]. Hence $\mathcal{I}_2$ and $\mathcal{I}_3$ in Eq. (3.8) indeed decay fast in η . The functions A , B , F , D_1 , and D_2 of ζ are familiar solutions of the following integral equations:

$$\begin{aligned}\mathcal{L}[\zeta_i A(\zeta)] &= -\zeta_i(\zeta^2 - 5/2), \quad \text{subsidiary condition: } \langle \zeta^2 A(\zeta) \rangle = 0, \\ \mathcal{L}[\zeta_{ij} B(\zeta)] &= -2\zeta_{ij}, \quad \mathcal{L}[\zeta_{ij} F(\zeta)] = \zeta_{ij} A(\zeta), \\ \mathcal{L}[(\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) D_1(\zeta) + \zeta_i \zeta_j \zeta_k D_2(\zeta)] &= \gamma_1(\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) - \zeta_i \zeta_j \zeta_k B(\zeta), \\ \text{subsidiary condition: } \langle 5\zeta^2 D_1(\zeta) + \zeta^4 D_2(\zeta) \rangle &= 0,\end{aligned}$$

where $\mathcal{L}[f] = -\nu(\zeta)f + \mathcal{C}[f]$ and $\zeta_{ij} = \zeta_i \zeta_j - (1/3)\zeta^2 \delta_{ij}$.

3.4 Numerical analysis

As mentioned at the end of Sec. 3.2.3, our numerical method is based on the integral form of Eqs. (3.4) and (3.5).

3.4.1 Integral formulation

Multiplied by $E(\zeta)$ and integrated with respect to η , both of the problems (3.4) and (3.5) are transformed into

$$\begin{aligned}\Phi(\eta, \mu, \zeta) &= G(\mu, \zeta) \exp\left(-\frac{\nu(\zeta)\eta}{\mu\zeta}\right) + \frac{1}{\mu\zeta} \int_0^\eta C[\Phi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds \\ &\quad + \frac{1}{\mu\zeta} \int_0^\eta C[\Psi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, \quad (\mu\zeta > 0),\end{aligned}\tag{3.10a}$$

$$\begin{aligned}\Phi(\eta, \mu, \zeta) &= \frac{1}{\mu\zeta} \int_\infty^\eta C[\Phi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds \\ &\quad + \frac{1}{\mu\zeta} \int_\infty^\eta C[\Psi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, \quad (\mu\zeta < 0),\end{aligned}\tag{3.10b}$$

with

$$\Phi(\eta, \mu, \zeta) \rightarrow 0, \quad \text{as } \eta \rightarrow \infty,\tag{3.10c}$$

$$\Psi(\eta, \mu, \zeta) = \begin{cases} -\frac{1}{\mu\zeta} \int_0^\eta I(s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, & (\mu\zeta > 0), \\ -\frac{1}{\mu\zeta} \int_\infty^\eta I(s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, & (\mu\zeta < 0). \end{cases}\tag{3.10d}$$

Here, Φ , C , I , and G should be read as $\Phi = \phi_\alpha E - \Psi$, $C[f] = \mathcal{C}[fE^{-1}]E$, $I = \mathcal{I}_\alpha E$, and $G = (-\sigma_\alpha^{(0)} - c_\alpha^{(0)}\zeta^2 + g_\alpha)E$ ($\alpha = 2, 3, 5$) for problem (3.4), while they should be read as $\Phi = \psi_\beta E - \Psi$, $C[f] = \mathcal{C}^S[fE^{-1}]E$, $I = \mathcal{I}_\beta^S E$, and $G = (-2b_\beta^{(1)} + g_\beta^S)E$ ($\beta = 4$) for problem (3.5). Remember that I decays fast in η , so does Ψ . In the above, the original solution $\phi_\alpha E$ or $\psi_\beta E$ is split into the given part Ψ and unknown part Φ . The equation for Φ is none other than the integral form of the inhomogeneous Boltzmann equation with $C[\Psi]$ being its inhomogeneous term. The conditions (3.4c) and (3.5c) are reduced to Eq. (3.10c), because $\Psi \rightarrow 0$ as $\eta \rightarrow \infty$. The condition (3.10c) is required, otherwise the constants $\sigma_\alpha^{(0)}$, $c_\alpha^{(0)}$, and $b_\beta^{(1)}$ are not determined [25].

Since C is the integral operator, $C[f]$ is mild even if its argument function f is not; thus the factor of steep variation of Φ in (μ, ζ) is picked up explicitly in the above integral form. When $\Phi = \phi_2 E - \Psi$ or $\phi_3 E - \Psi$, I contains ψ_2 or ψ_1 , the solution of the thermal- or shear-slip problem. In the case, the steep variation of Ψ in (μ, ζ) is, at a glance, less clear than that of Φ . However, since ψ_1 and ψ_2 are respectively the solution of Eq. (3.5) with (3.9), we can rewrite Ψ in a way that the factor of steep variation is explicit. Namely, in the case of $\Phi = \phi_2 E - \Psi$, Ψ can be written as

$$\Psi = -\frac{1-\mu^2}{2\mu}\zeta \left\{ (-2b_2^{(1)} + g_2^S)E(\zeta)\eta \exp\left(-\frac{\nu(\zeta)\eta}{\mu\zeta}\right) + \frac{1}{\mu\zeta} \int_0^\eta (\eta-s)C[\psi_2 E](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds \right\}, \quad (\mu\zeta > 0), \quad (3.11a)$$

$$\Psi = -\frac{1-\mu^2}{2\mu^2} \int_\infty^\eta (\eta-s)C[\psi_2 E](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, \quad (\mu\zeta < 0), \quad (3.11b)$$

where $C[\psi_2 E] = \mathcal{C}^S[\psi_2]E$. In the case of $\Phi = \phi_3 E - \Psi$, the above $(-2b_2^{(1)} + g_2^S)$ and ψ_2 are replaced by $-(1/2)(-2b_1^{(1)} + g_1^S)$ and $-(1/2)\psi_1$.

In this way, with the aid of the integral formulation, we can pick up the factor of steep variation in (μ, ζ) , which will be advantageous when the detailed information on the velocity distribution function is required, especially near the boundary.

3.4.2 Plan of numerical computation

In the computation for solving Eq. (3.10), we first prepare the accurate data of $C[\Psi]$ from Eq. (3.10d). Then, we abandon the condition (3.10c) and solve Eqs. (3.10a) and (3.10b) with G being replaced by $\tilde{G} \equiv (-\tilde{\sigma} - \tilde{c}\zeta^2 + g_\alpha)E$ [or $(-2\tilde{b} + g_\beta^S)E$], where $\tilde{\sigma}$ and $\tilde{c}$ (or $\tilde{b}$)

are given constants. Let us denote by $\tilde{\Phi}$ the (numerical) solution that is obtained with this process. The difference of $\tilde{\sigma}$ and $\tilde{c}$ (or $\tilde{b}$) from the desired $\sigma_\alpha^{(0)}$ and $c_\alpha^{(0)}$ (or $b_\beta^{(1)}$) results in the asymptote of $\tilde{\Phi}$ such that $\tilde{\Phi} \rightarrow (\sigma_\alpha^{(0)} - \tilde{\sigma})E + (c_\alpha^{(0)} - \tilde{c})\zeta^2 E$ [or $2(b_\beta^{(1)} - \tilde{b})E$] as $\eta \rightarrow \infty$. This property allows us to determine the constants $\sigma_\alpha^{(0)}$ and $c_\alpha^{(0)}$ (or $b_\beta^{(1)}$) by

$$\begin{aligned}\sigma_\alpha^{(0)} &= \tilde{\sigma} - 2\pi \int_0^\infty \int_{-1}^1 \zeta^2 (\zeta^2 - \frac{5}{2}) \tilde{\Phi}(\eta \rightarrow \infty, \mu, \zeta) d\mu d\zeta, \\ c_\alpha^{(0)} &= \tilde{c} + \frac{4\pi}{3} \int_0^\infty \int_{-1}^1 \zeta^2 (\zeta^2 - \frac{3}{2}) \tilde{\Phi}(\eta \rightarrow \infty, \mu, \zeta) d\mu d\zeta, \\ \left[\text{or } b_\beta^{(1)} = \tilde{b} + \pi \int_0^\infty \int_{-1}^1 \zeta^4 (1 - \mu^2) \tilde{\Phi}(\eta \rightarrow \infty, \mu, \zeta) d\mu d\zeta, \right]\end{aligned}$$

and, in turn, to determine Φ by

$$\Phi = \tilde{\Phi} - (\sigma_\alpha^{(0)} - \tilde{\sigma})E - (c_\alpha^{(0)} - \tilde{c})\zeta^2 E. \quad \left[\text{or } \Phi = \tilde{\Phi} - 2(b_\beta^{(1)} - \tilde{b})E. \right]$$

For the actual computation, we introduce a finite large distance d from the boundary, beyond which $\tilde{\Phi}$ may be regarded as that at infinity and Ψ may be discarded because of its fast decay in η (see the first paragraph of Sec. 3.4.1). Then, the problem for $\tilde{\Phi}$ is reduced to

$$\begin{aligned}\tilde{\Phi}(\eta, \mu, \zeta) &= \tilde{G}(\mu, \zeta) \exp(-\frac{\nu(\zeta)\eta}{\mu\zeta}) + \frac{1}{\mu\zeta} \int_0^\eta C[\tilde{\Phi}](s, \mu, \zeta) \exp(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}) ds \\ &\quad + \frac{1}{\mu\zeta} \int_0^\eta C[\Psi](s, \mu, \zeta) \exp(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}) ds, \quad (\mu\zeta > 0),\end{aligned}\tag{3.12a}$$

$$\begin{aligned}\tilde{\Phi}(\eta, \mu, \zeta) &= \tilde{\Phi}(d, -\mu, \zeta) \exp(\frac{\nu(\zeta)(d-\eta)}{\mu\zeta}) \\ &\quad + \frac{1}{\mu\zeta} \int_d^\eta C[\tilde{\Phi}](s, \mu, \zeta) \exp(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}) ds \\ &\quad + \frac{1}{\mu\zeta} \int_d^\eta C[\Psi](s, \mu, \zeta) \exp(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}) ds, \quad (\mu\zeta < 0).\end{aligned}\tag{3.12b}$$

Accordingly, ∞ 's in Eq. (3.10d) or (3.11) may be replaced by d because discarding Ψ beyond d implies discarding I beyond d . All the computations are thus reduced inside a finite region $0 \leq \eta \leq d$. The occurrence of $\tilde{\Phi}(d, -\mu, \zeta)$ on the right-hand side of Eq. (3.12b) comes from that $\tilde{\Phi}(d, \mu, \zeta) = \tilde{\Phi}(d, -\mu, \zeta)$ for d sufficiently large, thanks to its asymptotic form for $\eta \rightarrow \infty$. We solve the system (3.12) numerically by iteration for a certain fixed value of d . To be a little more specific, we evaluate $C[\tilde{\Phi}]$ with an initial guess of $\tilde{\Phi}$ and solve Eq. (3.12a) first. This gives the data $\tilde{\Phi}(d, -\mu, \zeta)$ in Eq. (3.12b), which allows us to solve Eq. (3.12b)

next. Then we refine the evaluation of $C[\tilde{\Phi}]$ with the updated $\tilde{\Phi}$, and repeat the same process until the numerical solution satisfies a criteria of convergence. The appropriateness of the value of d is judged a posteriori from the result.

3.4.3 Discretization and some details of numerical method

Thanks to the factor E , $\tilde{\Phi}$ is expected to decay rapidly in ζ . Thus, we truncate the region of ζ at $\zeta = Z(> 0)$ for a properly chosen constant Z and restrict the computation in the region $-1 \leq \mu \leq 1$, $0 \leq \zeta \leq Z$, and $0 \leq \eta \leq d$. Justification of the chosen value of Z is made a posteriori again from the resulting solution. Non-uniform discretization is made in order to capture the possible steep variation of $\tilde{\Phi}$. The grid in molecular velocity space (μ, ζ) is arranged two-fold: one is the grid for capturing the milder function $C[\tilde{\Phi}]$ (and $C[\Psi]$), while the other is that for capturing $\tilde{\Phi}$ (and Ψ) to compute $C[\tilde{\Phi}]$ (and $C[\Psi]$) enough accurately. The latter grid should be finer than the former. In the standard grid system, namely (S1,M1) in Sec. 3.7.1, we set $Z = 5.0$ and $d = 44.46$, and arrange 251 grid points in η space; the two-fold grid in μ - ζ space consists of 257×141 and 449×161 points. More details are found in Sec. 3.7.1.

For the spatial coordinate η and related integration, we arrange $2N_\eta + 1$ grid points, say $\eta^{(i)}$ ($i = 0, 1, \dots, 2N_\eta$), in the region $0 \leq \eta \leq d$:

$$0 = \eta^{(0)} < \eta^{(1)} < \dots < \eta^{(2N_\eta)} = d. \quad (3.13)$$

The functions $\tilde{\Phi}$ and Ψ are evaluated on these points. In carrying out the integration with respect to s in Eq. (3.12), $C[\tilde{\Phi}]$ and $C[\Psi]$ are interpolated with a piecewise quadratic function from their data on the grid points $\{\eta^{(i)}\}$.¹ The related integrals are commonly expressed as

$$T[F](\eta, \mu, \zeta) = \begin{cases} \frac{1}{\mu\zeta} \int_0^\eta C[F](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, & (\mu\zeta > 0), \\ \frac{1}{\mu\zeta} \int_d^\eta C[F](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, & (\mu\zeta < 0), \end{cases} \quad (3.14)$$

with $F = \tilde{\Phi}$ or Ψ . By the piecewise quadratic interpolation for $C[F]$ in s , $T[F]$ is computed as

$$T[F](\eta^{(i)}, \mu, \zeta) = \sum_{r=0}^{2N_\eta} S_{i,r}(\mu, \zeta) C[F](\eta^{(r)}, \mu, \zeta), \quad (3.15)$$

¹The quadratic interpolation cannot capture a possible logarithmic divergence of the gradient of $\tilde{\Phi}$ with respect to $\mu\zeta$ on the boundary for $\mu\zeta = 0_-$, which is discussed in Ref. [24]. However, it does capture the logarithmic divergence of the gradient of the Knudsen-layer functions with respect to η (see Ref. [24]), thanks to the integral formulation. See the last paragraph of Sec. 3.5.2.

with

$$S_{i,r}(\mu, \zeta) = \begin{cases} \frac{1}{\mu\zeta} \int_0^{\eta^{(i)}} Y_r^\eta(s) \exp\left(\frac{\nu(\zeta)(s - \eta^{(i)})}{\mu\zeta}\right) ds, & (\mu\zeta > 0), \\ \frac{1}{\mu\zeta} \int_d^{\eta^{(i)}} Y_r^\eta(s) \exp\left(\frac{\nu(\zeta)(s - \eta^{(i)})}{\mu\zeta}\right) ds, & (\mu\zeta < 0), \end{cases} \quad (3.16)$$

where Y 's are the following locally defined piecewise quadratic functions:

$$Y_{2r}^z(y) = \begin{cases} \frac{(y - z^{(2r+2)})(y - z^{(2r+1)})}{(z^{(2r)} - z^{(2r+2)})(z^{(2r)} - z^{(2r+1)})}, & (z^{(2r)} < y < z^{(2r+2)}), \\ \frac{(y - z^{(2r-2)})(y - z^{(2r-1)})}{(z^{(2r)} - z^{(2r-2)})(z^{(2r)} - z^{(2r-1)})}, & (z^{(2r-2)} < y < z^{(2r)}), \\ 0, & \text{otherwise,} \end{cases} \quad (3.17a)$$

$$Y_{2r+1}^z(y) = \begin{cases} \frac{(y - z^{(2r)})(y - z^{(2r+2)})}{(z^{(2r+1)} - z^{(2r)})(z^{(2r+1)} - z^{(2r+2)})}, & (z^{(2r)} < y < z^{(2r+2)}), \\ 0, & \text{otherwise.} \end{cases} \quad (3.17b)$$

The functional form of $S_{i,r}$ can be obtained explicitly, which enables us to perform delicate analyses with numerically obtained data. With the data $C[\tilde{\Phi}](\eta^{(i)}, \mu, \zeta)$ and $C[\Psi](\eta^{(i)}, \mu, \zeta)$, we can obtain $\tilde{\Phi}$ on the spatial grid points $\{\eta^{(i)}\}$, essentially by the simple summation (3.15).²

As already mentioned in the first paragraph of this section, we prepare the two-fold grid system for molecular velocity space. As the primary grid, we arrange $(4N_\mu + 1) \times (2N_\zeta + 1)$ points, say $(\mu^{(j)}, \zeta^{(k)})$ ($j = -2N_\mu, \dots, 2N_\mu$; $k = 0, \dots, 2N_\zeta$) in the region $-1 \leq \mu \leq 1$ and $0 \leq \zeta \leq Z$:

$$\begin{aligned} 0 &= \mu^{(0)} < \mu^{(1)} < \dots < \mu^{(2N_\mu-1)} < \mu^{(2N_\mu)} = 1, \\ \mu^{(-j)} &= -\mu^{(j)}, \quad (1 \leq j \leq 2N_\mu), \\ 0 &= \zeta^{(0)} < \zeta^{(1)} < \dots < \zeta^{(2N_\zeta)} = Z, \end{aligned}$$

which have the enough resolution for the mild function $C[F]$. Then $C[F](\eta^{(i)}, \mu, \zeta)$ is recovered accurately for any μ and ζ by the piecewise quadratic interpolation of $C[F]_{(i,j,k)} \equiv C[F](\eta^{(i)}, \mu^{(j)}, \zeta^{(k)})$. Hence $\tilde{\Phi}(\eta^{(i)}, \mu, \zeta)$ is obtained by Eq. (3.12) accurately for any μ and ζ . This applies also to $\Psi(\eta^{(i)}, \mu, \zeta)$.³

In order to obtain $C[F]_{(i,j,k)}$, we use the *numerical kernel method* that was first proposed in Ref. [18]. In this method, the argument function F is approximated by interpolation from the discretized data of F , so that a grid finer than the primary one is required for achieving

²In computing Ψ , the integrations in Eq. (3.11) with ∞ being replaced with d are performed in the same way. In the case, the piecewise quadratic interpolation is applied to $C[\psi_2 E]$.

³The primary grid should also have the enough resolution for $C[\psi_1 E]$ and $C[\psi_2 E]$ in Eq. (3.11) for the accurate computation of Ψ .

the enough resolution of F . Therefore, as the secondary grid, we prepare $(4N_M+1) \times (2N_\xi+1)$ points in μ - ζ space, say $(M^{(l)}, \xi^{(m)})$ ($l = -2N_M, \dots, 2N_M$; $m = 0, \dots, 2N_\xi$):

$$\begin{aligned} 0 &= M^{(0)} < M^{(1)} < \dots < M^{(2N_M-1)} < M^{(2N_M)} = 1, \\ M^{(-l)} &= -M^{(l)}, \quad (1 \leq l \leq 2N_M), \\ 0 &= \xi^{(0)} < \xi^{(1)} < \dots < \xi^{(2N_\xi)} = Z. \end{aligned}$$

Note that $N_M > N_\mu$ and $N_\xi > N_\zeta$ and that $F(\eta^{(i)}, M^{(l)}, \xi^{(m)})$ (for short, to be denoted by $F_{[i,l,m]}$ below) on the secondary grid is readily obtained as explained in the previous paragraph. In the present work, we adopt the piecewise quadratic interpolation for F :

$$F(\eta^{(i)}, M, \xi) = \sum_{m=0}^{2N_\xi} \sum_{l=0}^{2N_M} [F_{[i,l,m]} B_{l,m}^+(M, \xi) + F_{[i,-l,m]} B_{-l,m}^-(M, \xi)], \quad (3.18a)$$

with

$$B_{l,m}^\pm(M, \xi) = Y_l^M(M) \chi[0, 1](\pm M) Y_m^\xi(\xi) \chi[0, Z](\xi), \quad (3.18b)$$

$$\chi[a, b](y) = \begin{cases} 1 & \text{for } a \leq y \leq b, \\ 0 & \text{otherwise,} \end{cases} \quad (3.18c)$$

and compute $C[F]_{(i,j,k)}$ as

$$C[F]_{(i,j,k)} = \sum_{m=0}^{2N_\xi} \sum_{l=0}^{2N_M} (C_{j,k,l,m}^+ F_{[i,l,m]} + C_{j,k,-l,m}^- F_{[i,-l,m]}), \quad (3.19a)$$

$$C_{j,k,l,m}^\pm = C[B_{l,m}^\pm](\mu^{(j)}, \zeta^{(k)}). \quad (3.19b)$$

Note that, as the arguments of functions in Eq. (3.18), we use the integration variables $\xi(\equiv |\boldsymbol{\xi}|)$ and $M(\equiv \xi_i n_i / \xi)$, in place of ζ and μ , coming from $\boldsymbol{\xi}$ in the definition of $\mathcal{C}$ or C . The factor $\chi[0, 1](\pm M)$ is included in Eq. (3.18b) to reflect that F can be discontinuous at $M = 0$ on the boundary $\eta = 0$, as will be shown in Sec. 3.5.1. Hence, $F_{[i,l,m]}$ and $F_{[i,-l,m]}$ in Eq. (3.18a) [and Eq. (3.19a) soon below] should be considered different from each other, when $i = l = 0$. Note that $B_{l,m}^\pm$ vanishes for $l \leq 0$ and that the discrimination between $B_{l,m}^+$ and $B_{l,m}^-$ is effective when $l = 0$. Accordingly, $C_{j,k,l,m}^\pm$ corresponds to the integral kernel for the half-range of integration $M \gtrless 0$, and $C_{j,k,l,m}^\pm$ vanishes for $l \leq 0$. The computation of $C_{j,k,l,m}^\pm$, i.e., Eq. (3.19b), can be performed beforehand, independent of the iteration for solving Eq. (3.12). Incidentally, the integrations for this computation require careful treatments and some transformations to get rid of the singularity in the kernel k_1 .

However, we omit all the details on this issue; the interested reader is referred to Refs. [18] and [19].

Here are two remarks. (i) As is clear from its form, the collision kernel $k_1(\boldsymbol{\zeta}, \boldsymbol{\xi}) - k_2(\boldsymbol{\zeta}, \boldsymbol{\xi})$ is invariant under the transformation $(\boldsymbol{\zeta}, \boldsymbol{\xi}) \rightarrow (-\boldsymbol{\zeta}, -\boldsymbol{\xi})$. The symmetric arrangement of grid points in μ with respect to $\mu = 0$ enables the present numerical kernel $C_{j,k,l,m}^{\pm}$ to inherit this property, giving the relation $C_{j,k,l,m}^{-} = C_{-j,k,-l,m}^{+}$. This reduces the required memory capacity into half. (ii) If we have adopted a finite-difference method, we would need the secondary grid for the whole numerical solution process, which implies the size of the numerical kernel to be of $(N_M \times N_{\xi})^2$. The present approach economizes again the memory capacity by a factor of $(N_{\mu} \times N_{\zeta})/(N_M \times N_{\xi})$.

3.5 Numerical results

The results in the present section are obtained by the standard grid (S1,M1), unless otherwise stated.

3.5.1 Velocity distribution function

As illustrative examples, we show $\phi_2 E$ and $\phi_5 E$ at four spatial points $\eta = 0, 0.015, 0.58, 3.0$ in Figs. 3.1 and 3.2. The former is the case with nonzero Ψ in Eq. (3.10), while the latter is the case without it (or $\Psi = 0$). On the boundary $\eta = 0$, the discontinuity is commonly observed along the line $\mu\zeta = 0$. This line corresponds to the direction of molecular velocity that is tangential to the boundary. [Remind that the positive (or negative) $\mu\zeta$ corresponds to the velocity of molecules outgoing from (or incoming onto) the boundary.] The discontinuity vanishes immediately away from the boundary with keeping the other part almost unchanged. Accordingly, a continuous but steep variation part in the molecular velocity space appears near the boundary [see the panel (b) in both figures]. Such a local deformation can be understood by that the molecules coming from the boundary with small normal velocity ($\mu\zeta \sim 0$) would travel a long distance in the tangential direction before reaching the position of interest, thus more likely collide with others first to deform the part $\mu\zeta \sim 0$. As going away from the boundary, the distribution becomes milder and tends to vanish with keeping non-similar form between the positive and the negative $\mu\zeta$ part. The qualitative feature described above is common in both figures, irrespective of whether or not the inhomogeneous

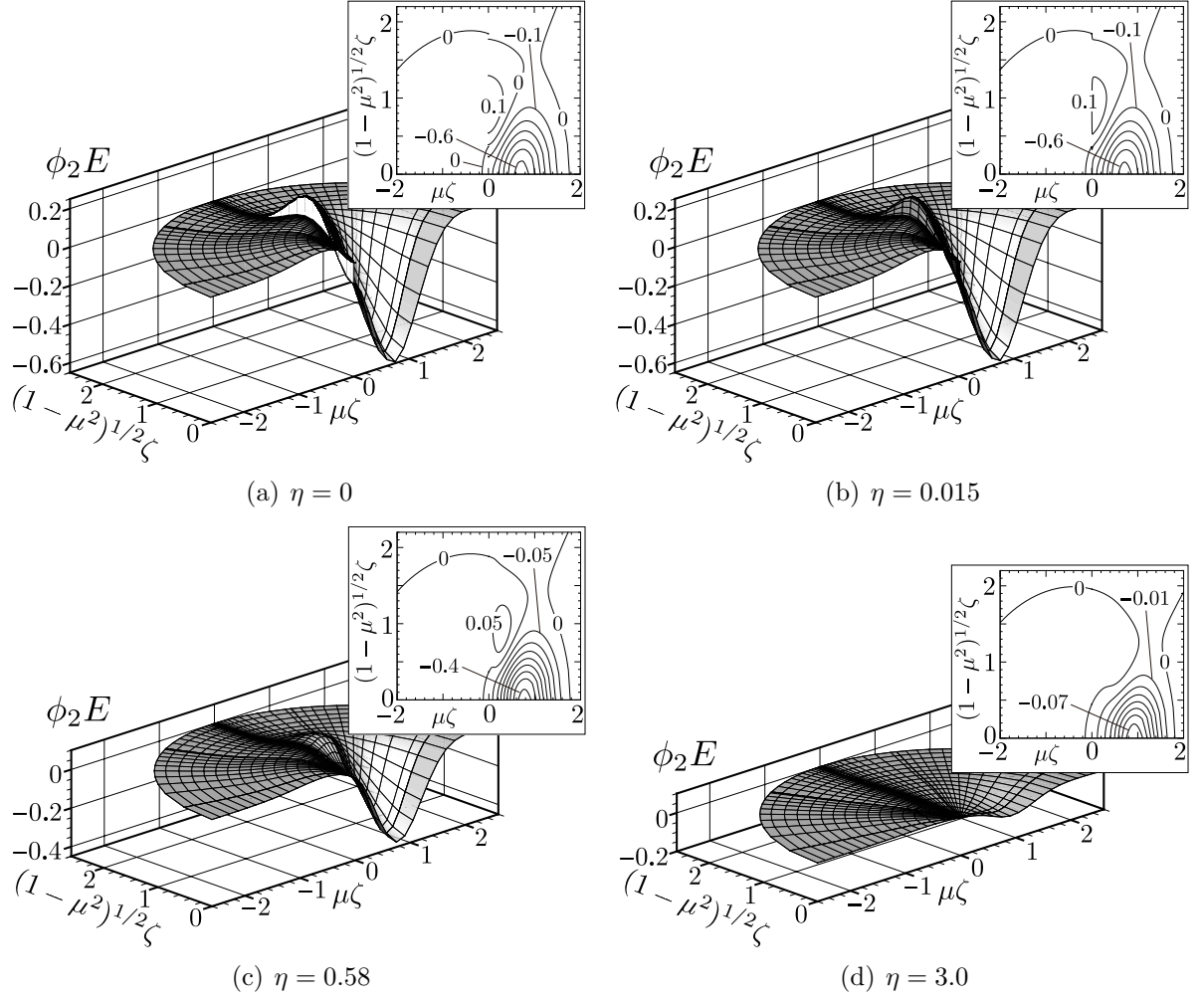


Figure 3.1: $\phi_2 E$ and its contour plots at four spatial points. (a) $\eta = 0$, (b) $\eta = 0.015$, (c) $\eta = 0.58$, and (d) $\eta = 3.0$. In the contour plots, the curves are drawn with the intervals 0.1 in (a) and (b), 0.05 in (c), and 0.01 in (d). The white vertical surface at $\mu\zeta = 0$ in (a) shows the discontinuity.

term appears in the problem (3.10).

3.5.2 Slip/jump coefficient and Knudsen-layer function

Slip/jump coefficients are shown in Table 3.1. The data in Ref. [13], which are obtained from the information about the first-order Knudsen layer by the theory of symmetry relation [22], are also shown for comparisons. Excellent agreement ensures the consistency of the present results to the theory.

The Knudsen-layer functions Ω 's, Θ 's, $Y_4^{(1)}$, and $H_4^{(1)}$ are shown in Fig. 3.3 and Table 3.2 (see Sec. 3.6 for $H_4^{(1)}$). Corresponding to the behavior of the velocity distribution functions in η , they decay fast as $\eta \rightarrow \infty$, mostly monotonically except for $\Theta_3^{(0)}$ and $\Theta_5^{(0)}$ (see Table 3.2).

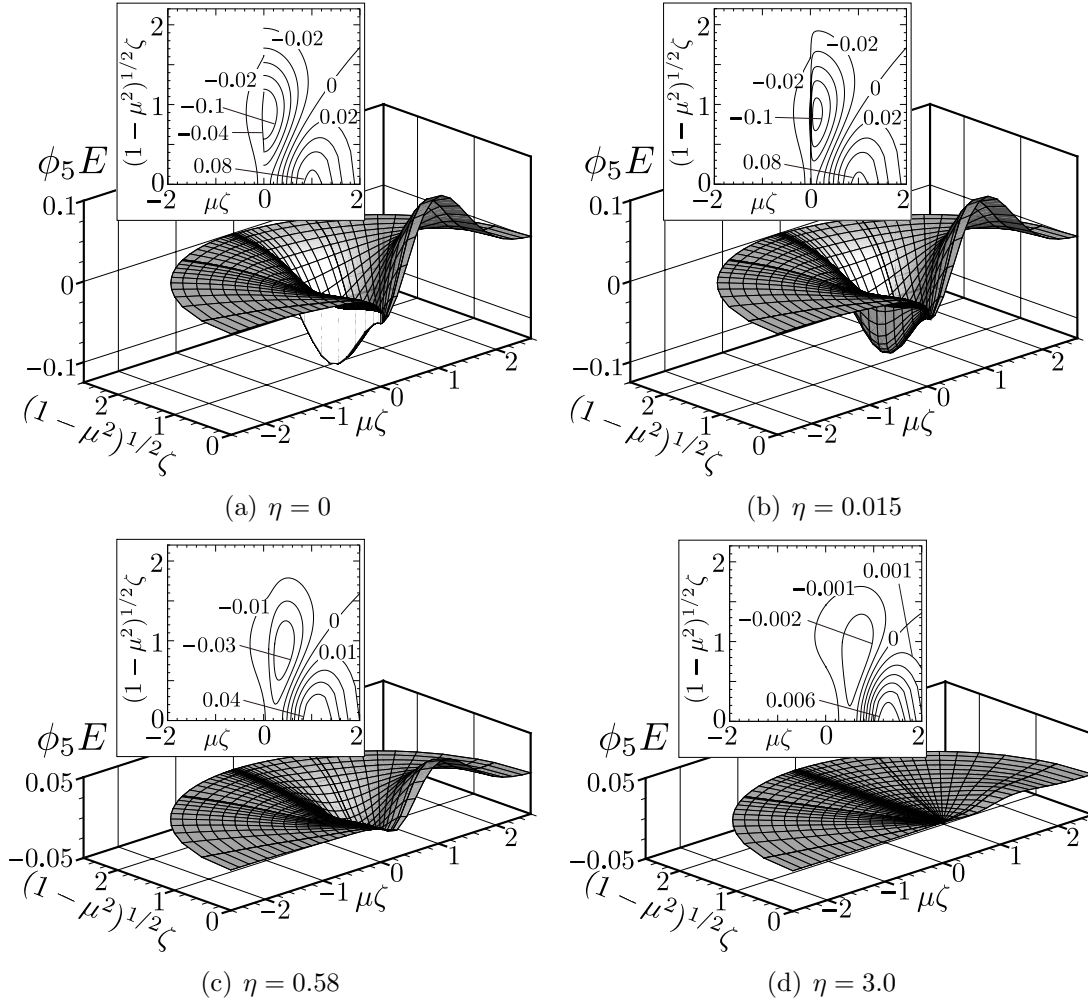


Figure 3.2: $\phi_5 E$ and its contour plots at four spatial points. (a) $\eta = 0$, (b) $\eta = 0.015$, (c) $\eta = 0.58$, and (d) $\eta = 3.0$. In the contour plots the curves are drawn with the intervals 0.02 in (a) and (b), 0.01 in (c), and 0.001 in (d). The white vertical surface at $\mu\zeta = 0$ in (a) shows the discontinuity.

From the table, the 90% thickness of the Knudsen layer is seen to be about $3 \sim 3.5$ for $\Omega_2^{(0)}$ and $\Theta_2^{(0)}$, about $2.5 \sim 3$ for $H_4^{(1)}$, and about $1.5 \sim 2$ for the others, measured in η , meaning about three mean-free-path thickness at most.

One interesting observation could be on the data of $c_2^{(0)}$, $\Omega_2^{(0)}$, and $\Theta_2^{(0)}$, as well as on the corresponding distribution function ϕ_2 . In the case of the BGK model under the diffuse reflection boundary condition, the former three all degenerate (or vanish) (see, e.g., Ref. [6]) in spite of the fact that ϕ_2 itself does not vanish [see Eqs. (3.8a) and (3.8b)]. The present results demonstrate that such a complete cancellation is not expected and the missing temperature jump and associated Knudsen layer indeed come out in general.

It is known that the Knudsen-layer function has the singularity $\eta \ln \eta$ on the boundary

Table 3.1: Slip/jump coefficients. The data in Ref. [13] obtained by the use of the symmetry relation are also shown for comparisons.

	$c_2^{(0)}$	$c_3^{(0)}$	$c_5^{(0)}$	$b_4^{(1)}$
Present results	-0.4993	0.00874	0.45957	-0.90393
Symmetry relation [13]	-0.4992	0.0087	0.4596	-0.9039

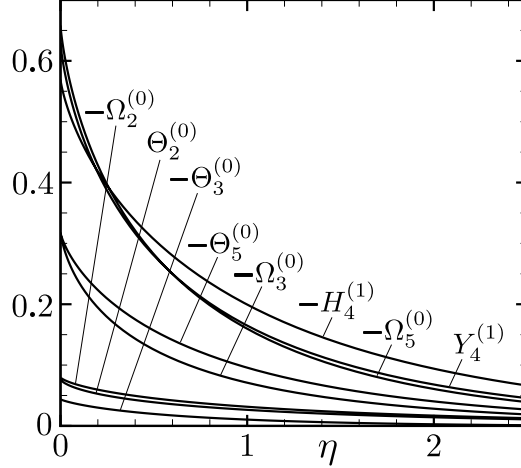


Figure 3.3: Knudsen-layer functions.

$\eta = 0$, which is another interest of the present work. This singularity that occurs on a plane boundary was first found in the study of the Rayleigh problem [27] on the basis of the BGK model. The same singularity has recently been confirmed to occur on the basis of the Boltzmann equation for hard-sphere molecules [24]. By the least-squares fitting of the curve $a + b\eta \ln \eta + c\eta$ to the numerical data near the boundary, we determine the coefficient b of the singularity $\eta \ln \eta$ for each Knudsen-layer function. The results are shown in the last row of Table 3.2. Incidentally, determining these coefficients is really challenging. Indeed, we could not get them by the finite-difference approach, even with the same computational size as Ref. [28]. The integral formulation enabled us to do it (see footnote 1).

3.6 Appendix A: Stress and heat flow

The stress tensor and heat-flow vector are also familiar fluid-dynamic quantities that become necessary, most typically, in computing the momentum and energy exchange with the body surface. Denoting the former by $p_0(\delta_{ij} + P_{ij})$ and the latter by $p_0(2RT_0)^{1/2}Q_i$, their Hilbert

Table 3.2: Knudsen-layer functions.

η	$-\Omega_2^{(0)}$	$\Theta_2^{(0)}$	$-\Omega_3^{(0)}$	$-\Theta_3^{(0)}$	$-\Omega_5^{(0)}$	$-\Theta_5^{(0)}$	$Y_4^{(1)}$	$-H_4^{(1)}$
0.00000	0.07965	0.07610	0.31803	0.04368	0.65823	0.31749	0.62894	0.56938
0.02348	0.07508	0.07019	0.28692	0.04108	0.60555	0.29707	0.58003	0.53685
0.05165	0.07153	0.06590	0.26406	0.03872	0.56413	0.28044	0.54166	0.51032
0.09881	0.06693	0.06064	0.23601	0.03547	0.51105	0.25858	0.49254	0.47522
0.15009	0.06290	0.05623	0.21276	0.03253	0.46540	0.23928	0.45030	0.44399
0.19315	0.05999	0.05316	0.19673	0.03038	0.43311	0.22536	0.42042	0.42128
0.30336	0.05380	0.04689	0.16463	0.02578	0.36657	0.19589	0.35876	0.37265
0.40911	0.04897	0.04220	0.14146	0.02224	0.31708	0.17323	0.31279	0.33467
0.58327	0.04252	0.03618	0.11301	0.01763	0.25470	0.14363	0.25459	0.28411
0.79673	0.03628	0.03059	0.08822	0.01340	0.19900	0.11603	0.20221	0.23582
0.98271	0.03187	0.02674	0.07228	0.01059	0.16263	0.09728	0.16769	0.20222
1.19037	0.02776	0.02323	0.05863	0.00814	0.13118	0.08051	0.13752	0.17148
1.41884	0.02400	0.02007	0.04715	0.00606	0.10458	0.06582	0.11167	0.14392
1.58214	0.02170	0.01814	0.04061	0.00488	0.08938	0.05719	0.09672	0.12738
1.84260	0.01856	0.01554	0.03230	0.00340	0.07008	0.04592	0.07747	0.10530
2.02594	0.01667	0.01399	0.02764	0.00259	0.05931	0.03945	0.06657	0.09234
2.51495	0.01263	0.01067	0.01860	0.00109	0.03855	0.02654	0.04506	0.06560
3.04221	0.00947	0.00807	0.01245	0.00019	0.02469	0.01748	0.03014	0.04586
3.48717	0.00748	0.00642	0.00901	-0.00022	0.01716	0.01236	0.02171	0.03412
4.06758	0.00554	0.00481	0.00603	-0.00049	0.01082	0.00790	0.01433	0.02336
4.91724	0.00362	0.00319	0.00345	-0.00059	0.00565	0.00412	0.00795	0.01356
6.06059	0.00207	0.00186	0.00172	-0.00050	0.00245	0.00171	0.00370	0.00662
8.07087	0.00081	0.00074	0.00056	-0.00028	0.00063	0.00034	0.00101	0.00193
10.06348	0.00032	0.00030	0.00020	-0.00013	0.00018	0.00005	0.00029	0.00058
13.81166	0.00006	0.00006	0.00003	-0.00003	0.00002	-0.00001	0.00003	0.00006
15.00856	0.00004	0.00003	0.00002	-0.00002	0.00001	-0.00001	0.00002	0.00003
20.05147	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
25.14669	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
$\eta \ln \eta^*$	0.0546	0.0809	0.405	0.0153	0.584	0.207	0.546	0.338

* The Knudsen-layer function has the singularity $\eta \ln \eta$ at $\eta = 0$. Its coefficient is shown in this row.

part and Knudsen-layer correction up to the second order in ε are summarized as follows:

$$P_{ijHm} = P_{Hm}\delta_{ij} - \gamma_1 \frac{\partial u_{iHm-1}}{\partial x_j} + \frac{1}{2}\gamma_3 \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j}, \quad (m = 0, 1, 2), \quad (3.20a)$$

$$Q_{iHm} = -\frac{5}{4}\gamma_2 \frac{\partial \tau_{Hm-1}}{\partial x_i} + \frac{1}{2}\gamma_3 \frac{\partial^2 u_{iHm-2}}{\partial x_j^2}, \quad (m = 0, 1, 2), \quad (3.20b)$$

and

$$P_{ijKm} = \frac{3}{2} \frac{\partial \tau_{Hm-1}}{\partial x_k} n_k (\delta_{ij} - n_i n_j) [\Omega_1^{(0)}(\eta) + \Theta_1^{(0)}(\eta)], \quad (m = 0, 1), \quad (3.21a)$$

$$P_{ijK2} n_i n_j = -3\bar{\kappa} \frac{\partial \tau_{H0}}{\partial x_i} n_i \int_{\eta}^{\infty} [\Omega_1^{(0)}(z) + \Theta_1^{(0)}(z)] dz, \quad (3.21b)$$

$$P_{ijK2} n_i t_j = \frac{3}{2} \left(\frac{\partial^2 \tau_{H0}}{\partial x_i \partial x_j} n_i t_j + \kappa_{ij} t_j \frac{\partial \tau_{H0}}{\partial x_i} \right) \int_{\eta}^{\infty} [\Omega_1^{(0)}(z) + \Theta_1^{(0)}(z)] dz, \quad (3.21c)$$

$$\begin{aligned}
Q_{iKm}t_i &= \frac{\overline{\partial u_{iHm-1}}}{\partial x_j} n_i t_j H_1^{(1)}(\eta) + \frac{\partial \tau_{Hm-1}}{\partial x_i} t_i H_2^{(1)}(\eta) \\
&+ \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} n_i t_j H_3^{(1)}(\eta) + \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j t_k H_4^{(1)}(\eta) \\
&+ \bar{\kappa} \frac{\overline{\partial u_{iHm-2}}}{\partial x_j} n_i t_j H_5^{(1)}(\eta) + \kappa_{ij} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_k t_i H_6^{(1)}(\eta) \\
&+ \kappa_{ij} t_j \frac{\partial \tau_{Hm-2}}{\partial x_i} H_7^{(1)}(\eta) + \bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} t_i H_8^{(1)}(\eta), \quad (m = 0, 1, 2), \tag{3.21d}
\end{aligned}$$

$$\begin{aligned}
Q_{iKm}n_i &= \left[\frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} (\delta_{ij} - n_i n_j) - 2\bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} n_i \right] \int_{\eta}^{\infty} H_2^{(1)}(z) dz \\
&- \frac{1}{2} \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j n_k \int_{\eta}^{\infty} H_1^{(1)}(z) dz, \quad (m = 0, 1, 2). \tag{3.21e}
\end{aligned}$$

Here the quantities with the subscript H in Eq. (3.21) denote their value on the boundary. The functions $H_1^{(1)}$, $H_2^{(1)}$, and $H_3^{(1)}$ have already been obtained in Refs. [19] and [20]. The present work newly provides the data of $H_4^{(1)}$, which are included in Fig. 3.3 and Table 3.2.

3.7 Appendix B: Data of computations

3.7.1 Grid system

Grid points for spatial coordinate η are arranged in the interval $[0, d]$ as

$$\eta^{(i)} = -20 \ln(1 - 0.7p^{(i)}) + 10^{-4} \frac{i}{2N}, \tag{3.22a}$$

$$p^{(i)} = \frac{26(i/2N)^4}{1 + 25(i/2N)^3}, \quad (i = 0, 1, \dots, 2N_{\eta}). \tag{3.22b}$$

The parameters N and N_{η} and the upper bound $d(= \eta^{(2N_{\eta})})$ for the grids S1, S2, and S3 are as follows: $(N, N_{\eta}, d) = (100, 125, 44.46)$ for S1, $(N, N_{\eta}, d) = (150, 188, 44.94)$ for S2, and $(N, N_{\eta}, d) = (100, 130, 53.04)$ for S3.

As to the μ and ζ spaces, subintervals $(0, \mu_I]$, $(\mu_I, \mu_{II}]$, and $(\mu_{II}, 1]$ in μ and $[0, \zeta_I]$, $(\zeta_I, \zeta_{II}]$, and $(\zeta_{II}, Z]$ in ζ are conveniently introduced for the easy control of grid points near $\mu = 0$ and $\zeta = 0$; the grid points are arranged as

$$\mu^{(j)} = \begin{cases} 0, & (j = 0), \\ 1 - \tanh\left(\frac{\pi}{2} \sinh(u^{(j)})\right), & (j = 1, \dots, 2N_{II}), \\ \left(\mu_{II}^{1/3} + (1 - \mu_{II}^{1/3}) \frac{j - 2N_{II}}{2N_{\mu} - 2N_{II}}\right)^3, & (j = 2N_{II} + 1, \dots, 2N_{\mu}), \\ -\mu^{(-j)}, & (j = -2N_{\mu}, \dots, -1), \end{cases} \tag{3.23a}$$

Table 3.3: Grid parameters for μ - ζ space.

Grid	Common		Primary grid					Secondary grid				
	μ_I	μ_{II}	N_I	N_{II}	N_μ	w		N_I	N_{II}	N_M	w	
	ζ_I	r	$\mathcal{N}_I$	$\mathcal{N}_{II}$	N_ζ	W		$\mathcal{N}_I$	$\mathcal{N}_{II}$	N_ξ	$\mathcal{N}$	W
	Z	ζ_{II}	$\mu^{(1)}$		$\zeta^{(1)}$			$M^{(1)}$		$\xi^{(1)}$		
M1	10^{-4}	1/125	0	16	64	-		48	64	112		1
	10^{-4}	1/10	0	16	70	70	-	10	26	80	80	1
	5.0	0.36	1.2×10^{-4}		1.6×10^{-4}			2.4×10^{-10}		5.1×10^{-10}		
M2	10^{-4}	1/125	0	16	64	-		64	80	128		1
	10^{-4}	1/10	0	16	70	70	-	10	26	80	80	1
	5.0	0.36	1.2×10^{-4}		1.6×10^{-4}			2.3×10^{-10}		5.1×10^{-10}		
M3	10^{-4}	1/125	0	16	64	-		48	64	112		1
	10^{-4}	1/10	0	22	94	94	-	14	36	108	108	1
	5.0	0.36	1.2×10^{-4}		1.4×10^{-4}			2.4×10^{-10}		3.9×10^{-10}		
M4	10^{-4}	1/125	0	22	86	-		48	70	134		1
	10^{-4}	1/10	0	16	70	70	-	10	26	80	80	1
	5.0	0.36	1.1×10^{-4}		1.6×10^{-4}			2.4×10^{-10}		5.1×10^{-10}		
M5	10^{-4}	1/125	5	21	69	5/24		53	69	117		53/48
	10^{-4}	1/10	5	21	75	75	5/24	12	28	82	82	12/10
	5.0	0.36	1.8×10^{-5}		1.8×10^{-5}			3.1×10^{-11}		9.3×10^{-12}		
M6	10^{-4}	1/125	0	16	64	-		48	64	112		1
	10^{-4}	1/10	0	16	73	70	-	10	26	83	80	1
	5.8	0.36	1.2×10^{-4}		1.6×10^{-4}			2.4×10^{-10}		5.1×10^{-10}		
M7	10^{-4}	1/125	0	12	48	-		36	48	84		1
	10^{-4}	1/10	0	12	48	48	-	7	19	55	55	1
	5.0	0.36	1.3×10^{-4}		1.8×10^{-4}			2.6×10^{-10}		7.5×10^{-10}		

with

$$u^{(j)} = \begin{cases} (2.69 - u_I)w(1 - \frac{j}{2N_I}) + u_I, & (j = 1, \dots, 2N_I), \\ (1 - \frac{j-2N_I}{2N_{II}-2N_I})u_I + \frac{j-2N_I}{2N_{II}-2N_I}u_{II}, & (j = 2N_I + 1, \dots, 2N_{II}), \end{cases} \quad (3.23b)$$

$$u_{I,II} = \sinh^{-1} \left(\frac{2}{\pi} \tanh^{-1}(1 - \mu_{I,II}) \right), \quad (3.23c)$$

and

$$\zeta^{(k)} = \begin{cases} 0, & (k = 0), \\ 1 - \tanh \left(\frac{\pi}{2} \sinh(v^{(k)}) \right), & (k = 1, \dots, 2\mathcal{N}_{II}), \\ 3.6v^{(k)} + 1.4(v^{(k)})^8, & (k = 2\mathcal{N}_{II} + 1, \dots, 2N_\zeta), \end{cases} \quad (3.24a)$$

with

$$v^{(k)} = \begin{cases} (2.69 - v_I)W(1 - \frac{k}{2\mathcal{N}_I}) + v_I, & (k = 1, \dots, 2\mathcal{N}_I), \\ v_I(1 - \frac{k-2\mathcal{N}_I}{2\mathcal{N}_{II}-2\mathcal{N}_I}) + v_{II}\frac{k-2\mathcal{N}_I}{2\mathcal{N}_{II}-2\mathcal{N}_I}, & (k = 2\mathcal{N}_I + 1, \dots, 2\mathcal{N}_{II}), \\ r + (1 - r)\frac{k-2\mathcal{N}_{II}}{2\mathcal{N}-2\mathcal{N}_{II}}, & (k = 2\mathcal{N}_{II} + 1, \dots, 2N_\zeta), \end{cases} \quad (3.24b)$$

$$v_{I,II} = \sinh^{-1} \left(\frac{2}{\pi} \tanh^{-1}(1 - \zeta_{I,II}) \right), \quad \zeta_{II} = 3.6r + 1.4r^8. \quad (3.24c)$$

Note that $\mu^{(\pm 2N_I)} = \pm \mu_I$, $\mu^{(\pm 2N_{II})} = \pm \mu_{II}$, $\mu^{(\pm 2N_\mu)} = \pm 1$, $\zeta^{(2\mathcal{N}_I)} = \zeta_I$, $\zeta^{(2\mathcal{N}_{II})} = \zeta_{II}$, and $\zeta^{(2N_\zeta)} = Z$. In Eqs. (3.23) and (3.24), the primary grid is supposed. For the secondary grid,

Table 3.4: Slip/jump coefficients $c_2^{(0)}$, $c_3^{(0)}$, $c_5^{(0)}$, and $b_4^{(1)}$ obtained by different grids.

Grid	$c_2^{(0)}$	$c_3^{(0)}$	$c_5^{(0)}$	$b_4^{(1)}$
(S1,M1)	-0.4992519	0.0087359	0.4595723	-0.9039300
(S1,M2)	-0.4992519	0.0087359	0.4595723	-0.9039300
(S1,M3)	-0.4992695	0.0087344	0.4595690	-0.9039293
(S1,M4)	-0.4992535	0.0087370	0.4595721	-0.9039303
(S1,M5)	-0.4992519	0.0087359	0.4595723	-0.9039300
(S1,M6)	-0.4992532	0.0087358	0.4595722	-0.9039301
(S1,M7)	-0.4991404	0.0087413	0.4595929	-0.9039335
(S2,M1)	-0.4992531	0.0087359	0.4595723	-0.9039288
(S3,M1)	-0.4992519	0.0087359	0.4595723	-0.9039300

$\mu^{(j)}$, N_μ , $\zeta^{(k)}$, and N_ζ should be replaced by $M^{(j)}$, N_M , $\xi^{(k)}$, and N_ξ , respectively. Although the common notation is used for $N_{\text{I,II}}$, w , $\mathcal{N}_{\text{I,II}}$, $\mathcal{N}$, and W , they may take different values between the primary and the secondary grid. The values of grid parameters for the μ and ζ spaces, as well as the resulting values of Z , ζ_{II} , $\mu^{(1)}$, $\zeta^{(1)}$, $M^{(1)}$, and $\xi^{(1)}$, are summarized in Table 3.3.

3.7.2 Measure of accuracy

The truncation of the ζ and η spaces is justified by checking the ratios $F(d, \cdot, \cdot)/F_{\text{max}}$ and $F(\cdot, \cdot, Z)/F_{\text{max}}$, where $F = |\phi_2 E|$, $|\phi_3 E|$, $|\phi_5 E|$, and $|\psi_4 E|$ and F_{max} is the respective maximum of F over the all grid points. For the standard grid (S1,M1) computation, the former and the latter are respectively less than 8.9×10^{-11} and 1.4×10^{-9} . The velocity distribution functions have decayed sufficiently at the truncated point.

The grid dependence of the computed slip/jump coefficients is shown in Table 3.4. The grid in ζ -space most affects the results, especially for $c_2^{(0)}$ [compare the results by (S1,M1), (S1,M3), and (S1,M7), where M3 (or M1) is the grid about twice (or 3/2) as many points as M7]. The accuracy down to the fourth or fifth decimal place is expected from the table. The comparisons among (S1,M1), (S3,M1), and (S1,M6) in the table show that the error due to the truncation of the ζ and η spaces is almost negligible.

The collision invariants are used to assess the accuracy of the collision integral computation, which is reduced to check the identities $C[(1, \mu\zeta, \zeta^2)E] = (1, \mu\zeta, \zeta^2)\nu E$ for $C = ECE^{-1}$ and $C[E] = \nu E$ for $C = EC^S E^{-1}$. With the standard grid M1, these identities are confirmed to hold within the error of 9.1×10^{-8} , 1.7×10^{-8} , 6.6×10^{-8} , and 8.9×10^{-9} respectively,

while the maximum values of $(1, \mu\zeta, \zeta^2)\nu E$ are 0.13, 0.064, and 0.062 respectively.

The mass, momentum, and energy balances offer another measure of accuracy. They are the following identities that are obtained from Eqs. (3.4a) and (3.5a) by the integration in molecular velocity space after multiplying the collision invariants:

$$\begin{aligned}\langle \mu\zeta\phi_2 \rangle &= \int_{\eta}^{\infty} Y_2^{(1)}(z)dz, \quad \langle \mu^2\zeta^2\phi_2 \rangle_+ = -\langle \mu^2\zeta^2\phi_2 \rangle_-, \\ \langle \mu\zeta(\zeta^2 - \frac{5}{2})\phi_2 \rangle &= \int_{\eta}^{\infty} H_2^{(1)}(z)dz, \end{aligned} \tag{3.25a}$$

$$\begin{aligned}\langle \mu\zeta\phi_3 \rangle &= -\frac{1}{2} \int_{\eta}^{\infty} Y_1^{(1)}(z)dz, \quad \langle \mu^2\zeta^2\phi_3 \rangle_+ = -\langle \mu^2\zeta^2\phi_3 \rangle_-, \\ \langle \mu\zeta(\zeta^2 - \frac{5}{2})\phi_3 \rangle &= -\frac{1}{2} \int_{\eta}^{\infty} H_1^{(1)}(z)dz, \end{aligned} \tag{3.25b}$$

$$\langle (\mu\zeta, \mu^2\zeta^2, \mu\zeta^3)\phi_5 \rangle_+ = -\langle (\mu\zeta, \mu^2\zeta^2, \mu\zeta^3)\phi_5 \rangle_-, \tag{3.25c}$$

$$\langle \mu(1 - \mu^2)\zeta^3\psi_4 \rangle_+ = -\langle \mu(1 - \mu^2)\zeta^3\psi_4 \rangle_-. \tag{3.25d}$$

Here $\langle \cdot \rangle_{\pm}$ is the half-range integral with respect to the molecular velocity defined by $\langle f(\zeta) \rangle_{\pm} = \langle f(\zeta)\chi[0, 1](\pm\mu) \rangle$. $H_1^{(1)}$ and $H_2^{(1)}$ above have already appeared in Eq. (3.21) and are defined as $H_{\beta}^{(1)}(\eta) = (1/2)\langle \zeta^2(\zeta^2 - 5/2)(1 - \mu^2)\psi_{\beta} \rangle$ ($\beta = 1, 2$). With the standard grid (S1,M1), the identities in Eq. (3.25a) hold within the error of 3.2×10^{-8} , 8.1×10^{-7} , and 6.8×10^{-7} , while the maxima of their left-hand side are 0.48, 0.11, and 1.3. In the case of Eq. (3.25b), the error is within 4.3×10^{-7} , 7.8×10^{-8} , and 1.3×10^{-6} , while the maxima of the l.h.s. are 0.11, 0.055, and 0.076. In the case of Eq. (3.25c), the error is within 6.4×10^{-7} , 6.3×10^{-7} , and 4.7×10^{-6} , while the maxima of the l.h.s. are 0.085, 0.071, and 0.29. In the case of Eq. (3.25d), the error is within 3.6×10^{-6} , while the maximum of the l.h.s. is 0.14.

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Chapter 4

Second-order Knudsen-layer analysis for the generalized slip-flow theory II: Curvature effects

4.1 Introduction

Study on the connection between the kinetic theory and the fluid-dynamics has a long history [1,2,3] and a number of important results have been obtained for a small or a vanishing limit of the Knudsen number, e.g., Refs. [4,5,6,7,8,9]. A systematic asymptotic theory has been established in the late 1960s and early 1970s [4,5] and developed further [6] since then. It provides not only the fluid-dynamic description in the bulk region but also the slip/jump boundary condition and the non-fluid-like correction in a thin layer adjacent to the boundary (the Knudsen layer) for small Knudsen numbers. We call the linear case of that theory the generalized slip-flow theory in the present paper. The generalized slip-flow theory contains rich information on the gas rarefaction effect, giving a fluid-dynamical interpretation to various phenomena occurring for small Knudsen numbers. It also motivates recent studies on time-dependent problems in its framework [10,11,12]. In the meantime, mathematical studies are also developing to include the argument of boundary condition for the fluid-dynamical equation [13,14,15].

The generalized slip-flow theory has been applied to various fundamental problems and was revealed to be practical. However, such applications have been limited mostly to the studies based on the Bhatnagar–Gross–Krook (BGK) [or Boltzmann–Krook–Welanders (BKW)] model equation [16,17]. This is due to lack of numerical data of the Knudsen layer at the second order of the Knudsen number for the original Boltzmann equation. We have recently obtained the required numerical data up to the second-order of the Knudsen number [18], except for the effects of boundary curvature, assuming the hard-sphere molecules and the diffuse reflection boundary condition.

In the present paper, we take a step further to complete the full set of numerical data

that are required in applications up to the second order of the Knudsen number expansion. They have been missing for nearly a half century in applying the theory to specific problems on the basis of the original Boltzmann equation. To be a little more specific, we focus on the effects of curvature of the boundary that manifest themselves at the second order of the Knudsen number. As will be clarified in the course of analyses, even though the macroscopic quantities are finite and well defined, the velocity distribution function diverges locally in the molecular velocity space in approaching or on the boundary. This feature forced us to separate the present work from the previous one [18]. The local but theoretical drawback behind the benefit of the expansion will also be discussed. We have developed the numerical method that handles such a singularity safely to complete the data required in the generalized slip-flow theory.

4.2 Generalized slip-flow theory: Outline and present concern

Let us consider a gas around smooth solid bodies which do not change in time their shape and position under the following assumptions: (i) There is no external force and the behavior of the gas is described by the Boltzmann equation for monatomic molecules; (ii) The gas molecules are reflected locally isotropically [6] on the surface of the solid bodies (no net flow across their surface); (iii) The deviation from the reference equilibrium state at rest with density ρ_0 and temperature T_0 is so small that the equation and the initial and the boundary condition can be linearized around that equilibrium state; (iv) The mean free path ℓ_0 of a molecule at the reference equilibrium state is much smaller than the characteristic length L of the physical system (i.e., the Knudsen number $\text{Kn} = \ell_0/L \ll 1$); (v) The time evolution is initiated by a slow change of the surroundings from the reference equilibrium state.

The time scale of the change t_0 in (v) is the same order as that of the viscous and the thermal diffusion, and we set it as $t_0 = (2/\sqrt{\pi})(L^2/\ell_0\sqrt{2RT_0})$, where R is the specific gas constant (the Boltzmann constant k divided by the mass of a molecule m). In the actual computations for the Knudsen-layer analysis, we assume the hard-sphere molecules in (i) and the diffuse reflection boundary condition in (ii). For hard-sphere molecules, $\ell_0 = [\sqrt{2}\pi d_m^2(\rho_0/m)]^{-1}$, where d_m is the diameter of a molecule.

The generalized slip-flow theory consists of first considering the overall behavior of the gas that changes in the scale of the characteristic length (and time) of the system and then introducing the correction in the vicinity of the boundary. The first part is conducted by the Hilbert (or Grad–Hilbert) expansion to yield a set of fluid-dynamic equations. The solution of this set is called the Hilbert part (or solution). The second part is conducted by the expansion after stretching the spatial coordinate in the direction normal to the boundary. This yields a set of slip/jump boundary condition and the associated correction to the fluid-dynamic solution near the boundary. The correction is called the Knudsen-layer correction, which is the present primary concern.

We shall denote by Lx_i the space coordinates, by t_0t the time, by $\rho_0(1 + \omega)$ the density of the gas, by $(2RT_0)^{1/2}u_i$ the flow velocity, by $T_0(1 + \tau)$ the temperature, by $p_0(1 + P)$ the pressure with $p_0 = \rho_0RT_0$, and by $(2RT_0)^{1/2}u_{iw}$ and $T_0(1 + \tau_w)$ the velocity and the temperature of the body surface. Since solid bodies change neither the shape nor the position, $u_{iw}n_i = 0$, where n_i is the unit vector normal to the surface, pointed to the gas. We denote the Hilbert part and the Knudsen-layer correction of the macroscopic quantity h ($h = \omega, u_i, \tau, P$) by h_H and h_K , respectively: $h = h_H + h_K$. Since the Knudsen number is small, we use $\varepsilon = (\sqrt{\pi}/2)\text{Kn}$ as a small parameter. The Hilbert part and the Knudsen-layer correction are obtained by a power series expansion in ε , which will be denoted as $h_H = h_{H0} + h_{H1}\varepsilon + h_{H2}\varepsilon^2 + \dots$ and $h_K = h_{K0} + h_{K1}\varepsilon + h_{K2}\varepsilon^2 + \dots$. Actually, the expansion of h_K starts from $O(\varepsilon)$, because no correction to the Hilbert solution is required at $O(1)$.

The resulting set of fluid-dynamic equations, their slip/jump boundary conditions, and the Knudsen-layer corrections up to the second order of the expansion in ε are summarized as follows: [10]

Fluid-dynamic equations

$$\frac{\partial P_{H0}}{\partial x_i} = 0, \quad (4.1a)$$

$$\frac{\partial u_{iHm}}{\partial x_i} + \frac{\partial \omega_{Hm-1}}{\partial t} = 0, \quad (4.1b)$$

$$\frac{\partial u_{iHm}}{\partial t} + \frac{1}{2} \frac{\partial P_{Hm+1}^*}{\partial x_i} - \frac{1}{2} \gamma_1 \frac{\partial^2 u_{iHm}}{\partial x_j^2} + \frac{1}{4} (\gamma_1 \gamma_{10} - 2\gamma_6) \frac{\partial^4 u_{iHm-2}}{\partial x_j^2 \partial x_k^2} = 0, \quad (4.1c)$$

$$\frac{\partial \tau_{Hm}}{\partial t} - \frac{2}{5} \frac{\partial P_{Hm}}{\partial t} - \frac{1}{2} \gamma_2 \frac{\partial^2 \tau_{Hm}}{\partial x_j^2} + \frac{1}{10} \left(\gamma_2 \gamma_3 - \frac{13}{2} \gamma_{11} \right) \frac{\partial^4 \tau_{Hm-2}}{\partial x_j^2 \partial x_k^2} = 0, \quad (4.1d)$$

$$P_{Hm+1}^* = P_{Hm+1} - \frac{1}{6}(\gamma_2\gamma_1 - 4\gamma_3)\frac{\partial^2\tau_{Hm-1}}{\partial x_j^2} + \frac{1}{5}\gamma_1\frac{\partial P_{Hm-1}}{\partial t}, \quad (4.1e)$$

$$P_{Hm} = \omega_{Hm} + \tau_{Hm}, \quad (4.1f)$$

where $m = 0, 1, 2$ and the quantities h_{H-1} and h_{H-2} ($h = \omega, u_i, \tau, P$) should be read as zero. The γ 's occurring in the equations are positive constants corresponding to the transport coefficients at the reference state; γ 's are all unity for the BGK model, while they are respectively $\gamma_1 = 1.270042427$, $\gamma_2 = 1.922284066$, $\gamma_3 = 1.947906335$, $\gamma_6 = 1.419423836$, $\gamma_{10} = 1.63607346$, and $\gamma_{11} = 2.7931173$ for hard-sphere molecules.

Slip/jump boundary condition and the Knudsen-layer correction

$$\begin{aligned} \begin{bmatrix} (u_{iHm} - u_{iwm})t_i \\ u_{iKm}t_i \end{bmatrix} &= \frac{\overline{\partial u_{iHm-1}}}{\partial x_j} n_i t_j \begin{bmatrix} b_1^{(1)} \\ Y_1^{(1)}(\eta) \end{bmatrix} + \frac{\partial \tau_{Hm-1}}{\partial x_i} t_i \begin{bmatrix} b_2^{(1)} \\ Y_2^{(1)}(\eta) \end{bmatrix} \\ &+ \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} n_i t_j \begin{bmatrix} b_3^{(1)} \\ Y_3^{(1)}(\eta) \end{bmatrix} + \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j t_k \begin{bmatrix} b_4^{(1)} \\ Y_4^{(1)}(\eta) \end{bmatrix} \\ &+ \bar{\kappa} \frac{\overline{\partial u_{iHm-2}}}{\partial x_j} n_i t_j \begin{bmatrix} b_5^{(1)} \\ Y_5^{(1)}(\eta) \end{bmatrix} + \kappa_{ij} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_k t_i \begin{bmatrix} b_6^{(1)} \\ Y_6^{(1)}(\eta) \end{bmatrix} \\ &+ \kappa_{ij} \frac{\partial \tau_{Hm-2}}{\partial x_i} t_j \begin{bmatrix} b_7^{(1)} \\ Y_7^{(1)}(\eta) \end{bmatrix} + \bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} t_i \begin{bmatrix} b_8^{(1)} \\ Y_8^{(1)}(\eta) \end{bmatrix}, \end{aligned} \quad (4.2a)$$

$$\begin{aligned} \begin{bmatrix} u_{iHm}n_i \\ u_{iKm}n_i \end{bmatrix} &= \frac{1}{2} \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j n_k \begin{bmatrix} \int_0^\infty Y_1^{(1)}(z) dz \\ -\int_\eta^\infty Y_1^{(1)}(z) dz \end{bmatrix} \\ &+ \left[2\bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} n_i - \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} (\delta_{ij} - n_i n_j) \right] \begin{bmatrix} \int_0^\infty Y_2^{(1)}(z) dz \\ -\int_\eta^\infty Y_2^{(1)}(z) dz \end{bmatrix}, \end{aligned} \quad (4.2b)$$

$$\begin{aligned} \begin{bmatrix} \tau_{Hm} - \tau_{wm} \\ \omega_{Km} \\ \tau_{Km} \end{bmatrix} &= \frac{\partial \tau_{Hm-1}}{\partial x_i} n_i \begin{bmatrix} c_1^{(0)} \\ \Omega_1^{(0)}(\eta) \\ \Theta_1^{(0)}(\eta) \end{bmatrix} + \frac{\partial u_{iHm-1}}{\partial x_i} \begin{bmatrix} c_5^{(0)} \\ \Omega_5^{(0)}(\eta) \\ \Theta_5^{(0)}(\eta) \end{bmatrix} \\ &+ \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} (\delta_{ij} - n_i n_j) \begin{bmatrix} c_2^{(0)} \\ \Omega_2^{(0)}(\eta) \\ \Theta_2^{(0)}(\eta) \end{bmatrix} + \frac{\partial^2 \tau_{Hm-2}}{\partial x_j^2} \begin{bmatrix} c_6^{(0)} \\ \Omega_6^{(0)}(\eta) \\ \Theta_6^{(0)}(\eta) \end{bmatrix} \\ &+ \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j n_k \begin{bmatrix} c_3^{(0)} \\ \Omega_3^{(0)}(\eta) \\ \Theta_3^{(0)}(\eta) \end{bmatrix} + \bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} n_i \begin{bmatrix} c_4^{(0)} \\ \Omega_4^{(0)}(\eta) \\ \Theta_4^{(0)}(\eta) \end{bmatrix}, \end{aligned} \quad (4.2c)$$

$$P_{Km} = \omega_{Km} + \tau_{Km}, \quad (4.2d)$$

where $m = 0, 1, 2$ and $\overline{f_{ij}} = f_{ij} + f_{ji} - (2/3)f_{kk}\delta_{ij}$ (δ_{ij} is the Kronecker delta). In Eq. (4.2), the quantities with the subscript H or w denote their values at the (dimensionless) surface

position x_{iw} , and t_i (or n_i) is a unit vector tangential (or normal) to the surface at x_{iw} . The quantities with the subscript K depend on η as well as x_{iw} , where η is the stretched spatial coordinate normal to the surface at x_{iw} such that the position x_i in the Knudsen layer is expressed by $x_i = x_{iw} + \varepsilon \eta n_i$. The surface velocity and temperature are also expanded in a power series of ε : $h_w = h_{w0} + h_{w1}\varepsilon + h_{w2}\varepsilon^2 + \dots$ ($h = u_i, \tau$). The effect of surface curvature occurs through the terms with κ 's defined by

$$\bar{\kappa} = \frac{1}{2}(\kappa_1 + \kappa_2), \quad \kappa_{ij} = \kappa_1 \ell_i \ell_j + \kappa_2 m_i m_j. \quad (4.3)$$

Here κ_1/L and κ_2/L are the principal curvatures of the boundary, with κ_1 and κ_2 being taken negative when the corresponding center of curvature lies on the gas side; ℓ_i and m_i are the direction cosines of the principal directions corresponding to κ_1 and κ_2 respectively. *It is those terms that we are interested in in the present paper.*

Many terms degenerate from Eqs. (4.1) and (4.2) when $m = 0, 1$. For instance, the boundary condition at $O(1)$, namely for $m = 0$, is none other than the non-slip/non-jump condition; the slip/jump may occur at the first or higher order of ε . The reader is referred to Refs. [6] and [10] for the discussions on the main features of the above system. The system can be solved from the lowest order to determine the time-dependent behavior of the gas under the considered situation, provided that the data of the slip/jump coefficients $c_1^{(0)} \sim c_6^{(0)}$, $b_1^{(1)} \sim b_8^{(1)}$, $\int_0^\infty Y_1^{(1)}(z)dz$, $\int_0^\infty Y_2^{(1)}(z)dz$ and the elemental (or component) Knudsen-layer functions $\Omega_1^{(0)}(\eta) \sim \Omega_6^{(0)}(\eta)$, $\Theta_1^{(0)}(\eta) \sim \Theta_6^{(0)}(\eta)$, $Y_1^{(1)}(\eta) \sim Y_8^{(1)}(\eta)$ are available. They are obtained by solving elemental (or component) half-space problems of the linearized Boltzmann equation, which may be homogeneous or inhomogeneous depending on that component.

Since the first publication of the generalized slip-flow theory [4,5], the complete set of the slip/jump coefficients and the associated Knudsen-layer corrections has been available only for the BGK model. For the original Boltzmann equation, a part of the information have been reported in the late 1980s and early 1990s [19,20,21], assuming the hard-sphere molecules and the diffuse reflection boundary condition. The present work reports the final step of our attempts that follows these and our recent works [22,10,18]. Thanks to the theory of symmetry relation [23], we have already obtained the complete set of slip/jump coefficients up to the second order of ε in Ref. [10]. However, the information of the Knudsen-layer structure is still incomplete, which motivates the present work. As will become clear in the

sequel, the numerical method developed in Refs. [24,25,18], which is based on the integral formulation of the Boltzmann equation, is really effective in the study of the Knudsen layer related to the curvature effects of boundary.

4.3 Knudsen-layer problems

Let us denote by $(2RT_0)^{1/2}\zeta$ the molecular velocity and introduce the notation $\zeta = |\zeta|$ and $\mu = \zeta_i n_i / \zeta$, where $-1 \leq \mu \leq 1$ and $0 \leq \zeta < \infty$. Then, the component problems for the Knudsen layer admit a similarity solution of three arguments (η, μ, ζ) and are reduced to the following two types of boundary-value problems by assuming the hard-sphere molecules and the diffuse reflection boundary condition:

$$\mu\zeta \frac{\partial \phi_\alpha}{\partial \eta} = -\nu(\zeta)\phi_\alpha + \mathcal{C}[\phi_\alpha] - \mathcal{I}_\alpha(\eta, \mu, \zeta), \quad (4.4a)$$

$$\phi_\alpha = -\sigma_\alpha^{(0)} - c_\alpha^{(0)}\zeta^2 + g_\alpha(\mu, \zeta), \quad (\mu\zeta > 0, \eta = 0), \quad (4.4b)$$

$$\phi_\alpha \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \quad (4.4c)$$

and

$$\mu\zeta \frac{\partial \psi_\beta}{\partial \eta} = -\nu(\zeta)\psi_\beta + \mathcal{C}^S[\psi_\beta] - \mathcal{I}_\beta^S(\eta, \mu, \zeta), \quad (4.5a)$$

$$\psi_\beta = -2b_\beta^{(1)} + g_\beta^S(\mu, \zeta), \quad (\mu\zeta > 0, \eta = 0), \quad (4.5b)$$

$$\psi_\beta \rightarrow 0, \quad \text{as } \eta \rightarrow \infty. \quad (4.5c)$$

Here

$$\nu(\zeta) = \frac{1}{2\sqrt{2}} \left[\exp(-\zeta^2) + \left(2\zeta + \frac{1}{\zeta} \right) \int_0^\zeta \exp(-\xi^2) d\xi \right];$$

$\mathcal{C}$ is an integral operator acting on a function of ζ to be defined soon later; and g_α , $\mathcal{I}_\alpha$, g_β^S , and $\mathcal{I}_\beta^S$ are given functions. $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta^S$ are supposed to decay fast in η . The solution ϕ_α (or ψ_β) ($\alpha = 1, \dots, 6$; $\beta = 1, \dots, 8$) is a function of η , μ , and ζ . It is determined together with the constants $\sigma_\alpha^{(0)}$ and $c_\alpha^{(0)}$ [or $b_\beta^{(1)}$] for every given $(\mathcal{I}_\alpha, g_\alpha)$ [or $(\mathcal{I}_\beta^S, g_\beta^S)$] [26]. The operator $\mathcal{C}$ is defined by

$$\mathcal{C}[\phi](\zeta) = \int [k_1(\zeta, \xi) - k_2(\zeta, \xi)] \phi(\xi) d\xi,$$

$$k_1(\zeta, \xi) = \frac{1}{\sqrt{2\pi}|\zeta - \xi|} \exp \left(-|\xi|^2 + \frac{|\xi \times \zeta|^2}{|\xi - \zeta|^2} \right),$$

$$k_2(\zeta, \xi) = \frac{|\zeta - \xi|}{2\sqrt{2}\pi} \exp(-|\xi|^2).$$

Thanks to its spherical and axial symmetry, $\mathcal{C}[\phi_\alpha]$ becomes a function of μ and ζ (and η) for the function $\phi_\alpha(\eta, \mu, \zeta)$. On the other hand, $\mathcal{C}^S$ is defined for functions of μ and ζ (and η) through $\mathcal{C}$, making use of its axial symmetry:

$$\zeta_i t_i \mathcal{C}^S[\psi_\beta] = \mathcal{C}[\zeta_i t_i \psi_\beta].$$

Once the solutions are obtained, $\Omega_\alpha^{(0)}(\eta)$, $\Theta_\alpha^{(0)}(\eta)$, $Y_\beta^{(1)}(\eta)$ occurring in Eq. (4.2) and $H_\beta^{(1)}(\eta)$ in Sec. 4.7 are obtained as their moment:

$$\Omega_\alpha^{(0)}(\eta) = \langle \phi_\alpha \rangle, \quad \Theta_\alpha^{(0)}(\eta) = \frac{2}{3} \langle (\zeta^2 - \frac{3}{2}) \phi_\alpha \rangle, \quad (4.6)$$

$$Y_\beta^{(1)}(\eta) = \frac{1}{2} \langle \zeta^2 (1 - \mu^2) \psi_\beta \rangle, \quad H_\beta^{(1)}(\eta) = \frac{1}{2} \langle \zeta^2 (1 - \mu^2) (\zeta^2 - \frac{5}{2}) \psi_\beta \rangle, \quad (4.7)$$

where $\langle f \rangle = \int f(\xi) E(|\xi|) d\xi$ with $E(z) = \pi^{-3/2} \exp(-z^2)$.

Our present concern is the case $\alpha = 4$ and $\beta = 5 \sim 8$, for which $(\mathcal{I}_\alpha, g_\alpha)$ and $(\mathcal{I}_\beta^S, g_\beta^S)$ are given as

$$\begin{aligned} \mathcal{I}_4 &= \zeta(1 - \mu^2) \frac{\partial \phi_1}{\partial \mu}(\eta, \mu, \zeta) - \zeta^2(1 - \mu^2) \psi_2(\eta, \mu, \zeta), \\ g_4 &= -4\mu\zeta \int_0^\infty Y_2^{(1)}(z) dz - b_2^{(1)} \zeta^2(1 - 3\mu^2) B(\zeta), \end{aligned} \quad (4.8a)$$

$$\mathcal{I}_5^S = \frac{1}{2} \zeta(1 - \mu^2) \frac{\partial \psi_1}{\partial \mu}(\eta, \mu, \zeta), \quad g_5^S = -2D_1(\zeta) - \frac{1}{2} \zeta^2(1 - \mu^2) D_2(\zeta), \quad (4.8b)$$

$$\begin{aligned} \mathcal{I}_6^S &= \frac{1}{2} \zeta(1 - \mu^2) \frac{\partial \psi_1}{\partial \mu}(\eta, \mu, \zeta) - \mu\zeta \psi_1(\eta, \mu, \zeta), \\ g_6^S &= -D_1(\zeta) - \frac{1}{2} \zeta^2(1 - 3\mu^2) D_2(\zeta), \end{aligned} \quad (4.8c)$$

$$\mathcal{I}_7^S = \frac{1}{2} \zeta(1 - \mu^2) \frac{\partial \psi_2}{\partial \mu}(\eta, \mu, \zeta) - \mu\zeta \psi_2(\eta, \mu, \zeta) + \phi_1(\eta, \mu, \zeta), \quad g_7^S = 0, \quad (4.8d)$$

$$\mathcal{I}_8^S = \frac{1}{2} \zeta(1 - \mu^2) \frac{\partial \psi_2}{\partial \mu}(\eta, \mu, \zeta), \quad g_8^S = 0. \quad (4.8e)$$

Here ϕ_1 is the solution of the problem (4.4) with $\mathcal{I}_1 = 0$ and $g_1 = \mu\zeta A(\zeta)$ (the so-called temperature-jump problem [19]), while ψ_1 and ψ_2 are respectively the solution of the problem (4.5) with $\mathcal{I}_1^S = 0$, $g_1^S = \mu\zeta B(\zeta)$, $\mathcal{I}_2^S = 0$, and $g_2^S = A(\zeta)$ (the so-called shear-slip and thermal-slip [27,6] problems). The pair $(b_2^{(1)}, Y_2^{(1)})$ is the slip coefficient and Knudsen-layer function associated with ψ_2 , which have already been obtained in Ref. [20]. Note that all of $\mathcal{I}_4$ and $\mathcal{I}_5^S \sim \mathcal{I}_8^S$ in Eq. (4.8) indeed decay fast in η because of the fast decay of ϕ_1 , ψ_1 ,

and ψ_2 . The functions A , B , D_1 , and D_2 of ζ are familiar solutions of the following integral equations:

$$\begin{aligned}
\mathcal{L}[\zeta_i A(\zeta)] &= -\zeta_i(\zeta^2 - 5/2), \quad \text{subsidiary condition: } \langle \zeta^2 A(\zeta) \rangle = 0, \\
\mathcal{L}[\zeta_{ij} B(\zeta)] &= -2\zeta_{ij}, \\
\mathcal{L}[(\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) D_1(\zeta) + \zeta_i \zeta_j \zeta_k D_2(\zeta)] \\
&= \gamma_1(\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) - \zeta_i \zeta_j \zeta_k B(\zeta), \\
&\quad \text{subsidiary condition: } \langle 5\zeta^2 D_1(\zeta) + \zeta^4 D_2(\zeta) \rangle = 0,
\end{aligned}$$

where $\mathcal{L}[f] = -\nu(\zeta)f + \mathcal{C}[f]$ and $\zeta_{ij} = \zeta_i \zeta_j - (1/3)\zeta^2 \delta_{ij}$.

The essential difficulty which newly appears is that the inhomogeneous term $\mathcal{I}_\alpha$ or $\mathcal{I}_\beta^S$ contains the derivative of the first order Knudsen-layer solution, i.e., $\partial\phi_1/\partial\mu$, $\partial\psi_1/\partial\mu$, or $\partial\psi_2/\partial\mu$, because ϕ_1 , ψ_1 , and ψ_2 behave like $\mu\zeta \ln|\mu\zeta|$ for $\mu\zeta \sim -0$ on the boundary and thus their derivatives logarithmically diverge [25]. Our approach to be developed in the sequel follows the method in Refs. [18] and [25] and, in addition, separates the difficult part essentially from the others, thereby dealing with the problem safely numerically.

4.4 Numerical analysis and singularity of the solution

Because Eqs. (4.4) and (4.5) are linear problems, they can be treated by first decomposing the sources $\mathcal{I}_\alpha$ (or $\mathcal{I}_\beta^S$) and g_α (or g_β^S) as we like and then superposing the individual results. In the present section, we shall concentrate on the cases that all the terms without $\partial\phi_1/\partial\mu$, $\partial\psi_1/\partial\mu$, and $\partial\psi_2/\partial\mu$ in those sources are absent. Hence, we drop the subscript α from Eq. (4.4) and put $\mathcal{I} = \zeta(1 - \mu^2)\partial\phi_1/\partial\mu$ and $g = 0$, while we put $\beta = A, B$ in Eq. (4.5) with $\mathcal{I}_A^S = \zeta(1 - \mu^2)\partial\psi_1/\partial\mu$, $\mathcal{I}_B^S = \zeta(1 - \mu^2)\partial\psi_2/\partial\mu$, $g_A^S = 0$ and $g_B^S = 0$. Note that $\psi_8 = \frac{1}{2}\psi_B$. The remaining parts $\bar{\phi}_4 \equiv \phi_4 - \phi$, $\bar{\psi}_{5,6} \equiv \psi_{5,6} - \frac{1}{2}\psi_A$, and $\bar{\psi}_7 \equiv \psi_7 - \frac{1}{2}\psi_B$ can be handled without difficulty by the method in our previous paper [18], the explanation of which is omitted here.

4.4.1 Integral formulation

Multiplied by $E(\zeta)$ and integrated with respect to η , both of the problems (4.4) and (4.5) are transformed into

$$\begin{aligned}\Phi(\eta, \mu, \zeta) = & G(\mu, \zeta) \exp\left(-\frac{\nu(\zeta)\eta}{\mu\zeta}\right) + \frac{1}{\mu\zeta} \int_0^\eta C[\Phi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds \\ & + \frac{1}{\mu\zeta} \int_0^\eta C[\Psi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, \quad (\mu\zeta > 0),\end{aligned}\quad (4.9a)$$

$$\begin{aligned}\Phi(\eta, \mu, \zeta) = & \frac{1}{\mu\zeta} \int_\infty^\eta C[\Phi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds \\ & + \frac{1}{\mu\zeta} \int_\infty^\eta C[\Psi](s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, \quad (\mu\zeta < 0),\end{aligned}\quad (4.9b)$$

with

$$\Phi(\eta, \mu, \zeta) \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \quad (4.9c)$$

$$\Psi(\eta, \mu, \zeta) = \begin{cases} -\frac{1}{\mu\zeta} \int_0^\eta I(s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, & (\mu\zeta > 0), \\ -\frac{1}{\mu\zeta} \int_\infty^\eta I(s, \mu, \zeta) \exp\left(\frac{\nu(\zeta)(s-\eta)}{\mu\zeta}\right) ds, & (\mu\zeta < 0). \end{cases} \quad (4.9d)$$

Here, Φ , C , I , and G should be read as $\Phi = \phi E - \Psi$, $C[f] = \mathcal{C}[fE^{-1}]E$, $I = \mathcal{I}E$, and $G = (-\sigma^{(0)} - c^{(0)}\zeta^2)E$ for problem (4.4), while they should be read as $\Phi = \psi_\beta E - \Psi$, $C[f] = \mathcal{C}^S[fE^{-1}]E$, $I = \mathcal{I}_\beta^S E$, and $G = -2b_\beta^{(1)}E$ for problem (4.5), where $\beta = A, B$. Remember that I decays fast in η , so does Ψ . In the above, the original solution ϕE , $\psi_A E$, or $\psi_B E$ is split into the given part Ψ and unknown part Φ . The equation for Φ is none other than the integral form of the inhomogeneous Boltzmann equation with $C[\Psi]$ being its inhomogeneous term. The conditions (4.4c) and (4.5c) are reduced to Eq. (4.9c), because $\Psi \rightarrow 0$ as $\eta \rightarrow \infty$. The condition (4.9c) is required, otherwise the constants $\sigma^{(0)}$, $c^{(0)}$, $b_A^{(1)}$, and $b_B^{(1)}$ are not determined [26].

Since C is the integral operator, $C[f]$ is milder than f . Nevertheless, because of the singular nature of I considered here, a special attention should be made in the present analysis. In fact, the most difficult part in the present work is the computation of Ψ and $C[\Psi]$, especially their values in approaching the boundary ($\eta \rightarrow 0$).

Using the facts that ϕ_1 is the solution of Eq. (4.4) with $\mathcal{I}_1 = 0$, $g_1 = \mu\zeta A(\zeta)$ and that ψ_1 and ψ_2 are the solutions of Eq. (4.5) with $\mathcal{I}_1^S = 0$, $g_1^S = \mu\zeta B(\zeta)$, $\mathcal{I}_2^S = 0$, $g_2^S = A(\zeta)$, Ψ

can be written as

$$\Psi = (1 - \mu^2)\zeta \left\{ -\left(\frac{\partial h_0}{\partial \mu} + \underbrace{\frac{\zeta \nu}{2\eta} \left|\frac{\eta}{\mu\zeta}\right|^2 h_0}_{\text{underbrace}}\right) \frac{\eta}{\mu\zeta} e^{-\frac{\nu\eta}{\mu\zeta}} + \frac{1}{|\mu\zeta|^2} \right. \\ \left. \times \int_0^\eta \underbrace{\left\{ \zeta \left(1 - \frac{\nu}{2} \left|\frac{\eta-s}{\mu\zeta}\right|\right) C[h] - |\mu\zeta| \frac{\partial C[h]}{\partial \mu} \right\}}_{\text{underbrace}} \left| \frac{\eta-s}{\mu\zeta} \right| e^{-\nu \left|\frac{\eta-s}{\mu\zeta}\right|} ds \right\}, \quad (\mu\zeta > 0), \quad (4.10a)$$

$$\Psi = -\frac{(1 - \mu^2)\zeta}{|\mu\zeta|^2} \int_\eta^\infty \underbrace{\left\{ \zeta \left(1 - \frac{\nu}{2} \left|\frac{\eta-s}{\mu\zeta}\right|\right) C[h] + |\mu\zeta| \frac{\partial C[h]}{\partial \mu} \right\}}_{\text{underbrace}} \left| \frac{\eta-s}{\mu\zeta} \right| e^{-\nu \left|\frac{\eta-s}{\mu\zeta}\right|} ds, \quad (\mu\zeta < 0), \quad (4.10b)$$

where h is $\phi_1 E$, $\psi_1 E$, or $\psi_2 E$, while h_0 its value at $\eta = 0$; accordingly $C[h] = \mathcal{C}[\phi_1]E$, $\mathcal{C}^S[\psi_1]E$, or $\mathcal{C}^S[\psi_2]E$. Note that $C[h]$ is a function of s , μ , and ζ and that the argument ζ of ν and E has been omitted just for conciseness in the above. The parts with the underbrace require a special attention in the computation, which we explain in Sec. 4.4.2.

4.4.2 Singular properties of Ψ

Let us introduce the following notation:

$$W[f] \equiv - (1 - \mu^2) \frac{\zeta^2 \nu}{2\eta} f(\mu, \zeta) \left| \frac{\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}}, \quad (\mu\zeta > 0), \quad (4.11a)$$

$$T[f] \equiv \begin{cases} \frac{(1-\mu^2)\zeta^2}{|\mu\zeta|^2} \int_0^\eta \left| \frac{\eta-s}{\mu\zeta} \right| \left(1 - \frac{\nu}{2} \left|\frac{\eta-s}{\mu\zeta}\right|\right) f(s, \mu, \zeta) e^{-\nu \left|\frac{\eta-s}{\mu\zeta}\right|} ds, & (\mu\zeta > 0), \\ -\frac{(1-\mu^2)\zeta^2}{|\mu\zeta|^2} \int_\eta^\infty \left| \frac{\eta-s}{\mu\zeta} \right| \left(1 - \frac{\nu}{2} \left|\frac{\eta-s}{\mu\zeta}\right|\right) f(s, \mu, \zeta) e^{-\nu \left|\frac{\eta-s}{\mu\zeta}\right|} ds, & (\mu\zeta < 0). \end{cases} \quad (4.11b)$$

The parts with the underbrace in Eq. (4.10) are then written as $W[h_0]$ and $T[C[h]]$, respectively; the equation (4.10) is recast as

$$\Psi = W[h_0] + T[C[h]] - (1 - \mu^2)\zeta \left\{ \frac{\partial h_0}{\partial \mu} \frac{\eta}{\mu\zeta} e^{-\frac{\nu\eta}{\mu\zeta}} \right. \\ \left. + \frac{1}{|\mu\zeta|} \int_0^\eta \frac{\partial C[h]}{\partial \mu} \left| \frac{\eta-s}{\mu\zeta} \right| e^{-\nu \left|\frac{\eta-s}{\mu\zeta}\right|} ds \right\}, \quad (\mu\zeta > 0), \quad (4.12a)$$

$$\Psi = T[C[h]] - \frac{(1 - \mu^2)\zeta}{|\mu\zeta|} \int_\eta^\infty \frac{\partial C[h]}{\partial \mu} \left| \frac{\eta-s}{\mu\zeta} \right| e^{-\nu \left|\frac{\eta-s}{\mu\zeta}\right|} ds, \quad (\mu\zeta < 0). \quad (4.12b)$$

We consider the behavior of $W[h_0]$ and $T[C[h]]$ in three limiting processes: (i) $\eta \rightarrow 0$ with $|\mu\zeta| (> 0)$ fixed, (ii) $|\mu\zeta| \rightarrow 0$ with $\eta (> 0)$ fixed, and (iii) $\eta, |\mu\zeta| \rightarrow 0$ with $z \equiv \left|\frac{\nu\eta}{\mu\zeta}\right| (> 0)$ fixed. Note that $z \rightarrow 0$ in the limit (i), while $z \rightarrow \infty$ in the limit (ii).

As is clear from its form, $W[h_0]$ behaves well in the limits (i) and (ii). It grows, however, in proportion to η^{-1} in the limit (iii). Therefore, $W[h_0]$ is singular for $\mu\zeta = +0$ and $\eta = 0$.

As to $T[C[h]]$, we need to pay attention to that, even though $C[h]$ behaves well with respect to μ and ζ , its derivative in s may diverge logarithmically [25,18] as $s \rightarrow 0$. Therefore,

as in Ref. [25], we put

$$C[h](s, \mu, \zeta) = a(\mu, \zeta) + b(\mu, \zeta)s \ln s + c(\mu, \zeta)s, \quad (4.13)$$

for small s and estimate the behavior of $T[C[h]]$ for $\eta \ll 1$. Then, for $\mu\zeta > 0$, we have

$$\begin{aligned} \frac{\nu^3 T[C[h]]}{(1 - \mu^2)\zeta^2} &= \frac{z^3}{2\eta} e^{-z} a(\mu, \zeta) - \left\{ \left(\frac{z^2}{2} + z + 1 \right) e^{-z} [\text{Ei}(z) - \ln z - \gamma] - \frac{z}{2} \right. \\ &\quad \left. + \underbrace{\left[\left(\frac{z^2}{2} + z + 1 \right) e^{-z} - 1 \right] \ln \eta - \frac{5}{2} (1 - e^{-z}) + \left(\frac{z^2}{2} + 2z \right) e^{-z}}_{\text{underbrace}} \right\} b(\mu, \zeta) \\ &\quad + \left\{ 1 - \left(\frac{z^2}{2} + z + 1 \right) e^{-z} \right\} c(\mu, \zeta), \end{aligned} \quad (4.14)$$

where $\text{Ei}(z) = \text{p.v.} \int_{-\infty}^z y^{-1} e^y dy$, ($z > 0$) and γ is the Euler constant ($\gamma = 0.577216\dots$). Taking into account the properties $\text{Ei}(z) - \gamma - \ln z = O(z)$ for $z \ll 1$ and $e^{-z} \text{Ei}(z) = z^{-1} + O(z^{-2})$ for $z \gg 1$, we see that all the terms are finite in the limit (i). In the limit (ii), the part with the underbrace approaches $-\ln \eta$; thus $T[C[h]]$ grows in proportion to $\ln \eta$ as $\eta \rightarrow 0$ at $\mu\zeta = +0$. In the limit (iii), the terms with a and b grow in proportion to η^{-1} and to $\ln \eta$, respectively. On the other hand, for $\mu\zeta < 0$, we have

$$\begin{aligned} \frac{\nu^3 T[C[h]]}{(1 - \mu^2)\zeta^2} &= -\frac{w^3 e^{-w}}{2|\eta - \eta_*|} a(\mu, \zeta) - \left\{ (z - 1) e^z \text{E}_1(z) - \ln \eta - \frac{z}{2} [z e^z \text{E}_1(z) - 1] \right. \\ &\quad \left. + (\ln \eta_*) e^{-w} \left[\left(\frac{z^2}{2} + \frac{y^2}{2} - z + 1 \right) (1 + y) - zy^2 \right] \right. \\ &\quad \left. + \frac{y}{2} [y e^y \text{E}_1(y) - 1] e^{-w} - e^{-w} [(y - 1)(1 + w) - \frac{w^2}{2}] e^y \text{E}_1(y) \right. \\ &\quad \left. - \frac{5}{2} (1 - e^{-w}) + e^{-w} \left(\frac{w^2}{2} + 2w \right) \right\} b(\mu, \zeta) \\ &\quad + \left\{ 1 - \left[1 + w + \frac{w^2}{2} (1 + z) + \frac{w^3}{2} \right] e^{-w} \right\} c(\mu, \zeta) \\ &\quad - \left| \frac{\nu}{\mu\zeta} \right|^2 \int_{\eta_*}^{\infty} \nu \left| \frac{\eta - s}{\mu\zeta} \right| \left(1 - \frac{\nu}{2} \left| \frac{\eta - s}{\mu\zeta} \right| \right) C[h](s, \mu, \zeta) e^{-\nu \left| \frac{\eta - s}{\mu\zeta} \right|} ds, \end{aligned} \quad (4.15)$$

where $\text{E}_1(z) = \int_z^{\infty} y^{-1} e^{-y} dy$, ($z > 0$), $w \equiv \nu \left| \frac{\eta_* - \eta}{\mu\zeta} \right|$, $y \equiv z + w = \left| \frac{\nu\eta_*}{\mu\zeta} \right|$, and η_* is a fixed positive constant such that $3\eta < \eta_* \ll 1$. In the above three limiting processes, w and y behave respectively as (i) $w \rightarrow y$ with $y(> 0)$ fixed, (ii) $w, y \rightarrow \infty$, and (iii) $w, y \rightarrow \infty$; and in all cases $\frac{z}{w}$ and $\frac{y}{w}$ are finite. The integral on the last line is seen to be bounded by

$$\frac{w}{\eta_* - \eta} \left(\frac{1}{2} w^2 + 2w + 2 \right) e^{-w} \max_{\eta \geq \eta_*} C[h],$$

and always remains finite in those limits. However, because of the properties $e^x \text{E}_1(x) + \gamma + \ln x = O(x \ln x)$ for $x \ll 1$ and $e^x \text{E}_1(x) = x^{-1} + O(x^{-2})$ for $x \gg 1$, it is seen that the part

with the underbrace approaches $\gamma + \ln |\frac{\nu}{\mu\zeta}|$ in the limit (i) and grows in proportion to $\ln \eta$ in the limit (iii). The limit (i) implies that $T[C[h]]$ grows with the rate $\ln |\mu\zeta|$ as $\mu\zeta \rightarrow -0$ on the boundary $\eta = 0$. In the limit (ii), it approaches $1 - \ln \eta$; thus $T[C[h]]$ grows in proportion to $\ln \eta$ as $\eta \rightarrow 0$ at $\mu\zeta = -0$.

To summarize, Ψ exhibits the singular behavior and may diverge for $\eta, \mu\zeta \sim 0$. To be specific, when $\mu\zeta < 0$, it diverges with the rate $\ln \eta$ as $\eta \rightarrow 0$ with $|\frac{\nu\eta}{\mu\zeta}|$ fixed and with the rate $\ln |\mu\zeta|$ as $\mu\zeta \rightarrow 0$ for $\eta = 0$. When $\mu\zeta > 0$, it diverges with the rate η^{-1} as $\eta \rightarrow 0$ with $|\frac{\nu\eta}{\mu\zeta}|$ fixed due to both $W[h_0]$ and $T[C[h]]$, especially the contribution from the part $a(\mu, \zeta)$ of $C[h]$ for the latter. When $\mu\zeta = \pm 0$, it grows with the rate $\ln \eta$ as $\eta \rightarrow 0$.

4.4.3 Influence of the singularity of Ψ on its moments and $C[\Psi]$

Due to the behavior studied in Sec. 4.4.2, $C[\Psi]$ and the moments of Ψ on the boundary should be considered carefully. We define them as the values in the limit $\eta \rightarrow 0$. Then, a special attention should be made to the singularity in the limit (iii); the order of the limit $\eta \rightarrow 0$ and the integration with respect to μ and ζ is not allowed to change naively. We have obtained the following identities, by which the moments of Ψ for $\eta = 0$ can be computed appropriately numerically:

$$\begin{aligned} \lim_{\eta \rightarrow 0} \int_{\mu\zeta > 0} \mathcal{P}(\mu, \zeta) W[h_0] d\zeta &= 2\pi \lim_{\eta \rightarrow 0} \int_0^\infty \int_0^1 \zeta^2 \mathcal{P}(\mu, \zeta) W[h_0] d\mu d\zeta \\ &= -2\pi \lim_{\eta \rightarrow 0} \int_0^\infty \int_0^1 (1 - \mu^2) \frac{\zeta^4}{2\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \mathcal{P}(\mu, \zeta) h_0(\mu, \zeta) d\mu d\zeta \\ &= -\pi \int_0^\infty \frac{\zeta^3}{\nu} \mathcal{P}(0, \zeta) h_0(0, \zeta) d\zeta = -\pi \int_0^\infty \frac{\zeta^3}{\nu} \mathcal{P}(0, \zeta) h(0, +0, \zeta) d\zeta, \end{aligned} \quad (4.16a)$$

$$\lim_{\eta \rightarrow 0} \int_{\mu\zeta < 0} \mathcal{P}(\mu, \zeta) T[C[h]] d\zeta = 2\pi \lim_{\epsilon \downarrow 0} \int_0^\infty \int_{-1}^{-\epsilon} \mathcal{P}(\mu, \zeta) \lim_{\eta \rightarrow 0} \zeta^2 T[C[h]] d\mu d\zeta, \quad (4.16b)$$

$$\begin{aligned} \lim_{\eta \rightarrow 0} \int_{\mu\zeta > 0} \mathcal{P}(\mu, \zeta) T[C[h]] d\zeta &= 2\pi \lim_{\eta \rightarrow 0} \int_0^\infty \int_0^1 \zeta^2 \mathcal{P}(\mu, \zeta) T[C[h]] d\mu d\zeta \\ &= \pi \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^\infty \int_0^1 \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\left| \frac{\nu\eta}{\mu\zeta} \right|} \mathcal{P}(\mu, \zeta) a(\mu, \zeta) \frac{(1 - \mu^2)\zeta^4}{\nu^3} d\mu d\zeta \\ &= \pi \int_0^\infty \frac{\zeta^3}{\nu^2} \mathcal{P}(0, \zeta) a(0, \zeta) d\zeta = \pi \int_0^\infty \frac{\zeta^3}{\nu^2} \mathcal{P}(0, \zeta) C[h](0, 0, \zeta) d\zeta, \end{aligned} \quad (4.16c)$$

where $\mathcal{P}$ is a regular function of its arguments [see Sec. 4.8 for the derivation of Eq. (4.16)]. Therefore, we have

$$\begin{aligned} \lim_{\eta \rightarrow 0} \int \mathcal{P}(\mu, \zeta) \Psi(\eta, \mu, \zeta) d\zeta &= 2\pi \lim_{\epsilon \downarrow 0} \int_0^\infty \int_{-1}^{-\epsilon} \mathcal{P}(\mu, \zeta) \zeta^2 \Psi(0, \mu, \zeta) d\mu d\zeta \\ &\quad + \pi \int_0^\infty \frac{\zeta^3}{\nu^2} \mathcal{P}(0, \zeta) \{C[h](0, 0, \zeta) - \nu h(0, +0, \zeta)\} d\zeta. \end{aligned} \quad (4.17)$$

As for $C[\Psi]$, it can be written in the form

$$C[\Psi](\eta, \mu, \zeta) = \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \xi^2 K(M, \xi, \theta; \mu, \zeta) \Psi(\eta, M, \xi) d\theta dM d\xi,$$

where θ is the angle between the projections of ξ and ζ onto the plane normal to their common polar direction ($M = \mu = 1$). The key difference of the above K from $\mathcal{P}$ is that it has a singularity of $|\xi - \zeta|^{-1}$. Fortunately, however, the singularity can be removed by transformation of integral variables, so that we obtain the identity similar to Eq. (4.17):

$$\begin{aligned} C[\Psi](0, \mu, \zeta) &\equiv \lim_{\eta \rightarrow 0} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \xi^2 K(M, \xi, \theta; \mu, \zeta) \Psi(\eta, M, \xi) d\theta dM d\xi \\ &= \lim_{\epsilon \downarrow 0} \int_0^\infty \int_{-1}^{-\epsilon} \int_0^{2\pi} K(M, \xi, \theta; \mu, \zeta) \xi^2 \Psi(0, M, \xi) d\theta dM d\xi \\ &\quad + \int_0^\infty \int_0^{2\pi} \frac{\xi^3}{2\nu^2} K(0, \xi, \theta; \mu, \zeta) \{C[h](0, 0, \xi) - \nu h(0, +0, \xi)\} d\theta d\xi. \end{aligned} \quad (4.18)$$

The required calculation becomes lengthy and we omit it in the present paper.

4.4.4 Discretization and some details of numerical method

Thanks to the factor E , both of Φ and Ψ are expected to decay rapidly in ζ , as well as in η . We thus truncate the ζ - and η -spaces at $\zeta = Z$ and $\eta = d$ for properly chosen positive constants Z and d and restrict the computation in the region $-1 \leq \mu \leq 1$, $0 \leq \zeta \leq Z$, and $0 \leq \eta \leq d$. The chosen values of Z and d are justified a posteriori from the results. Non-uniform discretization is made for each variables. The grid in molecular velocity space (μ, ζ) is arranged two-fold: one is the grid for capturing the milder function $C[\Phi]$ (and $C[\Psi]$), while the other is that for capturing Φ (and Ψ) to compute $C[\Phi]$ (and $C[\Psi]$) enough accurately. The latter grid should be finer than the former. In the case of the standard grid, i.e., (S1,M1) in Ref. [18], we set $Z = 5.0$ and $d = 44.46$, and arrange 251 grid points in η space; the two-fold grid in μ - ζ space consists of 257×141 and 449×161 points.¹ More details are found in Ref. [18].

¹Using the notation appearing soon below, $N_\eta = 125$, $N_\mu = 64$, $N_\zeta = 70$, $N_M = 112$, and $N_\xi = 80$.

For the spatial coordinate η , we arrange $2N_\eta + 1$ grid points, say $\eta^{(i)}$ ($i = 0, 1, \dots, 2N_\eta$), in the region $0 \leq \eta \leq d$:

$$0 = \eta^{(0)} < \eta^{(1)} < \dots < \eta^{(2N_\eta)} = d. \quad (4.19)$$

As the primary grid for the molecular velocity space, we arrange $(4N_\mu + 1) \times (2N_\zeta + 1)$ points, say $(\mu^{(j)}, \zeta^{(k)})$ ($j = -2N_\mu, \dots, 2N_\mu$; $k = 0, \dots, 2N_\zeta$) in the region $-1 \leq \mu \leq 1$ and $0 \leq \zeta \leq Z$:

$$\begin{aligned} 0 &= \mu^{(0)} < \mu^{(1)} < \dots < \mu^{(2N_\mu-1)} < \mu^{(2N_\mu)} = 1, \\ \mu^{(-j)} &= -\mu^{(j)}, \quad (1 \leq j \leq 2N_\mu), \\ 0 &= \zeta^{(0)} < \zeta^{(1)} < \dots < \zeta^{(2N_\zeta)} = Z, \end{aligned}$$

while, as the secondary grid, we arrange $(4N_M + 1) \times (2N_\xi + 1)$ points in μ - ζ space, say $(M^{(l)}, \xi^{(m)})$ ($l = -2N_M, \dots, 2N_M$; $m = 0, \dots, 2N_\xi$):

$$\begin{aligned} 0 &= M^{(0)} < M^{(1)} < \dots < M^{(2N_M-1)} < M^{(2N_M)} = 1, \\ M^{(-l)} &= -M^{(l)}, \quad (1 \leq l \leq 2N_M), \\ 0 &= \xi^{(0)} < \xi^{(1)} < \dots < \xi^{(2N_\xi)} = Z. \end{aligned}$$

Note that $N_M > N_\mu$ and $N_\xi > N_\zeta$. Once the reliable data of $C[\Psi]$ are obtained, the solution method for Eq. (4.9) is the same as that in Ref. [18]. Hence, we explain below how to obtain Ψ and $C[\Psi]$.

As is obvious from Eq. (4.9), in order to compute Φ , the information of $C[\Psi]$ on all grid points $(\eta^{(i)}, \mu^{(j)}, \zeta^{(k)})$ is required. For $\eta^{(i)}$ with $i \neq 0$, $C[\Psi]$ is computed in the same way as the computation of $C[\Phi]$, which is based on the piecewise quadratic interpolation of Ψ in the molecular velocity space (see also Ref. [18]). For $\eta^{(0)}$, however, $C[\Psi]$ is computed in accordance with Eq. (4.18) in Sec. 4.4.3, where $C[h](0, 0, \xi)$ is interpolated piecewise quadratically in ξ in the last integral. As to the first integral, the region of integration with respect to M is truncated from $-1 < M < -\epsilon$ with $\epsilon \rightarrow +0$ to $-1 < M < M^{(-2)}$ for the integration of $T[C[h]]$ in Ψ [see Eq. (4.12)]; no truncation is made for the integration of the reminder $\Psi - T[C[h]]$. In both integrations, $T[C[h]]$ or $\Psi - T[C[h]]$ is interpolated piecewise quadratically both in M and ξ . The truncation causes the error of $O(M^{(-2)} \ln |M^{(-2)}|)$, which can be controlled to be small enough by a proper arrangement of the secondary grid.

In computing the contribution from Ψ to the Knudsen-layer functions at $\eta = 0$, Eq. (4.17) is treated in the same way.

Finally, we turn to the computation of Ψ . As is clear from the previous paragraph, we have already handled the divergence of Ψ at $\eta = 0$ and do not have to worry about it. When computing Ψ by Eq. (4.12), we use the following piecewise interpolation with respect to s for the part $T[C[h]]$ in the range $0 < s \leq \eta^{(2\sigma)}$, where σ is a certain small natural number, [see Eq. (4.13)]:

$$C[h](s, \mu^{(j)}, \zeta^{(k)}) = a(\mu^{(j)}, \zeta^{(k)}) + b(\mu^{(j)}, \zeta^{(k)})s \ln s + c(\mu^{(j)}, \zeta^{(k)})s, \\ \text{for } \eta^{(2i)} < s \leq \eta^{(2i+2)}, \quad i = 0, 1, \dots, \sigma - 1.$$

Here the coefficients a , b , and c are determined in such a way that $C[h](s, \cdot, \cdot)$ takes the values $C[h](\eta^{(2i)}, \cdot, \cdot)$, $C[h](\eta^{(2i+1)}, \cdot, \cdot)$, and $C[h](\eta^{(2i+2)}, \cdot, \cdot)$ at $s = \eta^{(2i)}$, $\eta^{(2i+1)}$, and $\eta^{(2i+2)}$. As to the remaining part $\Psi - T[C[h]]$ in that range and Ψ in the remaining range, we use a piecewise quadratic interpolation with respect to s , as in Ref. [18]. The value of $\eta^{(2\sigma)}$ is about 0.1 in all of our computations.

4.5 Numerical results

The results in the present section are obtained by the standard grid (S1,M1) in Ref. [18], unless otherwise stated. The measure of accuracy of the present computations is given in Sec. 4.9. Further data, in addition to those covering the results in Refs. [10] and [18], are available from the Kyoto University Research Information Repository (<http://hdl.handle.net/2433/199811>).

4.5.1 Velocity distribution functions

As an illustrative example, we show $\psi_A E$ at three spatial points $\eta = 0.015, 0.58, 3.0$ in Fig. 4.1 and $\bar{\psi}_5 E$ at four spatial points $\eta = 0, 0.015, 0.58, 3.0$ in Fig. 4.2. It should be reminded that $\psi_5 = \frac{1}{2}\psi_A + \bar{\psi}_5$ (see the first paragraph of Sec. 4.4) and that ψ_A contains a diverging part at $\eta \sim 0$ and $\mu\zeta \sim 0$, while $\bar{\psi}_5$ does not. Due to this fact, the scale of the vertical axis is changed largely in Fig. 4.1, while it is not changed in Fig. 4.2. In Fig. 4.1, $\psi_A E$ on the boundary $\eta = 0$ is not drawn by the same fact.

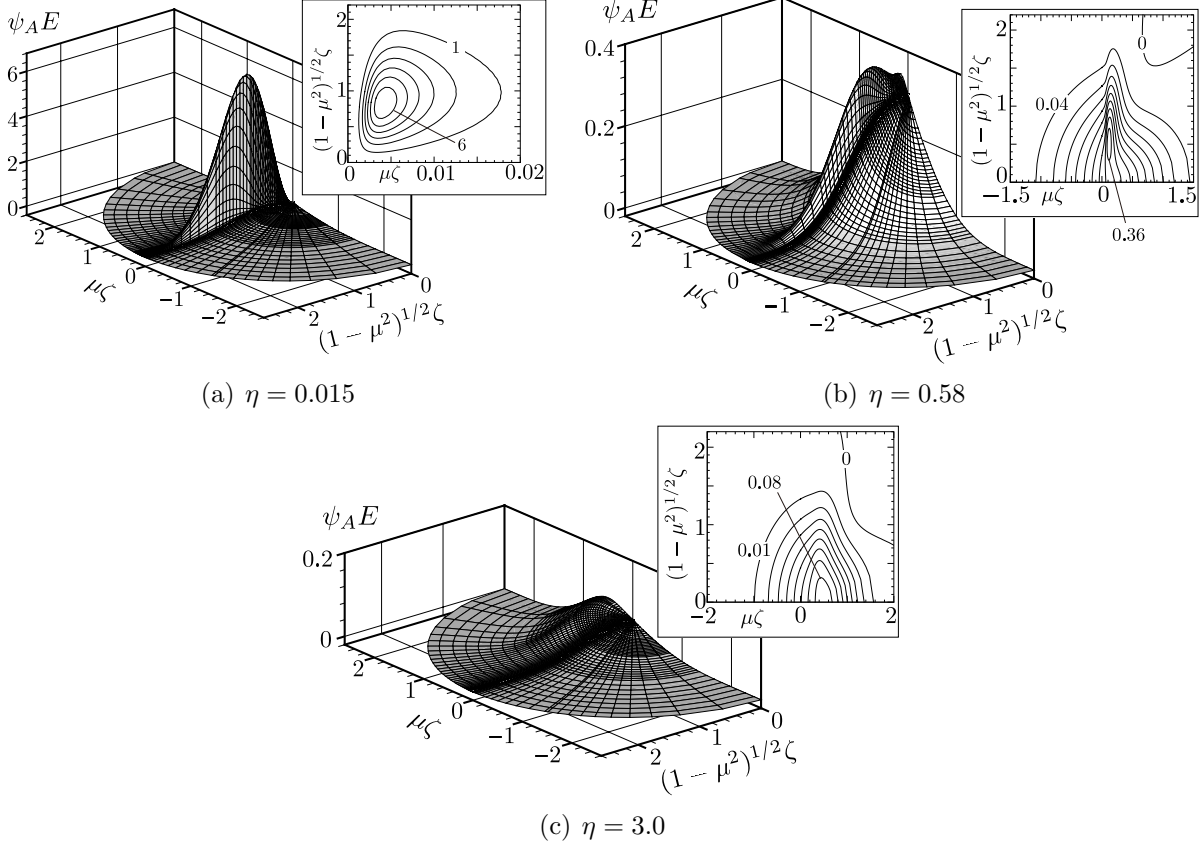


Figure 4.1: $\psi_A E$ and its contour plots at three spatial points. (a) $\eta = 0.015$, (b) $\eta = 0.58$, and (c) $\eta = 3.0$. In the contour plots, the curves are drawn with the interval of 1 in (a), 0.04 in (b), and 0.01 in (c).

In Fig. 4.2, the discontinuity is clearly observed along the line $\mu\zeta = 0$ on the boundary $\eta = 0$, which is the line corresponding to the molecular velocity tangential to the boundary (see Fig. 4.2a). The discontinuity vanishes immediately away from the boundary with keeping the other part almost unchanged. Accordingly, a continuous but steep variation appears in the molecular velocity space near the boundary (see Fig. 4.2b). These features are also observed in $\bar{\phi}_4$, $\bar{\psi}_6$, $\bar{\psi}_7$, and the solutions obtained in Ref. [18].

In order to see the singular behavior of the velocity distribution functions more closely, we show in Fig. 4.3 $\eta\phi$, $\eta\psi_A$, and $\eta\psi_B$ for small η as a function of $\mu/\eta (> 0)$. It is seen that the profiles of $\eta\phi$, $\eta\psi_A$, and $\eta\psi_B$ tend to converge individually as η decreases. This confirms that ϕ , ψ_A , and ψ_B diverge in proportion to η^{-1} for $\mu\zeta > 0$ in the limit (iii) of Sec. 4.4.2, due to the corresponding property of the part Ψ . Another close observation can be made for the limit (i) of Sec. 4.4.2 for $\mu\zeta < 0$. From Eq. (4.15), we see that for $|\mu| \ll 1$

$$T[C[h]](\eta = 0, \mu < 0, \zeta) = \frac{\zeta^2}{\nu^3} \left[\left(\frac{5}{2} - \gamma - \ln \frac{\nu}{|\mu\zeta|} \right) b(0, \zeta) + c(0, \zeta) \right] + O(\mu \ln |\mu|),$$

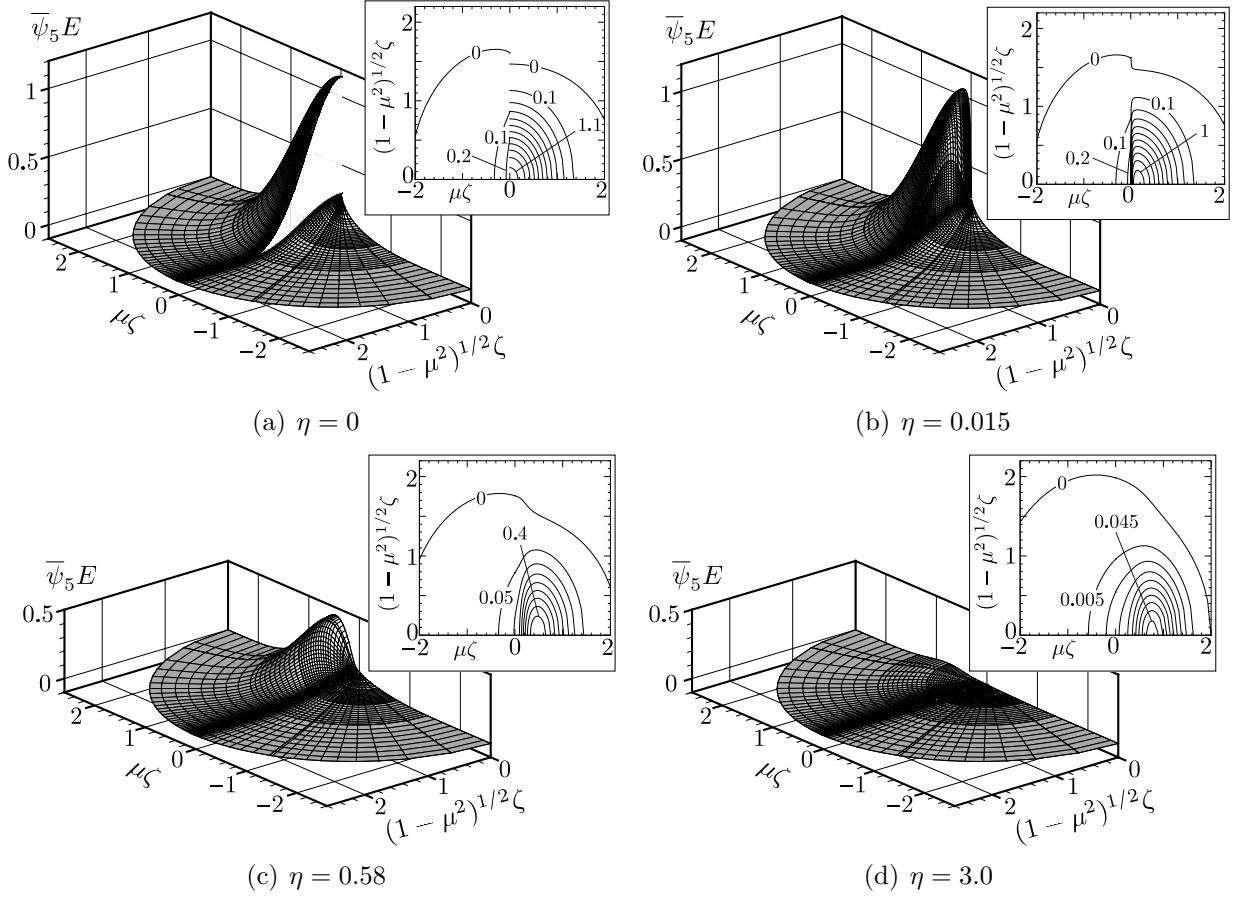


Figure 4.2: $\bar{\psi}_5 E$ and its contour plots at four spatial points. (a) $\eta = 0$, (b) $\eta = 0.015$, (c) $\eta = 0.58$, and (d) $\eta = 3.0$. In the contour plots the curves are drawn with the interval of 0.1 in (a) and (b), 0.05 in (c), and 0.005 in (d). The white vertical surface at $\mu\zeta = 0$ in (a) shows the discontinuity.

which in turn leads to the following asymptotic behavior:

$$\begin{aligned} \varphi(\eta = 0, \mu < 0, \zeta) \rightarrow (\varphi)_{asy} \equiv & \frac{\zeta^2}{\nu^3} \left[\left(\frac{5}{2} - \gamma - \ln \frac{\nu}{|\mu\zeta|} \right) b(0, \zeta) + c(0, \zeta) \right] \\ & - \frac{\zeta}{\nu^2} \frac{\partial C[h]}{\partial \mu}(0, 0, \zeta) + \frac{1}{\nu} C[\varphi](0, 0, \zeta), \quad \text{as } \mu \nearrow 0, \end{aligned} \quad (4.20)$$

where $\varphi = \phi E$, $\psi_A E$, $\psi_B E$ for $h = \phi_1 E$, $\psi_1 E$, $\psi_2 E$. Note that b and c depend on h . We show in Fig. 4.4 the numerical data of $\varphi(\eta = 0, \mu < 0, \zeta)$ for small $|\mu|$ by the solid line. It is clearly observed that, as $\mu \nearrow 0$, the numerical solution of ϕE , $\psi_A E$, or $\psi_B E$ on the boundary approaches $(\phi E)_{asy}$, $(\psi_A E)_{asy}$, or $(\psi_B E)_{asy}$, i.e., the dashed line. This again shows that our numerical method works consistently enough to capture the logarithmic divergence of the solution in $|\mu|$ for $\mu \nearrow 0$ on the boundary discussed in Sec. 4.4.2.

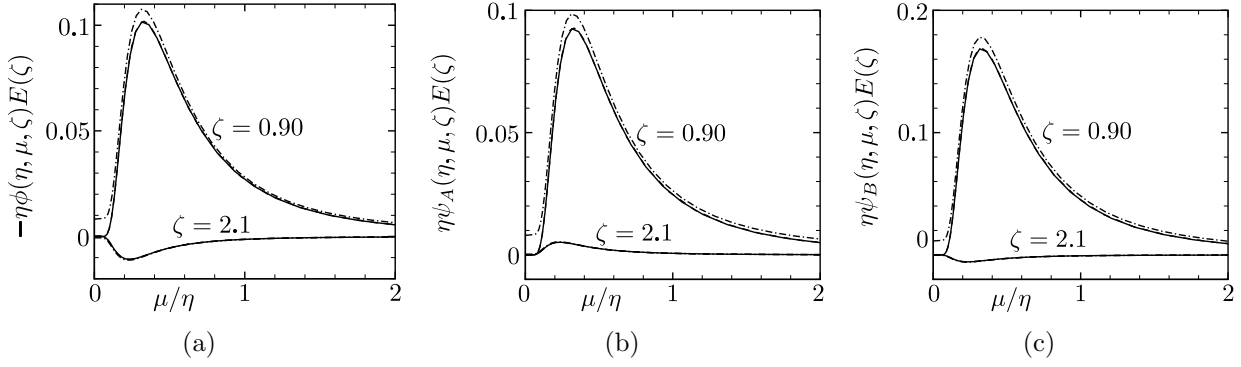


Figure 4.3: Behavior of ϕ , ψ_A , and ψ_B for small η . (a) $\eta\phi E$ vs μ/η , (b) $\eta\psi_A E$ vs μ/η , and (c) $\eta\psi_B E$ vs μ/η . Three different lines are drawn for each value of ζ in each panel: the solid line represents the result for $\eta = 7.3 \times 10^{-7}$, the dashed line that for $\eta = 1.4 \times 10^{-4}$, and the dash-dotted line that for $\eta = 1.5 \times 10^{-2}$. The dashed line almost agrees with the solid line and is invisible for $\zeta = 0.90$. Three lines almost agree with one another and the difference is invisible for $\zeta = 2.1$.

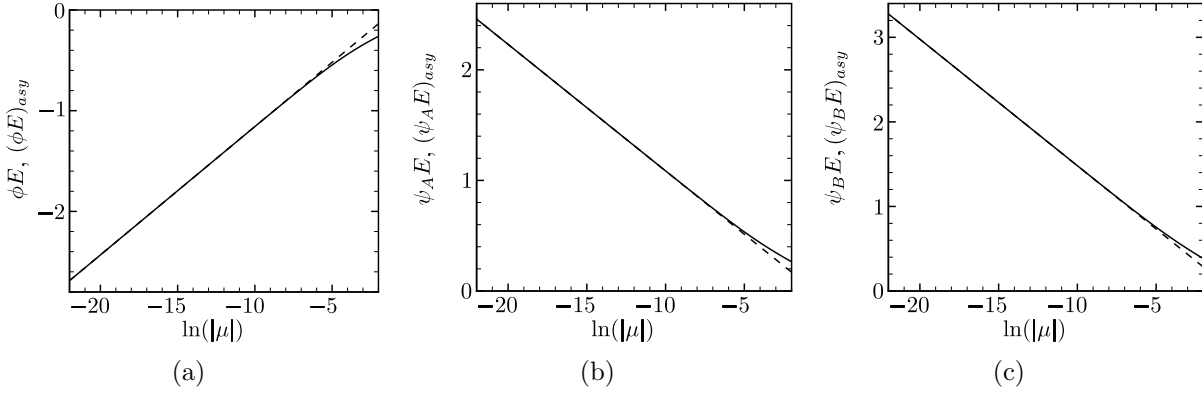


Figure 4.4: Behavior of ϕE , $\psi_A E$, and $\psi_B E$ for $\mu \lesssim 0$ on the boundary $\eta = 0$, with ζ being fixed ($\zeta = 0.90$). (a) ϕE , (b) $\psi_A E$, and (c) $\psi_B E$. The solid line represents the numerical solution of ϕE , $\psi_A E$, and $\psi_B E$, while the dashed line $(\phi E)_{asy}$, $(\psi_A E)_{asy}$, and $(\psi_B E)_{asy}$ predicted theoretically [see Eq. (4.20)].

4.5.2 Slip/jump coefficients and Knudsen-layer functions

Slip/jump coefficients are shown in Table 4.1. The data in Ref. [10], which are obtained from the information about the first-order Knudsen layer by the theory of symmetry relation [23], are also shown for comparisons. Excellent agreement ensures the consistency of the present results to the theory.

The Knudsen-layer functions $\Omega_4^{(0)}$, $\Theta_4^{(0)}$, Y 's, and H 's are shown in Fig. 4.5 and Table 4.2 [see Sec. 4.7 for H 's and Eq. (4.2) for the others]. Corresponding to the behavior of the velocity distribution functions in η , they decay fast as $\eta \rightarrow \infty$, mostly monotonically except for $\Omega_4^{(0)}$, $\Theta_4^{(0)}$, and $H_7^{(1)}$ (see Table 4.2). From the table, the 90% thickness of the Knudsen

Table 4.1: Slip/jump coefficients. The data in Ref. [10] obtained by the use of the symmetry relation [23] are also shown for comparisons.

	$c_4^{(0)}$	$b_5^{(1)}$	$b_6^{(1)}$	$b_7^{(1)}$	$b_8^{(1)}$
Present results	4.6180	-0.66012	0.24381	0.44728	-0.23353
Symmetry relation [10]	4.6181	-0.6601	0.2438	0.4472	-0.2336

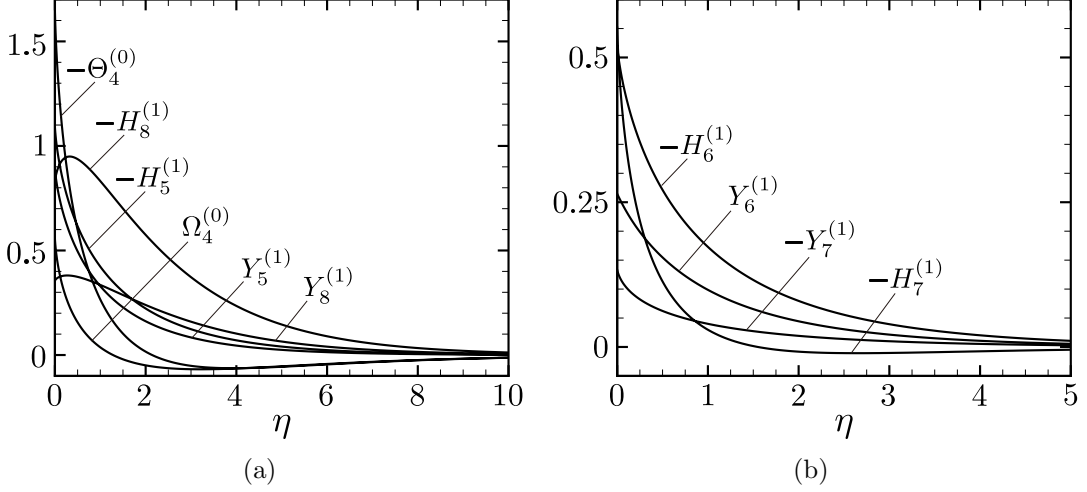


Figure 4.5: Knudsen-layer functions. (a) Knudsen-layer functions for ϕ_4 , ψ_5 , and ψ_8 , (b) those for ψ_6 and ψ_7 .

layer is seen to be about $4 \sim 5$ for $\Omega_4^{(0)}$, about $1 \sim 1.5$ for $\Theta_4^{(0)}$, about $2.5 \sim 3$ for $Y_5^{(1)}$, $H_5^{(1)}$, $Y_6^{(1)}$, $H_6^{(1)}$, and $Y_7^{(1)}$, about $0.5 \sim 1$ for $H_7^{(1)}$, and about $6 \sim 7$ for $Y_8^{(1)}$ and $H_8^{(1)}$, measured in η .

4.6 Concluding remarks

In the present paper, we have investigated the curvature effects in the generalized slip-flow theory [4,5,6], which occur at the second order of the Knudsen number expansion, and thereby complete the series of data required for its application to practical problems. This also completes our recent studies on the time-dependent slip-flow theory [10,18] for a slightly rarefied gas. In the course of analyses, we have also clarified that the curvature effects induce the local divergence singularity of the velocity distribution function, even though it is integrable to make macroscopic quantities well defined. We have obtained the precise information on the singularity and thereby have established a numerical method that handles it appropriately.

The occurrence of the local divergence above can be understood physically from the

Table 4.2: Knudsen-layer functions.

η	$\Omega_4^{(0)}$	$-\Theta_4^{(0)}$	$Y_5^{(1)}$	$-H_5^{(1)}$	$Y_6^{(1)}$	$-H_6^{(1)}$	$-Y_7^{(1)}$	$-H_7^{(1)}$	$Y_8^{(1)}$	$-H_8^{(1)}$
0.00000	0.55644	1.64676	0.89334	1.09051	0.26440	0.52114	0.13408	0.55360	0.35551	0.82712
0.02348	0.49619	1.49739	0.84419	1.03017	0.25697	0.49034	0.12204	0.47299	0.36279	0.85993
0.05165	0.45041	1.38089	0.80432	0.98148	0.24826	0.46490	0.11350	0.41545	0.36798	0.88355
0.09881	0.39296	1.23232	0.75158	0.91763	0.23478	0.43116	0.10319	0.34719	0.37368	0.90999
0.15009	0.34442	1.10501	0.70457	0.86124	0.22146	0.40119	0.09471	0.29286	0.37744	0.92837
0.19315	0.31052	1.01520	0.67030	0.82045	0.21118	0.37946	0.08886	0.25677	0.37929	0.93838
0.27263	0.25885	0.87696	0.61554	0.75576	0.19403	0.34503	0.08000	0.20477	0.38048	0.94786
0.30336	0.24169	0.83072	0.59662	0.73353	0.18795	0.33322	0.07705	0.18834	0.38032	0.94915
0.33632	0.22469	0.78471	0.57748	0.71110	0.18172	0.32133	0.07413	0.17248	0.37985	0.94935
0.40911	0.19136	0.69409	0.53874	0.66587	0.16896	0.29743	0.06838	0.14267	0.37784	0.94624
0.58327	0.12898	0.52281	0.46115	0.57572	0.14297	0.25031	0.05739	0.09170	0.36901	0.92453
0.79673	0.07444	0.37133	0.38637	0.48902	0.11774	0.20590	0.04734	0.05289	0.35324	0.88155
0.98271	0.03963	0.27370	0.33389	0.42799	0.10013	0.17539	0.04054	0.03135	0.33694	0.83648
1.19037	0.01031	0.19072	0.28550	0.37133	0.08407	0.14777	0.03442	0.01549	0.31729	0.78240
1.41884	-0.01360	0.12222	0.24175	0.31957	0.06978	0.12326	0.02899	0.00439	0.29495	0.72169
1.58214	-0.02669	0.08419	0.21530	0.28793	0.06127	0.10867	0.02574	-0.00085	0.27897	0.67881
1.84260	-0.04235	0.03766	0.17973	0.24482	0.05003	0.08934	0.02142	-0.00616	0.25400	0.61276
2.02594	-0.05037	0.01298	0.15869	0.21893	0.04351	0.07807	0.01887	-0.00836	0.23705	0.56858
2.51495	-0.06324	-0.03017	0.11479	0.16369	0.03027	0.05505	0.01361	-0.01064	0.19528	0.46183
2.61756	-0.06476	-0.03612	0.10739	0.15417	0.02809	0.05123	0.01272	-0.01071	0.18722	0.44156
3.04221	-0.06800	-0.05292	0.08182	0.12065	0.02073	0.03824	0.00967	-0.01023	0.15655	0.36544
3.26234	-0.06820	-0.05785	0.07121	0.10643	0.01776	0.03294	0.00841	-0.00969	0.14234	0.33068
3.48717	-0.06766	-0.06096	0.06188	0.09371	0.01519	0.02833	0.00730	-0.00905	0.12897	0.29825
3.94956	-0.06482	-0.06301	0.04651	0.07230	0.01106	0.02086	0.00547	-0.00762	0.10486	0.24051
4.06758	-0.06383	-0.06285	0.04328	0.06769	0.01021	0.01931	0.00509	-0.00726	0.09939	0.22753
5.04166	-0.05361	-0.05575	0.02407	0.03952	0.00534	0.01031	0.00281	-0.00465	0.06326	0.14297
6.06059	-0.04205	-0.04428	0.01321	0.02264	0.00276	0.00543	0.00153	-0.00277	0.03888	0.08708
7.11725	-0.03144	-0.03298	0.00716	0.01276	0.00141	0.00283	0.00082	-0.00158	0.02321	0.05169
8.07087	-0.02364	-0.02459	0.00415	0.00762	0.00078	0.00158	0.00047	-0.00094	0.01447	0.03212
10.06348	-0.01245	-0.01269	0.00135	0.00260	0.00023	0.00048	0.00015	-0.00031	0.00530	0.01175
15.00856	-0.00220	-0.00216	0.00009	0.00018	0.00001	0.00003	0.00001	-0.00002	0.00042	0.00093
20.05147	-0.00034	-0.00033	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00003	0.00007
25.14669	-0.00005	-0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
30.05092	-0.00001	-0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
35.11181	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

viewpoint of the boundary geometry. In the generalized slip-flow theory, the study of the Knudsen-layer structure is reduced to a series of spatially half-space problems. It implies that the boundary is approximated as if the boundary were flat at each stage of the analysis. As a result, the solution at the first order of the Knudsen number is forced to be discontinuous *on the boundary* in the tangential direction of molecular velocity, resulting in the divergence singularity at the second order through the inhomogeneous term. Actually, however, if the curvature exists, the behavior of that discontinuity dramatically changes. For instance, when the boundary is the surface of a convex body, that discontinuity exists but propagates into the gas region in the tangential directions of the boundary. Then, the implicit assumption of continuous solution in the analysis breaks down. The region with this breakdown, which is

thinner by one order of the Knudsen number in its thickness, is called the S-layer [6], which was discovered in Ref. [28] and was recognized later in Ref. [29] as the manifestation of the discontinuity propagation. The estimate of corrections to the generalized slip-flow theory in the S-layer can be found in Refs. [29] and [6]. When the boundary is a surface of concave body, the velocity distribution function becomes continuous even on the boundary. These facts near the convex/concave boundary illustrate that the flattened boundary treatment in the asymptotic analysis does not capture the very local structure of the velocity distribution function, when the curvature exists. Improvement of the local analytical drawback could be an interesting subject in the mathematical physics.

4.7 Appendix A: Stress and heat flow

The stress tensor and heat-flow vector are also familiar fluid-dynamic quantities that become necessary, most typically, in computing the momentum and energy exchange with the body surface. Denoting the former by $p_0(\delta_{ij} + P_{ij})$ and the latter by $p_0(2RT_0)^{1/2}Q_i$, their Hilbert part $h_H = h_{H0} + h_{H1}\varepsilon + \dots$ and Knudsen-layer correction $h_K = h_{K0} + h_{K1}\varepsilon + \dots$ ($h = P_{ij}, Q_i$) up to the second order in ε are summarized as follows:

$$P_{ijHm} = P_{Hm}\delta_{ij} - \gamma_1 \frac{\overline{\partial u_{iHm-1}}}{\partial x_j} + \frac{1}{2}\gamma_3 \frac{\overline{\partial^2 \tau_{Hm-2}}}{\partial x_i \partial x_j}, \quad (m = 0, 1, 2), \quad (4.21a)$$

$$Q_{iHm} = -\frac{5}{4}\gamma_2 \frac{\partial \tau_{Hm-1}}{\partial x_i} + \frac{1}{2}\gamma_3 \frac{\partial^2 u_{iHm-2}}{\partial x_j^2}, \quad (m = 0, 1, 2), \quad (4.21b)$$

and

$$P_{ijKm} = \frac{3}{2} \frac{\partial \tau_{Hm-1}}{\partial x_k} n_k (\delta_{ij} - n_i n_j) [\Omega_1^{(0)}(\eta) + \Theta_1^{(0)}(\eta)], \quad (m = 0, 1), \quad (4.22a)$$

$$P_{ijK2} n_i n_j = -3\bar{\kappa} \frac{\partial \tau_{H0}}{\partial x_i} n_i \int_{\eta}^{\infty} [\Omega_1^{(0)}(z) + \Theta_1^{(0)}(z)] dz, \quad (4.22b)$$

$$P_{ijK2} n_i t_j = \frac{3}{2} \left(\frac{\partial^2 \tau_{H0}}{\partial x_i \partial x_j} n_i t_j + \kappa_{ij} t_j \frac{\partial \tau_{H0}}{\partial x_i} \right) \int_{\eta}^{\infty} [\Omega_1^{(0)}(z) + \Theta_1^{(0)}(z)] dz, \quad (4.22c)$$

$$\begin{aligned} Q_{iKmt_i} = & \frac{\overline{\partial u_{iHm-1}}}{\partial x_j} n_i t_j H_1^{(1)}(\eta) + \frac{\partial \tau_{Hm-1}}{\partial x_i} t_i H_2^{(1)}(\eta) \\ & + \frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} n_i t_j H_3^{(1)}(\eta) + \frac{\partial}{\partial x_i} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_i n_j t_k H_4^{(1)}(\eta) \\ & + \bar{\kappa} \frac{\overline{\partial u_{iHm-2}}}{\partial x_j} n_i t_j H_5^{(1)}(\eta) + \kappa_{ij} \frac{\overline{\partial u_{jHm-2}}}{\partial x_k} n_k t_i H_6^{(1)}(\eta) \\ & + \kappa_{ij} t_j \frac{\partial \tau_{Hm-2}}{\partial x_i} H_7^{(1)}(\eta) + \bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} t_i H_8^{(1)}(\eta), \quad (m = 0, 1, 2), \end{aligned} \quad (4.22d)$$

$$Q_{iKm}n_i = \left[\frac{\partial^2 \tau_{Hm-2}}{\partial x_i \partial x_j} (\delta_{ij} - n_i n_j) - 2\bar{\kappa} \frac{\partial \tau_{Hm-2}}{\partial x_i} n_i \right] \int_{\eta}^{\infty} H_2^{(1)}(z) dz \\ - \frac{1}{2} \frac{\partial}{\partial x_i} \frac{\partial u_{jHm-2}}{\partial x_k} n_i n_j n_k \int_{\eta}^{\infty} H_1^{(1)}(z) dz, \quad (m = 0, 1, 2). \quad (4.22e)$$

Here the quantities with the subscript H in Eq. (4.22) denote their value on the boundary. The functions $H_1^{(1)} \sim H_4^{(1)}$ have already been obtained in [20,21,18]. The present work newly provides the data of $H_5^{(1)} \sim H_8^{(1)}$, which are included in Fig. 4.5 and Table 4.2.

4.8 Appendix B: Sketch of derivation of Eq. (4.16)

The non-trivial equality in Eq. (4.16a) is

$$\lim_{\eta \rightarrow 0} \int_0^{\infty} \int_0^1 (1 - \mu^2) \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \mathcal{P}(\mu, \zeta) h_0(\mu, \zeta) d\mu d\zeta = \int_0^{\infty} \frac{\zeta^3}{\nu} \mathcal{P}(0, \zeta) h_0(0, \zeta) d\zeta. \quad (4.23)$$

Once Eq. (4.23) is proved, the still non-trivial equality in Eq. (4.16c) is

$$2 \lim_{\eta \rightarrow 0} \int_0^{\infty} \int_0^1 \zeta^2 \mathcal{P}(\mu, \zeta) T[C[h]] d\mu d\zeta \\ = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^{\infty} \int_0^1 \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\left| \frac{\nu\eta}{\mu\zeta} \right|} \mathcal{P}(\mu, \zeta) a(\mu, \zeta) \frac{(1 - \mu^2)\zeta^4}{\nu^3} d\mu d\zeta. \quad (4.24)$$

We show below the outline of the proof for Eqs. (4.23), (4.24), and (4.16b).

Proof of Eq. (4.23) We first split the region of integration with respect to μ into $(0, \delta)$ and $(\delta, 1)$, where $0 < \delta < 1$ is a constant. Then for the second part, we see that

$$\left| \int_0^{\infty} \int_{\delta}^1 \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \{ (1 - \mu^2) \mathcal{P}(\mu, \zeta) h_0(\mu, \zeta) - \mathcal{P}(0, \zeta) h_0(0, \zeta) \} d\mu d\zeta \right| \\ \leq \int_0^{\infty} \int_{\delta}^1 \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \max_{\mu} (|\mathcal{P}(\mu, \zeta) h_0(\mu, \zeta)| + |\mathcal{P}(0, \zeta) h_0(0, \zeta)|) d\mu d\zeta \\ \leq \delta^{-3} \int_0^{\infty} \eta \zeta^2 \left| \frac{\nu\eta}{\zeta} \right| e^{-\frac{\nu\eta}{\zeta}} \max_{\mu} (|\mathcal{P}(\mu, \zeta) h_0(\mu, \zeta)| + |\mathcal{P}(0, \zeta) h_0(0, \zeta)|) d\zeta \\ \leq \delta^{-3} \eta \int_0^{\infty} \zeta^2 \max_{\mu} (|\mathcal{P}(\mu, \zeta) h_0(\mu, \zeta)| + |\mathcal{P}(0, \zeta) h_0(0, \zeta)|) d\zeta \rightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

On the other hand, by applying to $\mathcal{P}$ and h_0 the mean-value theorem with respect to μ , we have

$$\left| \int_0^{\infty} \int_0^{\delta} \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \{ (1 - \mu^2) \mathcal{P}(\mu, \zeta) h_0(\mu, \zeta) - \mathcal{P}(0, \zeta) h_0(0, \zeta) \} d\mu d\zeta \right| \\ \leq \int_0^{\infty} \int_0^{\delta} \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} (|\mathcal{P}' h_0(0, \zeta)| \mu + |\mathcal{P}(0, \zeta) h_0'(\mu)| \mu + |\mathcal{P} h_0| \mu^2 + |\mathcal{P}' h_0'| \mu^2) d\mu d\zeta$$

$$\begin{aligned}
&\leq \int_0^\infty \int_0^\delta \frac{\zeta^3}{\nu} \left| \frac{\nu\eta}{\mu\zeta} \right|^2 e^{-\frac{\nu\eta}{\mu\zeta}} \max_\mu (|\mathcal{P}'h_0(0, \zeta)| + |\mathcal{P}(0, \zeta)h'_0|) d\mu d\zeta \\
&\quad + \eta \int_0^\infty \int_0^\delta \zeta^2 \left| \frac{\nu\eta}{\mu\zeta} \right| e^{-\frac{\nu\eta}{\mu\zeta}} \max_\mu (|\mathcal{P}h_0| + |\mathcal{P}'h'_0|) d\mu d\zeta \\
&\leq \delta \int_0^\infty \frac{\zeta^3}{\nu} \max_\mu (|\mathcal{P}'h_0(0, \zeta)| + |\mathcal{P}(0, \zeta)h'_0|) d\zeta + \eta \delta \int_0^\infty \zeta^2 \max_\mu (|\mathcal{P}h_0| + |\mathcal{P}'h'_0|) d\zeta \\
&\quad \rightarrow \delta \int_0^\infty \frac{\zeta^3}{\nu} \max_\mu (|\mathcal{P}'h_0(0, \zeta)| + |\mathcal{P}(0, \zeta)h'_0|) d\zeta, \quad \text{as } \eta \rightarrow 0,
\end{aligned}$$

where $\mathcal{P}'$ (or h'_0) is the partial derivative of $\mathcal{P}$ (or h_0) with respect to μ at $\mu = \mu_P$ (or μ_h), and μ_P (or μ_h) is a certain value in the interval $[0, \mu]$ that depends on μ and ζ . Therefore, we have

$$\begin{aligned}
&\lim_{\eta \rightarrow 0} \int_0^\infty \int_0^1 (1 - \mu^2) \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \mathcal{P}(\mu, \zeta) h_0(\mu, \zeta) d\mu d\zeta \\
&= \lim_{\eta \rightarrow 0} \int_0^\infty \int_0^1 \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \mathcal{P}(0, \zeta) h_0(0, \zeta) d\mu d\zeta + O(\delta) \\
&= \lim_{\eta \rightarrow 0} \int_0^\infty \frac{\zeta^3}{\nu} \left(1 + \frac{\nu\eta}{\zeta}\right) e^{-\frac{\nu\eta}{\zeta}} \mathcal{P}(0, \zeta) h_0(0, \zeta) d\zeta + O(\delta) \\
&= \lim_{\eta \rightarrow 0} \int_0^\infty \frac{\zeta^3}{\nu} e^{-\frac{\nu\eta}{\zeta}} \mathcal{P}(0, \zeta) h_0(0, \zeta) d\zeta + O(\delta). \quad (4.25)
\end{aligned}$$

Now, in the most right-hand side of Eq. (4.25), we can change the order of the limit and integration, because

$$\begin{aligned}
&\left| \int_0^\infty \frac{\zeta^3}{\nu} (1 - e^{-\frac{\nu\eta}{\zeta}}) \mathcal{P}(0, \zeta) h_0(0, \zeta) d\zeta \right| \leq \int_0^\infty \frac{\zeta}{\nu} (1 - e^{-\frac{\nu\eta}{\zeta}}) \zeta^2 |\mathcal{P}(0, \zeta) h_0(0, \zeta)| d\zeta \\
&\leq \eta \int_0^\infty \zeta^2 |\mathcal{P}(0, \zeta) h_0(0, \zeta)| d\zeta \rightarrow 0, \quad \text{as } \eta \rightarrow 0.
\end{aligned}$$

Here, we have used the fact that $0 \leq x^{-1}(1 - e^{-x\eta}) \leq \eta$ for $x \geq 0$. Therefore, we finally arrive at

$$\lim_{\eta \rightarrow 0} \int_0^\infty \int_0^1 (1 - \mu^2) \frac{\zeta^4}{\nu^2} \frac{1}{\eta} \left| \frac{\nu\eta}{\mu\zeta} \right|^3 e^{-\frac{\nu\eta}{\mu\zeta}} \mathcal{P}(\mu, \zeta) h_0(\mu, \zeta) d\mu d\zeta = \int_0^\infty \frac{\zeta^3}{\nu} \mathcal{P}(0, \zeta) h_0(0, \zeta) d\zeta + O(\delta).$$

In the course of estimates, δ is a positive constant, arbitrary as far as smaller than unity; thus it can be made small as we wish, which proves Eq. (4.23).

Proof of Eq. (4.24) We use the expression (4.13) of $C[h]$ for small η and its consequence (4.14) for $\mu\zeta > 0$, namely

$$\frac{\nu^3 T[C[h]]}{(1 - \mu^2)\zeta^2} = \frac{1}{\eta} h_1(z) a(\mu, \zeta) + h_2(z) b(\mu, \zeta) \ln \eta + h_3(z) b(\mu, \zeta) + h_2(z) c(\mu, \zeta),$$

where

$$\begin{aligned} h_1(z) &= \frac{1}{2}z^3e^{-z}, \quad h_2(z) = 1 - \left(\frac{1}{2}z^2 + z + 1\right)e^{-z}, \\ h_3(z) &= -\left(\frac{1}{2}z^2 + z + 1\right)e^{-z}[\text{Ei}(z) - \ln z - \gamma] + \frac{1}{2}z + \frac{5}{2}(1 - e^{-z}) - \left(\frac{1}{2}z^2 + 2z\right)e^{-z}; \end{aligned}$$

and $h_1 \sim h_3$ are $O(z^3)$ for $z \ll 1$ and $O(1)$ for $z \gg 1$ (remember that $z = \frac{\nu\eta}{\mu\zeta}$). Then,

$$\begin{aligned} 2 \int_0^\infty \int_0^1 \zeta^2 \mathcal{P}(\mu, \zeta) T[C[h]] d\mu d\zeta &= 2 \int_0^\infty \int_0^1 \frac{\zeta^4}{\nu^3} (1 - \mu^2) \mathcal{P}(\mu, \zeta) \left[\frac{1}{\eta} h_1(z) a(\mu, \zeta) \right. \\ &\quad \left. + h_2(z) b(\mu, \zeta) \ln \eta + h_3(z) b(\mu, \zeta) + h_2(z) c(\mu, \zeta) \right] d\mu d\zeta. \end{aligned} \quad (4.26)$$

For $f(z)$ that follows the same estimate for both $z \ll 1$ and $z \gg 1$ as $h_1 \sim h_3$ and for $Q(\mu, \zeta)$ that is regular and decays rapidly in ζ , we have

$$\begin{aligned} \left| \int_0^\infty \int_0^1 \frac{\zeta^4}{\nu^3} (1 - \mu^2) f(z) Q(\mu, \zeta) d\mu d\zeta \right| &\leq \int_0^\infty \frac{\zeta^4}{\nu^3} \max_\mu |Q| \int_0^1 |f(z)| d\mu d\zeta \\ &\leq \eta \int_0^\infty \frac{\zeta^3}{\nu^2} \max_\mu |Q| \int_{\frac{\nu\eta}{\zeta}}^\infty z^{-2} |f(z)| dz d\zeta \leq \eta \int_0^\infty \frac{\zeta^3}{\nu^2} \max_\mu |Q| d\zeta \int_0^\infty z^{-2} |f(z)| dz. \end{aligned}$$

Therefore, taking the limit $\eta \rightarrow 0$, all the terms, except the first, on the right-hand side of Eq. (4.26) are seen to vanish. We are left with

$$\lim_{\eta \rightarrow 0} 2 \int_0^\infty \int_0^1 \zeta^2 \mathcal{P}(\mu, \zeta) T[C[h]] d\mu d\zeta = 2 \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^\infty \int_0^1 \frac{\zeta^4}{\nu^3} (1 - \mu^2) \mathcal{P}(\mu, \zeta) h_1(z) a(\mu, \zeta) d\mu d\zeta,$$

which is none other than Eq. (4.24).

Proof of Eq. (4.16b) We use again the expression (4.13) of $C[h]$ for small η and its consequence (4.15) for $\mu\zeta < 0$. We first rewrite Eq. (4.15) as

$$\begin{aligned} T[C[h]] &= [f_1(w, \eta, \eta_*) a(\mu, \zeta) + f_2(z, \eta) b(\mu, \zeta) + f_3(w, z, y, \eta_*) b(\mu, \zeta) \\ &\quad + f_4(w, z) c(\mu, \zeta) + f_5(\mu, \zeta, \eta, \eta_*)] \frac{(1 - \mu^2) \zeta^2}{\nu^3}, \end{aligned} \quad (4.27)$$

$$f_1 = -\frac{w^3 e^{-w}}{2|\eta - \eta_*|}, \quad (4.28)$$

$$f_2 = \ln \eta - (z - 1)e^z \text{E}_1(z) + \frac{z}{2}[ze^z \text{E}_1(z) - 1], \quad (4.29)$$

$$\begin{aligned} f_3 &= -(\ln \eta_*)e^{-w} \left[\left(\frac{z^2}{2} + \frac{y^2}{2} - z + 1 \right) (1 + y) - zy^2 \right] - \frac{y}{2}[ye^y \text{E}_1(y) - 1]e^{-w} \\ &\quad + e^{-w} \left[(y - 1)(1 + w) - \frac{w^2}{2} \right] e^y \text{E}_1(y) + \frac{5}{2}(1 - e^{-w}) - e^{-w} \left(\frac{w^2}{2} + 2w \right), \end{aligned} \quad (4.30)$$

$$f_4 = 1 - \left(1 + w + \frac{w^2}{2}(1 + z) + \frac{w^3}{2} \right) e^{-w}, \quad (4.31)$$

$$f_5 = - \left| \frac{\nu}{\mu\zeta} \right|^2 \int_{\eta_*}^{\infty} \nu \left| \frac{\eta-s}{\mu\zeta} \right| \left(1 - \frac{\nu}{2} \left| \frac{\eta-s}{\mu\zeta} \right| \right) C[h](s, \mu, \zeta) e^{-\nu \left| \frac{\eta-s}{\mu\zeta} \right|} ds, \quad (4.32)$$

and consider the limit (i) in Sec. 4.4.2 (remember that $y = \left| \frac{\nu\eta_*}{\mu\zeta} \right|$ and $w = y - z$). Then, in this limit we have

$$\begin{aligned} T[C[h]] &\rightarrow [f_1(y, 0, \eta_*)a(\mu, \zeta) + f_2(0, 0)b(\mu, \zeta) + f_3(y, 0, y, \eta_*)b(\mu, \zeta) \\ &\quad + f_4(y, 0)c(\mu, \zeta) + f_5(\mu, \zeta, 0, \eta_*)] \frac{(1 - \mu^2)\zeta^2}{\nu^3} \equiv T[C[h]]_0. \end{aligned}$$

Here a special attention should be paid to the limit for f_2 . It yields $f_2(0, 0) = -\gamma - \ln \nu + \ln |\mu\zeta|$. The limits for the others are straightforward. As a result, $\int_{\mu\zeta < 0} \mathcal{P}(\mu, \zeta) T[C[h]]_0 d\zeta$ converges, because the integrand diverges only at the rate $\ln |\mu|$. Thanks to the regularity in ζ , the integral is identical to $2\pi \lim_{\epsilon \downarrow 0} \int_0^\infty \int_{-1}^{-\epsilon} \mathcal{P}(\mu, \zeta) T[C[h]]_0 \zeta^2 d\mu d\zeta$, so that all we have to do is just to show that

$$\left| \int_{\mu\zeta < 0} \mathcal{P}(\mu, \zeta) (T[C[h]] - T[C[h]]_0) d\zeta \right| \rightarrow 0, \quad \text{as } \eta \rightarrow 0. \quad (4.33)$$

To prove this, we estimate the differences occurring in $T[C[h]] - T[C[h]]_0$. Firstly,

$$\begin{aligned} \Delta f_1 &\equiv |a| |f_1(w, \eta, \eta_*) - f_1(y, 0, \eta_*)| = \frac{y^3 e^{-y}}{2\eta_*} \left| 1 - \left(1 - \frac{\eta}{\eta_*} \right)^2 e^{\frac{\eta}{\eta_*} y} \right| |a| \\ &\leq \frac{y^3 e^{-y}}{2\eta_*} \left| 1 - e^{\frac{\eta}{\eta_*} y} + \frac{\eta}{\eta_*} \left(2 - \frac{\eta}{\eta_*} \right) e^{\frac{\eta}{\eta_*} y} \right| |a| \leq [C_1 (e^{\frac{\eta}{\eta_*} y} - 1) + C_2 \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*} y}] |a| e^{-y/2}, \end{aligned} \quad (4.34)$$

$$\begin{aligned} \Delta f_3 &\equiv |b| |f_3(w, z, y, \eta_*) - f_3(y, 0, y, \eta_*)| \\ &= |b| \left\{ (\ln \eta_*) e^{-y} \left\{ \left(\frac{y^2}{2} + 1 \right) (1 + y) - e^{\frac{\eta}{\eta_*} y} \left[\left(\frac{y^2}{2} \frac{\eta^2}{\eta_*^2} + \frac{y^2}{2} - \frac{\eta}{\eta_*} y + 1 \right) (1 + y) - \frac{\eta}{\eta_*} y^3 \right] \right\} \right. \\ &\quad \left. + \left(\frac{5}{2} - \frac{y}{2} \right) e^{-y} (1 - e^{\frac{\eta}{\eta_*} y}) + E_1(y) \left\{ 1 + [(y-1)(1 + |1 - \frac{\eta}{\eta_*}| y) \right. \right. \\ &\quad \left. \left. - \frac{y^2}{2} |1 - \frac{\eta}{\eta_*}|^2 - \frac{y^2}{2}] e^{\frac{\eta}{\eta_*} y} \right\} - \frac{y}{2} e^{-y} [e^{\frac{\eta}{\eta_*} y} |1 - \frac{\eta}{\eta_*}| (|1 - \frac{\eta}{\eta_*}| y + 4) - (y + 4)] \right\} \\ &\leq |b| \left\{ |\ln \eta_*| (1 + y)^3 (e^{\frac{\eta}{\eta_*} y} - 1 + \frac{2\eta}{\eta_*} y e^{\frac{\eta}{\eta_*} y}) e^{-y} + (3 + y) (e^{\frac{\eta}{\eta_*} y} - 1) e^{-y} \right. \\ &\quad \left. + [e^{\frac{\eta}{\eta_*} y} - 1 + \frac{\eta}{\eta_*} y (1 + y) e^{\frac{\eta}{\eta_*} y}] E_1(y) + \frac{y}{2} (y + 4) (e^{\frac{\eta}{\eta_*} y} - 1 + \frac{2\eta}{\eta_*} e^{\frac{\eta}{\eta_*} y}) e^{-y} \right\} \\ &\leq |b| [C_1 (e^{\frac{\eta}{\eta_*} y} - 1) + C_2 \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*} y}] e^{-y/2}, \end{aligned} \quad (4.35)$$

$$\begin{aligned} \Delta f_4 &\equiv |c| |f_4(w, z) - f_4(y, 0)| = |c| e^{-y} |y \{ 1 - \frac{1}{2} y [y - 2 + \frac{\eta}{\eta_*} (1 - 2y) + \frac{\eta^2}{\eta_*^2} y] \\ &\quad + \frac{1}{2} y^2 (3 - 3 \frac{\eta}{\eta_*} + \frac{\eta^2}{\eta_*^2}) \} \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*} y} + (1 + y + \frac{1}{2} y^2 + \frac{1}{2} y^3) (1 - e^{\frac{\eta}{\eta_*} y})| \\ &\leq |c| (1 + y)^2 [6y \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*} y} + (1 + y) (e^{\frac{\eta}{\eta_*} y} - 1)] e^{-y} \end{aligned}$$

$$\leq |c| [C_1(e^{\frac{\eta}{\eta_*}y} - 1) + C_2 \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*}y}] e^{-y/2}, \quad (4.36)$$

$$\begin{aligned} \Delta f_5 &\equiv |f_5(\mu, \zeta, \eta, \eta_*) - f_5(\mu, \zeta, 0, \eta_*)| \\ &= \left| \frac{\nu}{\mu\zeta} \right|^3 \left| \int_{\eta_*}^{\infty} \left\{ (s - \eta) \left(1 - \frac{\nu}{2} \frac{s - \eta}{|\mu\zeta|} \right) e^{\frac{\nu\eta}{|\mu\zeta|}} - s \left(1 - \frac{\nu}{2} \frac{s}{|\mu\zeta|} \right) \right\} e^{-\frac{\nu s}{|\mu\zeta|}} C[h](s, \mu, \zeta) ds \right| \\ &\leq \left| \frac{\nu}{\mu\zeta} \right|^3 \int_{\eta_*}^{\infty} \left| s \left(1 - \frac{\nu}{2} \frac{s}{|\mu\zeta|} \right) \left(e^{\frac{\nu\eta}{|\mu\zeta|}} - 1 \right) - \eta \left(1 - \frac{\nu}{2} \frac{2s - \eta}{|\mu\zeta|} \right) e^{\frac{\nu\eta}{|\mu\zeta|}} \right| e^{-\frac{\nu s}{|\mu\zeta|}} ds \max_{s \geq \eta_*} |C[h]| \\ &\leq \left| \frac{\nu}{\mu\zeta} \right|^3 \int_{\eta_*}^{\infty} \left\{ s \left(1 + \frac{\nu}{2} \frac{s}{|\mu\zeta|} \right) \left(e^{\frac{\nu\eta}{|\mu\zeta|}} - 1 \right) + \eta \left(1 + \frac{\nu s}{|\mu\zeta|} \right) e^{\frac{\nu\eta}{|\mu\zeta|}} \right\} e^{-\frac{\nu s}{|\mu\zeta|}} ds \max_{s \geq \eta_*} |C[h]| \\ &= \frac{y}{\eta_*} \left\{ \frac{1}{2} (2 + y)^2 (e^{\frac{\eta}{\eta_*}y} - 1) + \frac{\eta}{\eta_*} y (2 + y) e^{\frac{\eta}{\eta_*}y} \right\} e^{-y} \max_{s \geq \eta_*} |C[h]| \\ &\leq [C_1(e^{\frac{\eta}{\eta_*}y} - 1) + C_2 \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*}y}] e^{-y/2} \max_{s \geq \eta_*} |C[h]|, \end{aligned} \quad (4.37)$$

where C_1 and C_2 are positive constants, common to Δf_1 and $\Delta f_3 \sim \Delta f_5$, and the arguments of $a(\mu, \zeta)$, $b(\mu, \zeta)$, and $c(\mu, \zeta)$ are omitted for conciseness. In the above estimates, we have used that $z = (\eta/\eta_*)y$ and $0 < \eta/\eta_* < 1/3 (< 1)$. Therefore, for $i = 1, 3 \sim 5$, we have

$$\begin{aligned} &\left| \int_{\mu\zeta < 0} \mathcal{P}(\mu, \zeta) \Delta f_i \frac{(1 - \mu^2)\zeta^2}{\nu^3} d\zeta \right| \leq 2\pi \int_{\mu\zeta < 0} |\mathcal{P}(\mu, \zeta)| \Delta f_i \frac{\zeta^4}{\nu^3} d\mu d\zeta \\ &\leq 2\pi \int_{\mu\zeta < 0} \tilde{P}(\mu, \zeta) (C_1 |e^{\frac{\eta}{\eta_*}y} - 1| + C_2 \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*}y}) e^{-y/2} \frac{\zeta^4}{\nu^3} d\mu d\zeta \\ &= 2\pi \eta_* \int_0^{\infty} \max_{\mu} |\tilde{P}| \frac{\zeta^3}{\nu^2} \int_{\frac{\eta_*\nu}{\zeta}}^{\infty} (C_1 |e^{\frac{\eta}{\eta_*}y} - 1| + C_2 \frac{\eta}{\eta_*} e^{\frac{\eta}{\eta_*}y}) y^{-2} e^{-y/2} dy d\zeta \\ &\leq 2\pi \eta_* \int_0^{\infty} \max_{\mu} |\tilde{P}| \frac{\zeta^5}{\nu^4} \frac{1}{\eta_*^2} \frac{2C_1(e^{\frac{\eta\nu}{\zeta}} - 1) + 2\frac{\eta}{\eta_*}(C_2 e^{\frac{\eta\nu}{\zeta}} + 2C_1)}{1 - 2\frac{\eta}{\eta_*}} e^{-\frac{\eta_*\nu}{2\zeta}} d\zeta \\ &\leq \frac{12\pi}{\eta_*} \int_0^{\infty} \max_{\mu} |\tilde{P}| \frac{\zeta^5}{\nu^4} [C_1(e^{\frac{\eta\nu}{\zeta}} - 1)e^{-\frac{\eta_*\nu}{2\zeta}} + \frac{\eta}{\eta_*}(C_2 + 2C_1)] d\zeta \\ &= \frac{12\pi}{\eta_*} \left\{ \int_0^{\sqrt{\eta}} + \int_{\sqrt{\eta}}^{\infty} \right\} \max_{\mu} |\tilde{P}| \frac{\zeta^5}{\nu^4} C_1(e^{\frac{\eta\nu}{\zeta}} - 1)e^{-\frac{\eta_*\nu}{2\zeta}} d\zeta + O(\eta) \\ &\leq \frac{12\pi}{\eta_*} C_1 \int_0^{\sqrt{\eta}} \max_{\mu} |\tilde{P}| \frac{\zeta^5}{\nu^4} (e^{\frac{\eta\nu}{\zeta}} - 1)e^{-\frac{\eta_*\nu}{2\zeta}} d\zeta \\ &\quad + \frac{12\pi}{\eta_*} C_1 \int_{\sqrt{\eta}}^{\infty} \max_{\mu} |\tilde{P}| \frac{\zeta^5}{\nu^4} (e^{\eta c_1(1 + \frac{1}{\sqrt{\eta}})} - 1) d\zeta + O(\eta) = O(\sqrt{\eta}) \rightarrow 0, \quad \text{as } \eta \rightarrow 0, \end{aligned}$$

where $\tilde{P}(\mu, \zeta) = \max(|a|, |b|, |c|, \max_{s \geq \eta_*} |C[h]|) |\mathcal{P}|$. In the last line we have used that there exist positive constants c_0 and c_1 s.t. $c_0(1 + \zeta) < \nu(\zeta) < c_1(1 + \zeta)$. As to the remaining

$$\Delta f_2 \equiv |f_2(z, \eta) - f_2(0, 0)| = |\ln z - (z - 1)e^z E_1(z) + \frac{1}{2} z [ze^z E_1(z) - 1] + \gamma|,$$

we have, as in the case of Eq. (4.24),

$$\int_{\mu\zeta < 0} |\mathcal{P}(\mu, \zeta) b(\mu, \zeta) \Delta f_2| \frac{(1 - \mu^2)\zeta^2}{\nu^3} d\zeta \leq 2\pi \int_0^{\infty} \frac{\zeta^4}{\nu^3} \max_{\mu} |\tilde{P}| \int_0^1 \Delta f_2 d\mu d\zeta$$

Table 4.3: Data for the decay assessment of the velocity distribution function at the edge of the computational region.

F	$\max F(\cdot, \cdot, Z) $	$\max F(d, \cdot, \cdot) $	$\max F $
$\bar{\phi}_4 E$	4.0×10^{-10}	6.3×10^{-10}	2.0
$\bar{\psi}_5 E$	3.8×10^{-11}	5.3×10^{-13}	1.1
$\bar{\psi}_6 E$	3.1×10^{-11}	4.7×10^{-12}	0.36
$\bar{\psi}_7 E$	4.1×10^{-11}	4.6×10^{-10}	0.95

F	$\max_{i \geq 1} F(\eta^{(i)}, \cdot, Z) $	$\max F(d, \cdot, \cdot) $	$\max_{i \geq 1} F(\eta^{(i)}, \cdot, \cdot) $
ϕE	2.6×10^{-4}	5.2×10^{-9}	1.7×10^5
$\psi_A E$	1.3×10^{-5}	8.0×10^{-12}	1.3×10^5
$\psi_B E$	8.7×10^{-5}	6.5×10^{-11}	2.8×10^5

$$= 2\pi\eta \int_0^\infty \frac{\zeta^3}{\nu^2} \max_\mu |\tilde{P}| \left(\int_{\frac{\nu\eta}{\zeta}}^\infty z^{-2} \Delta f_2 dz \right) d\zeta \leq 2\pi\eta \int_0^\infty \frac{\zeta^3}{\nu^2} \max_\mu |\tilde{P}| \left(\int_{c_0\eta}^\infty z^{-2} \Delta f_2 dz \right) d\zeta.$$

Then, using the fact that Δf_2 is $O(z)$ for $z \ll 1$ and $O(\ln z)$ for $z \gg 1$, we see that the first and the second term of the following splitting

$$\int_{c_0\eta}^\infty z^{-2} \Delta f_2 dz = \int_{c_0\eta}^{c_0\eta_*} z^{-2} \Delta f_2 dz + \int_{c_0\eta_*}^\infty z^{-2} \Delta f_2 dz,$$

are $O(\ln \eta)$ and $O(1)$, respectively. And finally we obtain

$$\int_{\mu\zeta < 0} |\mathcal{P}(\mu, \zeta) b(\mu, \zeta) \Delta f_2| \frac{(1 - \mu^2)\zeta^2}{\nu^3} d\zeta \leq O(\eta \ln \eta) \rightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

This completes the proof of Eq. (4.33), thus that of Eq. (4.16b).

4.9 Appendix C: Data of computations and measure of accuracy

For the check of numerical accuracy, besides the grids S1–S3 and M1–M7 in Ref. [18], we have introduced a new grid S4 for η space, which is defined by setting $(N, N_\eta) = (200, 250)$ in Eq. (B.1) of Ref. [18] and is twice as fine as the standard spatial grid S1.

The truncation of the ζ and η spaces is justified by confirming the sufficient decay of the velocity distribution function at $\eta = d$ and $\zeta = Z$. Table 4.3 shows the results in the case of the standard grid (S1, M1), for which $d = 44.46$ and $Z = 5.0$. The sufficient decay is actually observed. One may think that the decay at $\zeta = Z$ would not be enough for ϕE , $\psi_A E$, and $\psi_B E$, when compared with the others. Actually, however, a small extension of Z improves a

Table 4.4: Slip/jump coefficients $c_4^{(0)}$ and $b_5^{(1)} \sim b_8^{(1)}$ obtained by different grids.

Grid	$c_4^{(0)}$	$b_5^{(1)}$	$b_6^{(1)}$	$b_7^{(1)}$	$b_8^{(1)}$
(S1,M1)	4.6180185	-0.6601218	0.2438061	0.4472751	-0.2335314
(S1,M2)	4.6180185	-0.6601218	0.2438061	0.4472751	-0.2335314
(S1,M3)	4.6180492	-0.6601221	0.2438051	0.4472728	-0.2335330
(S1,M4)	4.6180143	-0.6601211	0.2438071	0.4472741	-0.2335313
(S1,M5)	4.6180185	-0.6601218	0.2438061	0.4472751	-0.2335314
(S1,M6)	4.6180221	-0.6601220	0.2438061	0.4472750	-0.2335315
(S1,M7)	4.6178471	-0.6601224	0.2438091	0.4472922	-0.2335224
(S2,M1)	4.6180153	-0.6601231	0.2438053	0.4472744	-0.2335334
(S3,M1)	4.6180185	-0.6601218	0.2438061	0.4472750	-0.2335314
(S4,M1)	4.6180148	-0.6601233	0.2438052	0.4472743	-0.2335337
(S4,M3)	4.6180455	-0.6601236	0.2438041	0.4472720	-0.2335353

lot. For instance, $\max_{i \geq 1} |\phi_4 E(\eta^{(i)}, \cdot, Z)|$ improves from 2.6×10^{-4} with (S1,M1) to 3.9×10^{-8} with (S1,M6) (remind that $\phi_4 = \bar{\phi}_4 + \phi$), where the arrangement of grid points is common between M1 and M6 for $0 \leq \zeta \leq 5.0$ and M6 covers a wider region, i.e., $0 \leq \zeta \leq 5.8$. As will be shown below, the difference of the results between (S1,M1) and (S1,M6) is negligible, at least at the level of the macroscopic quantities like the slip/jump coefficients.

Although the results in Sec. 4.5 are obtained by the splitting of solution explained at the beginning of Sec. 4.4, we have also solved ϕ_4 and $\psi_5 \sim \psi_8$ directly without the splitting. The results are hardly different from each other. Indeed, the difference of the results between the two manners in the Knudsen-layer functions is less than 7.9×10^{-10} in the case of grid (S1,M1). Therefore, we have examined the grid dependence of the results by the computation without splitting only. The grid dependence of the computed slip/jump coefficients is shown in Table 4.4. The grid in ζ -space most affects the results, especially for $c_4^{(0)}$, $b_7^{(1)}$, and $b_8^{(1)}$ [compare the results by (S1,M1), (S1,M3), and (S1,M7), where M3 (or M1) is the grid about twice (or 3/2) as many points as M7]. The accuracy down to the fourth or fifth decimal place is expected from the table. M2 is the grid that refines M1 only in the range that M is small. The comparison between (S1,M1) and (S1,M2) in the table shows that M1 is fine enough for small M . The comparisons among (S1,M1), (S3,M1), and (S1,M6) in the table show that the error due to the truncation of the ζ and η spaces is almost negligible.

The accuracy of the collision integral computation has already been assessed in Ref. [18] by checking the identities $C[(1, \mu\zeta, \zeta^2)E] = (1, \mu\zeta, \zeta^2)\nu E$ for $C = ECE^{-1}$ and $C[E] = \nu E$ for

$C = EC^SE^{-1}$. With the standard grid M1, these identities are confirmed to hold within the error of 9.1×10^{-8} , 1.7×10^{-8} , 6.6×10^{-8} , and 8.9×10^{-9} respectively, while the maximum values of $(1, \mu\zeta, \zeta^2)\nu E$ are 0.13, 0.064, and 0.062 respectively. [18]

The mass, momentum, and energy balances offer another measure of accuracy. They are the following identities that are obtained from Eqs. (4.4a) and (4.5a) by the integration in molecular velocity space after multiplying the collision invariants:

$$\langle \mu\zeta\phi_4 \rangle = -2 \int_{\eta}^{\infty} Y_2^{(1)}(z)dz, \quad \langle \mu^2\zeta^2\phi_4 \rangle = -\frac{3}{2} \int_{\eta}^{\infty} [\Omega_1^{(0)}(z) + \Theta_1^{(0)}(z)]dz, \\ \langle \mu\zeta(\zeta^2 - \frac{5}{2})\phi_4 \rangle = -2 \int_{\eta}^{\infty} H_2^{(1)}(z)dz, \quad (4.38a)$$

$$\langle \mu(1 - \mu^2)\zeta^3\psi_7 \rangle = \frac{3}{2} \int_{\eta}^{\infty} [\Omega_1^{(0)}(z) + \Theta_1^{(0)}(z)]dz, \quad (4.38b)$$

$$\langle \mu(1 - \mu^2)\zeta^3\psi_i \rangle_+ = -\langle \mu(1 - \mu^2)\zeta^3\psi_i \rangle_-, \quad (i = 5, 6, 8). \quad (4.38c)$$

Here $\langle \cdot \rangle_{\pm}$ is the half-range integral with respect to the molecular velocity defined by $\langle f(\zeta) \rangle_{\pm} = \langle f(\zeta)\chi[0, 1](\pm\mu) \rangle$. With the standard grid (S1,M1), the identities in Eq. (4.38a) hold within the error of 6.0×10^{-7} , 3.2×10^{-6} , and 5.0×10^{-6} , while the maxima of their left-hand side are 0.96, 0.29, and 2.6. In the case of Eq. (4.38b), the error is within 5.1×10^{-6} , while the maximum of the l.h.s. is 0.29. In the case of Eq. (4.38c), the error is within 7.6×10^{-7} , 3.2×10^{-6} , or 7.2×10^{-7} ($i = 5, 6$, or 8), while the maximum of the l.h.s. are 0.27, 0.12, or 0.13 ($i = 5, 6$, or 8).

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Conclusion

In the present thesis, we have developed the generalized slip-flow theory for unsteady problems, including the complete set of numerical data required in practical applications for the original Boltzmann equation.

In Chapter 1, the behavior of a slightly rarefied monatomic gas between two parallel plates whose temperature grows slowly and linearly in time has been investigated. At first, it has been shown that this problem is equivalent to a boundary-value problem for the steady linearized Boltzmann equation with a volumetric heating source, recently studied by Radtke, Hadjiconstantinou, Takata, and Aoki who focus on the parabolic temperature profile and related second-order jump. Secondly, a systematic asymptotic analysis for small Knudsen numbers has been carried out. Analysis of the Knudsen-layer correction shows that the second-order temperature jump is indeed determined by a new half-space problem which does not occur in the generalized slip-flow theory for steady problems. This means that the second-order temperature jump considered in this chapter is a new one which can occur only in unsteady systems and explains why the jump coefficient observed by RHTA does not agree with any of the jump coefficients in the existing theory. Finally, numerical computations have been carried out to determine the coefficient of the new second-order temperature jump and the structure of the related Knudsen layer for the BGK model and hard-sphere molecules. The jump coefficient was also calculated by the use of the theory of symmetry relation. The values obtained by the two different methods agree well. Moreover, the present asymptotic solution for the behavior of the gas agrees well with the results that RHTA have obtained by the low variance stochastic method.

In Chapter 2, a time-dependent problem of a slightly rarefied monoatomic gas which is perturbed slowly and slightly from a reference equilibrium state at rest has been studied on the basis of the linearized Boltzmann equation. By a systematic asymptotic analysis, we have obtained a set of fluid-dynamic-type equations and its appropriate boundary conditions, formulas of the Knudsen-layer correction, and a series of formulated Knudsen-layer problems up to the second order of the Knudsen number. It has been shown that (i) the compressibility of the gas manifests itself from the leading order in the energy equation and from the first

order in the continuity equation; (ii) although the momentum equation is the Stokes equation, it contains a double Laplacian of the leading order flow velocity as a source term at the second order; (iii) a double Laplacian source term also appears in the energy equation at the second order; (iv) the slip and jump conditions and the Knudsen-layer correction are the same as those in the time-independent case up to the first order, and the difference occurs at the second order in the jump conditions as the terms of the divergence of the first-order flow velocity and of the Laplacian of the leading-order temperature. Numerical data of all the slip and jump coefficients for a hard-sphere gas under the diffuse reflection boundary condition, most of which have not been available, have been completed not by solving the Knudsen-layer problems but by the use of the theory of symmetric relation.

In Chapter 3, numerical analyses of the second-order Knudsen layer have been carried out on the basis of the linearized Boltzmann equation for hard-sphere molecules under the diffuse reflection boundary condition. We have constructed a new approach that makes use of the integral formulation of the Boltzmann equation, which is effective in dealing with the singularity of the solution. The spatial singularity of the Knudsen-layer correction on the boundary has been captured successfully by the approach. The values of slip and jump coefficients obtained in this computation agree well with those previously obtained by the use of the theory of symmetric relation. Numerical data of the Knudsen-layer correction, except for the curvature effects, have been obtained.

In Chapter 4, numerical analyses of the remaining Knudsen-layer problems which are related to the curvature effects have been conducted. In the course of analyses, it has been clarified that the curvature effects induce the local divergence singularity of the velocity distribution function, even though it is integrable to make macroscopic quantities well defined. We have obtained the precise information on the singularity and thereby have established a numerical method that handles it appropriately. The values of slip and jump coefficients obtained in this computation agree well with those previously obtained by the use of the theory of symmetric relation. Numerical data of the Knudsen-layer correction for the curvature effects have been obtained, thereby the required numerical data in the generalized slip-flow theory have been completed for the original Boltzmann equation, assuming a hard-sphere gas and the diffuse reflection boundary condition.

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List of publications

Chapter 1

S. Takata, K. Aoki, M. Hattori, and N. G. Hadjiconstantinou, “Parabolic temperature profile and second-order temperature jump of a slightly rarefied gas in an unsteady two-surface problem,” *Phys. Fluids* **24**, 032002 (2012).

Chapter 2

S. Takata and M. Hattori, “Asymptotic theory for the time-dependent behavior of a slightly rarefied gas over a smooth solid boundary,” *J. Stat. Phys.* **147**, 1182 (2012).

Chapter 3

M. Hattori and S. Takata, “Second-order Knudsen-layer analysis for the generalized slip-flow theory I,” *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)* **10**, 423 (2015).

Chapter 4

M. Hattori and S. Takata, “Second-order Knudsen-layer analysis for the generalized slip-flow theory II: Curvature effects,” *J. Stat. Phys.* **161**, 1010 (2015).

Additional information

The content of Chapter 1 is an author produced version of a paper published in Physics of Fluids. The final publication is available at AIP via <http://dx.doi.org/10.1063/1.3691262>. The content of Chapters 2 and 4 is an author produced version of papers published in Journal of Statistical Physics. The final publications are available at Springer via <http://dx.doi.org/10.1007/s10955-012-0512-z> and <http://dx.doi.org/10.1007/s10955-015-1364-0>, respectively.