

Nonlinear Nonautonomous Differential Equations

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Introduction.

Let X be a real Banach space with norm $\|\cdot\|$ and let $C = C([-r, 0]; X)$, $0 \leq r < \infty$, be the Banach space of all continuous functions from $[-r, 0]$ into X . We denote the norm of $\phi \in C$ by $\|\phi\|_C$, i.e., $\|\phi\|_C = \sup_{\theta \in [-r, 0]} \|\phi(\theta)\|$.

This paper is concerned with the abstract nonlinear functional differential equation

$$\begin{aligned} (FDE; \phi)_s \quad & u'(t) + A(t)u(t) \ni F(t, u_t), \quad t \in [s, T] \quad (s \geq 0) \\ & u_s = \phi, \end{aligned}$$

where $u: [-r, T] \rightarrow X$ is the unknown function; $\{A(t); t \in [0, T]\}$ is a given family of operators in X ; $F: [0, T] \times C \rightarrow X$ is a given function; ϕ is given in C . The symbol u_t denotes the function $u_t(\theta) = u(t+\theta)$, $\theta \in [-r, T]$.

We assume that the following conditions (A.1) – (A.4) hold:

(A.1) There exists a constant α_0 such that for each $t \in [0, T]$, $A(t) + \alpha_0$ is accretive and $R(I + \lambda A(t)) = X$ for $0 < \lambda < 1/\max(0, \alpha_0)$.

(A.2) There are a continuous function $h: [0, T] \rightarrow X$ which is of bounded variation on $[0, T]$, and a monotone increasing continuous function $L_1: [0, \infty) \rightarrow [0, \infty)$ such that

$$\|A_\lambda(t)x - A_\lambda(\tau)x\| \leq \|h(t) - h(\tau)\| L_1(\|x\|)(1 + \|A_\lambda(\tau)x\|)$$

for $0 < \lambda < 1/\max(0, \alpha_0)$, $t, \tau \in [0, T]$ and $x \in X$, where

$$J_\lambda(t) = (I + \lambda A(t))^{-1} \text{ and } A_\lambda(t) = \lambda^{-1}(I - J_\lambda(t)).$$

(A.3) There exists a constant $\beta_0 > 0$ such that for $\phi, \psi \in C$ and $t \in [0, T]$, $\|F(t, \phi) - F(t, \psi)\| \leq \beta_0 \|\phi - \psi\|_C$.

(A.4) There are a continuous function $k:[0,T] \rightarrow X$ which is of bounded variation on $[0,T]$, and a monotone increasing function $L_2:[0,\infty) \rightarrow [0,\infty)$ such that for $t, \tau \in [0,T]$ and $\phi \in C$,

$$\|F(t, \phi) - F(\tau, \phi)\| \leq \|k(t) - k(\tau)\| L_2(\|\phi\|_C).$$

The purpose of this paper is to show the existence of a generalized solution of $(FDE; \phi)_S$. In particular, in case X is reflexive, we show that the generalized solution is the strong solution of $(FDE; \phi)_S$.

Recently, Kartsatos [6] has proved the existence of the evolution operator associated with $(FDE; \phi)_S$ under the following conditions (B.2) and (B.3) instead of (A.2), (A.3) and (A.4).

(B.2) There exists an increasing continuous function $L:[0,\infty) \rightarrow [0,\infty)$ such that for all $\lambda > 0$, $x \in X$, $t, \tau \in [0,T]$,

$$\|A_\lambda(t)x - A_\lambda(\tau)x\| \leq |t - \tau| L(\|x\|)(1 + \|A_\lambda(\tau)x\|).$$

(B.3) There exists a positive constant b such that

$$\|F(\tau, f_1) - F(t, f_2)\| \leq b(|t - \tau| + \|f_1 - f_2\|_C)$$

for every $t, \tau \in [0,T]$, $f_1, f_2 \in C$.

In order to apply the method of successive approximations to $(FDE; \phi)_S$, he essentially used conditions (B.2) and (B.3) which imply that $A_\lambda(t)x$ and $F(t, f)$ are Lipschitz continuous in t . However this method does not seem to be directly applicable under (A.1) - (A.4). Also, it has not been proved that the generalized solutions in the sense of Kartsatos are weak solutions, except on a small interval in which they are Lipschitz continuous. (For a refined definition of weak solutions, see Definition 2.)

Now, in order to improve these points, we use the nonlinear evolution operator theory of Crandall and Pazy [2] as the main

tool for solving $(FDE;\phi)_S$. Various author have so far considered $(FDE;\phi)_S$ under different setting in nonlinear operator theory. (For example, see [3,4,10].)

This paper consists of three sections. In section 1, we recall the nonlinear evolution operator theory. In section 2, we show that the existence of generalized solutions of $(FDE;\phi)_S$ and it is represented as the uniform limit of a sequence of strong solutions of the approximating equations for $(FDE;\phi)_S$ involving the Yosida approximations. Finally, in section 3, we investigate some properties of generalized solutions and consider weak solutions and give the existence for strong solutions of $(FDE;\phi)_S$ when X is reflexive.

1. Basic concept of nonlinear evolution operator theory

We discuss briefly some concepts in the nonlinear evolution operator theory. Let Y be a Banach space with $\|\cdot\|_Y$. A family $\{V(t,s); 0 \leq s \leq t \leq T\}$ of operators $V(t,s): Y \rightarrow Y$ is said to be a family of operators, if

$$V(t,t)y = y \text{ for all } y \in Y \text{ and } t \in [0,T],$$

$$V(t,r)V(r,s) = V(t,s) \text{ for } 0 \leq s \leq r \leq t \leq T.$$

Let $\{V(t,s); 0 \leq s \leq t \leq T\}$ be an evolution operator and define the operator $B(t)$ by

$$D(B(t)) = \{y \in Y; \lim_{h \rightarrow 0^+} (1/h)(V(t+h,t)y - y) \text{ exists}\}$$

$$-B(t)y = \lim_{h \rightarrow 0^+} (1/h)(V(t+h,t)y - y) \text{ for } y \in D(B(t)).$$

If $D(B(t))$ is non-empty for each $t \geq 0$, then the family $-B(t)$ is said to be the infinitesimal generator of $V(t,s)$.

Consider the problem $(FDE; \phi)_s$. Suppose that for every $\phi \in C$ and $s \geq 0$, $(FDE; \phi)_s$ has the unique solution $u(s, \phi)(\cdot)$ and that $A(t)$ and F are continuous. Then one can find that the infinitesimal generator of the evolution operator $V(t, s)$, defined by $V(t, s)\phi = u_t(s, \phi)$ is given by

$$(1.1) \quad \begin{aligned} D(\hat{A}(t)) &= \{\phi \in C; \phi' \in C, \phi(0) \in D(A(t)), \\ &\quad \phi'(0) + A(t)\phi(0) \ni F(t, \phi)\} \\ \hat{A}(t)\phi &= -\phi'. \end{aligned}$$

Conversely, given the family $A(t)$, we shall prove that under suitable conditions on $A(t)$ and F , $A(t)$ generates an evolution operator $V(t, s)$ such that $V(t, s)\phi$ gives the segments of a solution of $(FDE; \phi)_s$. This will rely on the following result due to Crandall - Pazy [2].

A subset B of $Y \times Y$ is in class $\mathcal{A}(\omega)$ if for each $\lambda > 0$ such that $\lambda\omega > 1$ and each pair $[y_i, z_i] \in B$, $i=1, 2$, we have

$$(1.2) \quad \|(y_1 + \lambda z_1) - (y_2 + \lambda z_2)\|_Y \geq (1 - \lambda\omega) \|y_1 - y_2\|_Y.$$

B is called accretive if $B \in \mathcal{A}(0)$. Also, (1.2) implies that $(I + \lambda B)^{-1}$ exists on $R(I + \lambda B)$ and is a Lipschitzian with constant $(1 - \lambda\omega)^{-1}$. Let $B \in \mathcal{A}(\omega)$ and $R(I + \lambda B) = Y$ for all $0 < \lambda \leq \lambda_0$.

Define $|By|$ by $|By| = \lim_{\lambda \rightarrow 0+} \|B_\lambda y\|_Y$, where $J_\lambda = (I + \lambda B)^{-1}$ and $B_\lambda = \lambda^{-1}(I - J_\lambda)$. (Note that this limit exists, although it may be infinite.) For such B we define $\hat{D}(B) = \{y \in Y; |By| < \infty\}$ which is called a generalized domain of B .

Theorem 1. (Crandall-Pazy). Let $T > 0$ and ω be real number and assume that $B(t)$ satisfies the following conditions:

$$(C.1) \quad B(t) \in \mathcal{A}(\omega) \text{ for } 0 \leq t \leq T,$$

(C.2) $R(I + \lambda B(t)) = Y$ for $0 \leq t \leq T$ and $0 < \lambda < \lambda_0$, where $\lambda_0 > 0$ and $\lambda_0 \omega < 1$,

(C.3) There are a continuous function $f: [0, T] \rightarrow Y$ which is of bounded variation on $[0, T]$, and a monotone increasing function $L: [0, \infty) \rightarrow [0, \infty)$ such that

$$\|B_\lambda(t)y - B_\lambda(\tau)y\|_Y \leq \|f(t) - f(\tau)\|_Y L(\|y\|_Y)(1 + \|B_\lambda(\tau)y\|_Y)$$

for $0 < \lambda < \lambda_0$, $0 \leq t, \tau \leq T$ and $y \in Y$.

Then

$$(1.3) \quad V(t, s)y = \lim_{n \rightarrow \infty} \prod_{i=1}^n (I + (\frac{t-s}{n})B(s + i(\frac{t-s}{n})))^{-1}y$$

exists for $y \in \overline{D(B(t))}$ and $0 \leq s < t \leq T$. The $V(t, s)$ defined by (1.3) for $0 \leq s < t \leq T$ and by $V(t, t) = I$ for $0 \leq t \leq T$ is an evolution operator on $\overline{D(B(t))}$.

2. On the existence of generalized solutions of $(FDE; \phi)_s$

We define for each $t \in [0, T]$ an operator $\hat{A}(t): D(\hat{A}(t)) \subset C \rightarrow C$ by (1.1).

Proposition 1. Suppose that conditions (A.1)-(A.4) hold. If $\{\hat{A}(t); t \in [0, T]\}$ is the family of operators defined in C by (1.1), then there exists a family of nonlinear evolution operators $V(t, s): \overline{D(\hat{A}(t))} \subset C \rightarrow C$ such that for all $\phi \in \overline{D(\hat{A}(t))}$

$$(2.1) \quad V(t, s)\phi = \begin{cases} \lim_{n \rightarrow \infty} \prod_{i=1}^n (I + (\frac{t-s}{n})\hat{A}(s + i(\frac{t-s}{n})))^{-1}\phi & 0 \leq s < t \leq T, \\ \phi & 0 \leq s = t \leq T. \end{cases}$$

Proof. We are going to apply Theorem 1 for $B(t) = \hat{A}(t)$ and $Y = C$. Under assumptions (A.1) and (A.3) we can apply [11, Proposition 1] to show that $\hat{A}(t) \in \hat{A}(\omega_0)$ for $t \in [0, T]$ and $R(I + \lambda \hat{A}(t)) = C$ for $0 < \lambda < 1/\omega_0$, where $\omega_0 = \max(0, \alpha_0 + \beta_0)$. Thus

conditions (C.1) and (C.2) hold for $\hat{A}(t)$. Next, by using the same argument as in [4, Theorems 12 and 13] and the inequality

$$\|h(t) - h(\tau)\| + \|k(t) - k(\tau)\| \leq |g(t) - g(\tau)|, \text{ where } g(t) = \text{Var}([0, t]; h) + \text{Var}([0, t]; k) \text{ and } \text{Var}([0, t]; h) \text{ denotes the total variation of } h \text{ on } [0, t], \text{ we will show that } \hat{A}(t) \text{ satisfies (C.3) with } B(t) = \hat{A}(t) \text{ and } f(t) = g(t)I, \text{ where } I \text{ denotes the identity in } X. \text{ To this end, set } \phi(t, \cdot) = (I + \lambda A(t))^{-1} \psi, \psi \in C. \text{ Then we have } \phi(t, \theta) = e^{\theta/\lambda} \phi(t, 0) + \int_{\theta}^0 \frac{1}{\lambda} e^{-(s-\theta)/\lambda} \psi(s) ds, \text{ and by } \phi(t, \cdot) \in D(\hat{A}(t)), \text{ we have } \phi(t, 0) = \psi(0) + \lambda \phi'(t, 0) = \psi(0) - \lambda A(t) \phi(t, 0) + \lambda F(t, \phi(t, \cdot)), \text{ i.e., } \phi(t, 0) = (I + \lambda A(t))^{-1} (\psi(0) + \lambda F(t, \phi(t, \cdot))).$$

Now, for $0 < \lambda < 1$ with $\lambda \omega_0 < 1/2$,

$$\begin{aligned} \|\phi(t, \cdot) - \phi(\tau, \cdot)\|_C &= \|\phi(t, 0) - \phi(\tau, 0)\| \\ &= \|(I + \lambda A(t))^{-1} (\psi(0) + \lambda F(t, \phi(t, \cdot))) \\ &\quad - (I + \lambda A(\tau))^{-1} (\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)))\| \\ &\leq \lambda(1 - \lambda \alpha_0)^{-1} \|F(t, \phi(t, \cdot)) - F(\tau, \phi(\tau, \cdot))\| \\ &\quad + \lambda \|h(t) - h(\tau)\| L_1(\|\psi(0) + \lambda F(\tau, \phi(\tau, \cdot))\|) \\ &\quad \times (1 + \|A_\lambda(\tau)(\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)))\|). \end{aligned}$$

But,

$$\begin{aligned} &\|A_\lambda(\tau)(\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)))\| \\ &= \lambda^{-1} \|\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)) - J_\lambda(\tau)(\psi(0) + \lambda F(\tau, \phi(\tau, \cdot)))\| \\ &\leq \|\hat{A}_\lambda(\tau)\psi\|_C + \|F(\tau, \phi(\tau, \cdot))\|, \end{aligned}$$

which implies that

$$\begin{aligned} &\|\phi(t, \cdot) - \phi(\tau, \cdot)\|_C \\ &\leq \lambda(1 - \lambda \alpha_0)^{-1} [\beta_0 \|\phi(t, \cdot) - \phi(\tau, \cdot)\|_C \\ &\quad + \|k(t) - k(\tau)\| L_2(\|\phi(\tau, \cdot)\|_C)] + \end{aligned}$$

$$+ \lambda \|h(t) - h(\tau)\|_{L_1(\|\psi(0)\| + \lambda \|F(\tau, \phi(\tau, \cdot))\|)} \\ \times (1 + \|\hat{A}_\lambda(\tau)\psi\|_C + \|F(\tau, \phi(\tau, \cdot))\|).$$

Thus there exists a constant K_1 such that

$$(2.2) \quad \|\phi(t, \cdot) - \phi(\tau, \cdot)\|_C \\ \leq K_1 \lambda |g(t) - g(\tau)| [1 + \|\hat{A}_\lambda(\tau)\psi\|_C] [L_2(\|\phi(\tau, \cdot)\|_C) \\ + (1 + \|F(\tau, \phi(\tau, \cdot))\|) L_1(\|\psi(0)\| + \lambda \|F(\tau, \phi(\tau, \cdot))\|)].$$

Suppose that $\chi \in C$ and $\phi_0 \in D(\hat{A}(0))$. Then $\|F(\tau, \chi)\| \leq$

$$\beta_0 [\|\chi\|_C + \|\phi_0\|_C] + \|k(\tau) - k(0)\|_{L_2(\|\phi_0\|_C)} + \|F(0, \phi_0)\|$$

and hence $\|F(\tau, \chi)\|$ is bounded by an increasing function of $\|\chi\|_C$. It remains to prove that $\|\phi(\tau, \cdot)\|_C \leq L_3(\|\psi\|_C)$ for some monotone increasing function L_3 . From (2.2), $\|\phi(\tau, \cdot)\|_C \leq$

$$\|\phi(0, \cdot)\|_C + K_1 \lambda |g(\tau) - g(0)| [1 + \|\hat{A}_\lambda(0)\psi\|_C] \times \\ \times [L_2(\|\phi(0, \cdot)\|_C) + \\ + (1 + \|F(0, \phi(0, \cdot))\|) L_1(\|\psi(0)\| + \lambda \|F(0, \phi(0, \cdot))\|)].$$

However $\lambda \|\hat{A}_\lambda(0)\psi\|_C = \|\psi - \hat{J}_\lambda(0)\psi\|_C \leq \|\psi\|_C + \|\phi(0, \cdot)\|_C$

and if $\phi_0 \in D(\hat{A}(0))$ then

$$\|\phi(0, \cdot)\|_C = \|(1 + \lambda \hat{A}(0))^{-1} \psi\|_C \\ \leq (1 - \lambda \omega_0)^{-1} [\|\psi - \phi_0\|_C + \lambda \|\hat{A}(0)\phi_0\|_C] + \|\phi_0\|_C \\ \leq K_2 [\|\psi\|_C + \|\phi_0\|_C + \|\hat{A}(0)\phi_0\|_C] \text{ for some } K_2,$$

which implies that

$\|\phi(0, \cdot)\|_C$ is bounded by a monotone increasing function of $\|\psi\|_C$. Thus $\hat{A}(t)$ satisfies (C.3) with $B(t) = \hat{A}(t)$ and $f(t) = g(t)I$. Therefore, the conclusion of the proposition follows from Theorem 1. Q.E.D.

Note that, as was proved in [5], $\hat{D}(\hat{A}(t))$ is independent of t because $\hat{A}(t)$ satisfies (C.3) and also $\hat{D}(A(t))$ is independent of t because of (A.2). In what follows, $\hat{D}_0$ and $\hat{D}$ stand for a generalized domain of $\hat{A}(0)$ and $A(0)$, respectively.

As in [3, Proposition 1], we have the following

Proposition 2. Suppose that conditions (A.1)-(A.4) hold.

If $u(s, \phi)(\cdot)$ for each $\phi \in \hat{D}_0$ and $s \geq 0$ is defined by

$$(2.3) \quad u(s, \phi)(t) = \begin{cases} \phi(t-s) & s-r \leq t \leq s, \\ (V(t, s)\phi)(0) & s \leq t \leq T, \end{cases}$$

where $V(t, s)$ is as constructed by Proposition 1, then

$u(s, \phi)(\cdot) \in C([s-r, T]; X)$ and $V(t, s)\phi = u_t(s, \phi)$ for $t \in [s, T]$.

Remark. We introduce the following stronger conditions than (A.1) and (A.2):

(A.1)' There exists a constant $\alpha_1 > 0$ such that for $x, y \in X$,
 $\|A(t)x - A(t)y\| \leq \alpha_1 \|x - y\|$.

(A.2)' There are a continuous function $h: [0, T] \rightarrow X$ which is of bounded variation on $[0, T]$ and a monotone increasing continuous function $L_4: [0, \infty) \rightarrow [0, \infty)$ such that

$$\|A(t)x - A(\tau)x\| \leq \|h(t) - h(\tau)\| L_4(\|x\|)(1 + \|A(\tau)x\|)$$

for all $t, \tau \in [0, T]$ and $x \in X$.

Since (A.1)' and (A.2)' imply (A.1) and (A.2), Propositions 1 and 2 hold, although (A.1) and (A.2) are replaced by (A.1)' and (A.2)'.

Next, we recall the following expression for $\hat{D}_0$.

Lemma 1 ([4, Theorem 10]). Let $A(t)$ and $F(t, \phi)$ satisfy conditions (A.1) and (A.3). Then

$$\hat{D}_0 = \{\phi \in C; \phi \text{ is Lipschitz continuous function and } \phi(0) \in \hat{D}\}.$$

Remark. If ϕ is Lipschitz continuous function and $\phi(0) \in \hat{D}$, then the function defined by (2.3) is a Lipschitzian. In fact, for such ϕ , by [2, Proposition 2.3] and Lemma 1, there exists a constant K such that for $0 \leq s \leq t, \tau \leq T$, $\|V(t,s)\phi - V(\tau,s)\phi\|_C \leq K|t - \tau|$. So that our assertion holds.

Definition 1. A function $u(s,\phi)(\cdot) \in C([-r,T];X)$ is said to be a strong solution of $(FDE;\phi)_s$ if it is an absolutely continuous function which is differentiable a.e. on $[s,T]$ and satisfies $(FDE;\phi)_s$ a.e. on $[s,T]$.

We shall first prove the following uniqueness result for strong solutions of $(FDE;\phi)_s$.

Proposition 3. Assume that $\{A(t); t \in [0,T]\}$ and $F:[0,T] \times C \rightarrow X$ satisfy conditions (A.1) and (A.3). Then there exists at most one strong solution of $(FDE;\phi)_s$.

Proof. Let $u(s,\phi)(t)$ and $v(s,\phi)(t)$ be two strong solutions of $(FDE;\phi)_s$. Then $\|u(s,\phi)(t) - v(s,\phi)(t)\|$ is differentiable a.e. t and $(d/dt) \|u(s,\phi)(t) - v(s,\phi)(t)\|$

$$= [u(s,\phi)(t) - v(s,\phi)(t), u'(s,\phi)(t) - v'(s,\phi)(t)]_-$$

$$\leq [u(s,\phi)(t) - v(s,\phi)(t), F(t, u_t(s,\phi)) - F(t, v_t(s,\phi))]_+$$

$$- [u(s,\phi)(t) - v(s,\phi)(t), F(t, u_t(s,\phi)) - u'(s,\phi)(t)$$

$$- F(t, v_t(s,\phi)) + v'(s,\phi)(t)]_+.$$

By $A(t) \in A(\alpha_0)$ and (A.3), we obtain that

$$(d/dt) \|u(s,\phi)(t) - v(s,\phi)(t)\|$$

$$\leq (\alpha_0 + \beta_0) \|u_t(s,\phi) - v_t(s,\phi)\|_C \quad \text{a.e. } t \in [s,T],$$

which yields that for $t \in [s, T]$,

$$\sup_{\theta \in [s-r, t]} \|u(s, \phi)(\theta) - v(s, \phi)(\theta)\| \leq \begin{cases} (\alpha_0 + \beta_0) \int_0^t \sup_{\theta \in [s-r, \tau]} \|u(s, \phi)(\theta) - v(s, \phi)(\theta)\| d\tau & \text{if } \alpha_0 + \beta_0 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Grownwall's inequality, we have that

$$\sup_{\theta \in [s-r, T]} \|u(s, \phi)(\theta) - v(s, \phi)(\theta)\| = 0, \text{ i.e., } u(s, \phi) = v(s, \phi).$$

Q.E.D.

We next prove the existence of strong solutions to $(FDE; \phi)_s$ under stronger conditions than those in Propositions 1 and 2.

Proposition 4. Suppose that conditions $(A.1)'$, $(A.2)'$, $(A.3)$ and $(A.4)$ hold. If $u(s, \phi)(\cdot)$ is the function defined by (2.3), then $u(s, \phi)(\cdot) \in C^1([s-r, T]; X)$ and satisfies

$$(2.4) \quad u'(s, \phi)(t) + A(t)(u(s, \phi)(t)) = F(t, u_t(s, \phi))$$

for $t \in [s, T]$ and for all $\phi \in \text{Lip} \equiv \{\phi \in C; \phi \text{ is Lipschitz continuous}\}$.

Proof. By Remark after Proposition 2, $\{V(t, s); 0 \leq s \leq t \leq T\}$ defined by (2.1) is an evolution operator. We approximate $V(t, s)$ by the evolution operator $V_\lambda(t, s)$ generated by $\hat{A}_\lambda(t) = \hat{A}(t)\hat{J}_\lambda(t) = \lambda^{-1}(I - \hat{J}_\lambda(t))$. From [2, Lemma 4.2], we see that for $\phi \in \bar{D}_0$, $\lim_{\lambda \rightarrow 0^+} V_\lambda(t, s)\phi = V(t, s)\phi$ uniformly in $t \in [s, T]$.

Also, the approximate problem

$$u'(t) + \hat{A}_\lambda(t)u_\lambda(t) = 0, \quad t \in [s, T], \quad u_\lambda(s) = \phi,$$

has a unique continuously differentiable solution $u_\lambda(t) = V_\lambda(t, s)\phi$.

Hence, we have that

$$V_\lambda(t, s)\phi = \phi - \int_s^t \hat{A}_\lambda(\tau)V_\lambda(\tau, s)\phi d\tau = \phi - \int_s^t \hat{A}(\tau)\hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi d\tau.$$

Taking account of the definition of $D(\hat{A}(\tau))$, we obtain that

$$(2.5) \quad (V_\lambda(t, s)\phi)(0) = \phi(0) - \int_s^t [A(\tau)(\hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi)(0) - F(\tau, \hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi)] d\tau.$$

Now, by (A.1)' and (A.3), we see that

$$\begin{aligned} I_1 &= \int_s^t \|A(\tau)(\hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi)(0) - A(\tau)(V(\tau, s)\phi)(0)\| d\tau \\ &\leq \alpha_1 \int_s^t \|\hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi - V(\tau, s)\phi\|_C d\tau \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_s^t \|F(\tau, \hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi) - F(\tau, V(\tau, s)\phi)\| d\tau \\ &\leq \beta_0 \int_s^t \|\hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi - V(\tau, s)\phi\|_C d\tau. \end{aligned}$$

Let $\phi \in \hat{D}_0$; note here that $\phi \in \text{Lip}$ by $D(A(t)) = X$ and Lemma 1.

For each $\tau \in [s, T]$, we have for λ with $\lambda\omega_1 < 1$,

$$\begin{aligned} I_3 &= \|\hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi - V(\tau, s)\phi\|_C \\ &\leq (1 - \lambda\omega_1)^{-1} \|V_\lambda(\tau, s)\phi - V(\tau, s)\phi\|_C \\ &\quad + \|\hat{J}_\lambda(\tau)V(\tau, s)\phi - V(\tau, s)\phi\|_C, \text{ where } \omega_1 = \alpha_1 + \beta_0. \end{aligned}$$

By [2, Proposition 2.4], $V(\tau, s)\phi \in \hat{D}_0$ for $\phi \in \hat{D}_0$. This implies that the second term of the above inequality tends to zero as $\lambda \rightarrow 0+$.

Hence $I_3 \rightarrow 0$ as $\lambda \rightarrow 0+$.

Next, we note that

$$\begin{aligned} (2.6) \quad \|\hat{J}_\lambda(\tau)V_\lambda(\tau, s)\phi\|_C &\leq (1 - \lambda\omega_1)^{-1} \|V_\lambda(\tau, s)\phi - \phi\|_C + \|\hat{J}_\lambda(\tau)\phi\|_C. \end{aligned}$$

Since (C.3) is satisfied with $B(t) = \hat{A}(t)$ and $f(t) = g(t)I$, it follows that

$$\begin{aligned}
(2.7) \quad & \| \hat{J}_\lambda(\tau)\phi \|_C \\
& \leq \| \hat{J}_\lambda(s)\phi \|_C + \lambda |g(\tau) - g(s)| L(\|\phi\|_C) (1 + \| \hat{A}_\lambda(s)\phi \|_C) \\
& \leq \lambda \| \hat{A}_\lambda(s)\phi \|_C + \|\phi\|_C \\
& \quad + \lambda |g(\tau) - g(s)| L(\|\phi\|_C) (1 + \| \hat{A}_\lambda(s)\phi \|_C) \\
& \leq \lambda (1 - \lambda \omega_1)^{-1} \| \hat{A}(s)\phi \|_C + \|\phi\|_C \\
& \quad + \lambda |g(\tau) - g(s)| L(\|\phi\|_C) (1 + (1 - \lambda \omega_1)^{-1} \| \hat{A}(s)\phi \|_C).
\end{aligned}$$

Besides, since $\lim_{\lambda \rightarrow 0+} \{ \sup_{\tau \in [s, T]} \| V_\lambda(\tau, s)\phi - V(\tau, s)\phi \|_C \} = 0$, we see that there exists λ_1 such that if $0 < \lambda \leq \lambda_1$, $\sup_{\tau \in [s, T]} \| V_\lambda(\tau, s)\phi - V(\tau, s)\phi \|_C < 1$. Thus it follows from (2.6) and (2.7) that $\sup_{0 < \lambda < \lambda_1} (\sup_{\tau \in [s, T]} \| \hat{J}_\lambda(\tau) V_\lambda(\tau, s)\phi \|_C)$ is bounded. By the Lebesgue's dominated convergence theorem, we obtain that $I_1 \rightarrow 0$ and $I_2 \rightarrow 0$ as $\lambda \rightarrow 0+$. Therefore, letting $\lambda \rightarrow 0+$ in (2.5) yields (2.4). Q.E.D.

Remark. In general setting $u(s, \phi)(\cdot)$ defined by (2.3) need not have a strong derivative. We may have regard the function $u(s, \phi)(t)$ as a generalized solution of $(FDE; \phi)_s$ and investigate the meaning of generalized solutions. For convenience, the function $u(s, \phi)(t)$ defined by (2.3) is called a generalized solution.

Now, we consider the approximate problem

$$\begin{aligned}
(FDE; \phi)_s^\beta \quad & u_\beta'(t) + A_\beta(t)u_\beta(t) = F(t, u_{\beta t}) \quad t \in [s, T] \\
& u_{\beta s} = \phi,
\end{aligned}$$

where $A_\beta(t)$ is the Yosida approximation of $A(t)$.

We define $\hat{A}^\beta(t): D(\hat{A}^\beta(t)) \subset C \rightarrow C$ by

$$\hat{A}^\beta(t)\phi = -\phi'$$

$$D(\hat{A}^\beta(t)) = \{ \phi \in C; \phi' \in C, \phi'(0) + A_\beta(t)\phi(0) = F(t, \phi) \}.$$

Clearly $A_\beta(t)$ satisfies the conditions of Proposition 4 with $\alpha_1 = \beta^{-1}(1 + (1 - \beta\alpha_0)^{-1})$; see [2, Lemma 1.2]. Therefore, there exists a family of nonlinear evolution operators $\{V_\beta(t,s); 0 \leq s \leq t \leq T\}$ generated by $\hat{A}^\beta(t)$. If $u_\beta(s,\phi)(\cdot)$ is defined by

$$u_\beta(s,\phi)(t) = \begin{cases} \phi(t-s) & s-r \leq t \leq s, \\ (V_\beta(t,s)\phi)(0) & s \leq t \leq T, \end{cases}$$

then $u_\beta(s,\phi)(t)$ is the strong solution of $(FDE;\phi)_S^\beta$ and by Proposition 2, $V_\beta(t,s)\phi = u_{\beta t}(s,\phi)$ for $s \leq t \leq T$ and $\phi \in \text{Lip}$. By the proof of [2, Lemma 4.2], $\lim_{\beta \rightarrow 0+} (1 + \lambda A_\beta(t))^{-1}x = (1 + \lambda A(t))^{-1}x$ for $x \in X$ and sufficiently small λ . Thus, by [10, Lemma 3.2], we obtain that $\lim_{\beta \rightarrow 0+} (1 + \lambda \hat{A}^\beta(t))^{-1}\phi = (1 + \lambda \hat{A}(t))^{-1}\phi$ for $\phi \in C$ and small λ . Also, it follows from [2, Lemma 4.1] that $A_\beta(t)$ satisfies (A.1) and (A.2) uniformly in β , sufficiently small and hence $\hat{A}^\beta(t)$ satisfies (C.1)-(C.3) uniformly in β , sufficiently small. (To speak more carefully, by the same way as Proposition 1, we have that

$$\begin{aligned} & \|\phi_\beta(t,\cdot) - \phi_\beta(\tau,\cdot)\|_C \\ & \leq K_3 \lambda |g(t) - g(\tau)| [1 + \|\hat{A}_\lambda^\beta(\tau)\psi\|_C] [L_2(\|\phi_\beta(\tau,\cdot)\|_C) + \\ & \quad + (1 + \|F(\tau,\phi_\beta(\tau,\cdot))\|) L_1(\|\psi(0)\| + \lambda \|F(\tau,\phi_\beta(\tau,\cdot))\|)], \end{aligned}$$

where $\phi_\beta(t,\cdot) = (1 + \lambda \hat{A}^\beta(t))^{-1}\psi$, $\psi \in C$,

and if $\chi \in C$ and $\phi_0 \in D(\hat{A}(0))$ then

$\|F(\tau,\chi)\|$ is bounded by an increasing function of $\|\chi\|_C$.

Now, in this case, we must prove that

$$(2.8) \quad \|\phi_\beta(\tau,\cdot)\|_C \leq L_5(\|\psi\|_C)$$

for some monotone increasing function L_5 . However, since

$$\lim_{\beta \rightarrow 0+} (1 + \lambda \hat{A}^\beta(t))^{-1}\phi = (1 + \lambda \hat{A}(t))^{-1}\phi \quad \text{for all } \phi \in C,$$

$\|\phi_\beta(\tau,\cdot)\|_C \leq \|\phi(\tau,\cdot)\|_C + 1$ for all small β . Therefore,

using $\|\phi(\tau, \cdot)\|_C \leq L_3(\|\psi\|_C)$ (see, Proposition 1), (2.8) is proved and hence $\hat{A}^\beta(t)$ satisfies (C.3) uniformly in β .) We can apply the Crandall-Pazy approximation theorem [2, Theorem 4.1] to give $\lim_{\beta \rightarrow 0+} V_\beta(t, s)\phi = V(t, s)\phi$ for all $\phi \in \hat{D}_0$. Therefore, by Proposition 2 and Lemma 1, we have that

Theorem 2. Let $\phi \in \text{Lip}$ with $\phi(0) \in \hat{D}$. Suppose that $\{A(t); t \in [0, T]\}$ and $F: [0, T] \times C \rightarrow X$ satisfy conditions (A.1)-(A.4). If $u(s, \phi)(\cdot)$ is a generalized solution of $(FDE; \phi)_s$ then $u(s, \phi)(t) = \lim_{\beta \rightarrow 0+} u_\beta(s, \phi)(t)$ uniformly in $t \in [s, T]$, where $u_\beta(s, \phi)(\cdot)$ is the strong solution of $(FDE; \phi)_s^\beta$.

3. Properties for generalized solutions and existence of weak solutions and strong solutions.

Our first result in this section is on the comparison of two generalized solutions.

Theorem 3. Let $\phi_i \in \text{Lip}$ with $\phi_i(0) \in \hat{D}$ for $i = 1, 2$. If $u(s, \phi_i)(\cdot)$ is a generalized solution of $(FDE; \phi_i)_s$, then we have

$$(3.1) \quad e^{-\alpha_0 t} \|u(s, \phi_1)(t) - u(s, \phi_2)(t)\| \\ - e^{-\alpha_0 \tau} \|u(s, \phi_1)(\tau) - u(s, \phi_2)(\tau)\| \\ \leq \int_\tau^t e^{-\alpha_0 \xi} [u(s, \phi_1)(\xi) - u(s, \phi_2)(\xi), F(\xi, u_\xi(s, \phi_1)) - F(\xi, u_\xi(s, \phi_2))]_+ d\xi$$

for $s \leq \tau \leq t \leq T$, where the symbol $[x, y]_+$ is defined by

$$[x, y]_+ = \lim_{\lambda \rightarrow 0+} \lambda^{-1} (\|x + \lambda y\| - \|x\|) \quad \text{for } x, y \in X.$$

Proof. Let $u_\beta(s, \phi_i)(t)$ be the strong solution of $(FDE; \phi_i)_s^\beta$ such that $\lim_{\beta \rightarrow 0+} u_\beta(s, \phi_i)(t) = u(s, \phi_i)(t)$ uniformly for $t \in [s, T]$.

Then $\| u_\beta(s, \phi_1)(t) - u_\beta(s, \phi_2)(t) \|$ is differentiable a.e. $t \in [s, T]$ and $(d/dt) \| u_\beta(s, \phi_1)(t) - u_\beta(s, \phi_2)(t) \|$

$$= [u_\beta(s, \phi_1)(t) - u_\beta(s, \phi_2)(t), -A_\beta(t)(u_\beta(s, \phi_1)(t)) + F(t, u_{\beta_t}(s, \phi_1)) + A_\beta(t)(u_\beta(s, \phi_2)(t)) - F(t, u_{\beta_t}(s, \phi_2))]_-.$$

where $[x, y]_- = -[x, -y]_+$.

Since $[x - y, A_\beta(t)x - A_\beta(t)y]_+ \leq -\alpha_0(1 - \beta\alpha_0)^{-1} \|x - y\|$, it follows that

$$\begin{aligned} & (d/dt) \| u_\beta(s, \phi_1)(t) - u_\beta(s, \phi_2)(t) \| \\ & \leq \alpha_0(1 - \beta\alpha_0)^{-1} \| u_\beta(s, \phi_1)(t) - u_\beta(s, \phi_2)(t) \| \\ & + [u_\beta(s, \phi_1)(t) - u_\beta(s, \phi_2)(t), F(t, u_{\beta_t}(s, \phi_1)) - F(t, u_{\beta_t}(s, \phi_2))]_+. \end{aligned}$$

Integrating the above inequality, we have for $s \leq \tau \leq t \leq T$,

$$\begin{aligned} & \| u_\beta(s, \phi_1)(t) - u_\beta(s, \phi_2)(t) \| - \| u_\beta(s, \phi_1)(\tau) - u_\beta(s, \phi_2)(\tau) \| \\ & \leq \alpha_0(1 - \beta\alpha_0)^{-1} \int_\tau^t \| u_\beta(s, \phi_1)(\xi) - u_\beta(s, \phi_2)(\xi) \| d\xi \\ & + \int_\tau^t [u_\beta(s, \phi_1)(\xi) - u_\beta(s, \phi_2)(\xi), F(\xi, u_{\beta_\xi}(s, \phi_1)) - F(\xi, u_{\beta_\xi}(s, \phi_2))]_+ d\xi. \end{aligned}$$

Letting $\beta \rightarrow 0+$ in this inequality, we see that for $s \leq \tau \leq t \leq T$,

$$\begin{aligned} (3.2) \quad & \| u(s, \phi_1)(t) - u(s, \phi_2)(t) \| - \| u(s, \phi_1)(\tau) - u(s, \phi_2)(\tau) \| \\ & \leq \alpha_0 \int_\tau^t \| u(s, \phi_1)(\xi) - u(s, \phi_2)(\xi) \| d\xi \\ & + \int_\tau^t [u(s, \phi_1)(\xi) - u(s, \phi_2)(\xi), F(\xi, u_\xi(s, \phi_1)) - F(\xi, u_\xi(s, \phi_2))]_+ d\xi. \end{aligned}$$

By the standard argument one can prove that (3.2) implies (3.1).

(For example, see [9].)

Q.E.D.

The following theorem gives the existence of integral solutions.

Theorem 4. Let $u(s, \phi)(\cdot)$ be a generalized solution of $(FDE; \phi)_s$. Then the following inequality holds:

$$(3.4) \quad e^{-\alpha_0 t} \|u(s, \phi)(t) - x\| - e^{-\alpha_0 \tau} \|u(s, \phi)(\tau) - x\| \\ \leq \int_{\tau}^t e^{-\alpha_0 \xi} \{ [u(s, \phi)(\xi) - x, F(\xi, u_{\xi}(s, \phi)) - y]_+ + \theta(\xi, r) \} d\xi$$

for $s \leq \tau \leq t$, $[x, y] \in A(r)$, $r \in [0, T]$,

where $\theta(\xi, r) = L_1(\|x\|) \|h(\xi) - h(r)\| (1 + \|y\|)$.

Proof. Let $u(s, \phi)(\cdot)$ be a generalized solution of $(FDE; \phi)_s$. By Theorem 2, $\lim_{\beta \rightarrow 0+} u_{\beta}(s, \phi)(t) = u(s, \phi)(t)$ uniformly for $t \in [s, T]$, where $u_{\beta}(s, \phi)(t)$ is the strong solution of $(FDE; \phi)_s^{\beta}$. Let $[x, y] \in A(r)$ and set $x_{\beta} = x + \beta y$. Note that $x = J_{\beta}(r)x_{\beta}$ and $y = A_{\beta}(r)x_{\beta}$, where $J_{\beta}(r)$ and $A_{\beta}(r)$ are the resolvent and the Yosida approximation of $A(r)$, respectively. Then

$$\begin{aligned} & (d/dt) \|u_{\beta}(s, \phi)(t) - x_{\beta}\| \\ &= [u_{\beta}(s, \phi)(t) - x_{\beta}, -A_{\beta}(t)(u_{\beta}(s, \phi)(t)) + F(t, u_{\beta t}(s, \phi))]_{-} \\ &\leq [u_{\beta}(s, \phi)(t) - x_{\beta}, -A_{\beta}(t)(u_{\beta}(s, \phi)(t)) + y]_{-} \\ &\quad + [u_{\beta}(s, \phi)(t) - x_{\beta}, F(t, u_{\beta t}(s, \phi)) - y]_{+} \\ &\leq \beta^{-1} (\|u_{\beta}(s, \phi)(t) - x_{\beta} + \beta(-A_{\beta}(t)(u_{\beta}(s, \phi)(t)) + y)\| \\ &\quad - \|u_{\beta}(s, \phi)(t) - x_{\beta}\|) + [u_{\beta}(s, \phi)(t) - x_{\beta}, F(t, u_{\beta t}(s, \phi)) - y]_{+} \\ &= \beta^{-1} (\|J_{\beta}(t)(u_{\beta}(s, \phi)(t)) - J_{\beta}(r)x_{\beta}\| - \|u_{\beta}(s, \phi)(t) - x_{\beta}\|) \\ &\quad + [u_{\beta}(s, \phi)(t) - x_{\beta}, F(t, u_{\beta t}(s, \phi)) - y]_{+} \\ &\leq L_1(\|x_{\beta}\|) \|h(t) - h(r)\| (1 + \|y\|) \\ &\quad + \alpha_0(1 - \beta\alpha_0)^{-1} \|u_{\beta}(s, \phi)(t) - x_{\beta}\| \\ &\quad + [u_{\beta}(s, \phi)(t) - x_{\beta}, F(t, u_{\beta t}(s, \phi)) - y]_{+} \text{ by } A(t) \in \mathcal{A}(\alpha_0) \text{ and (A.2).} \end{aligned}$$

Integrating these inequality over $[\tau, t] \subset [s, T]$,

$$\begin{aligned}
 & \|u_\beta(s, \phi)(t) - x_\beta\| - \|u_\beta(s, \phi)(\tau) - x_\beta\| \\
 \leq & \int_\tau^t \{L_1(\|x_\beta\|) \|h(\xi) - h(r)\| (1 + \|y\|) \\
 & + \alpha_0(1 - \beta\alpha_0)^{-1} \|u_\beta(s, \phi)(\xi) - x_\beta\| \\
 & + [u_\beta(s, \phi)(\xi) - x_\beta, F(\xi, u_{\beta\xi}(s, \phi)) - y]_+ \} d\xi.
 \end{aligned}$$

Letting $\beta \rightarrow 0+$, we see that for $s \leq \tau \leq t \leq T$,

$$\begin{aligned}
 (3.5) \quad & \|u(s, \phi)(t) - x\| - \|u(s, \phi)(\tau) - x\| \\
 \leq & \int_\tau^t \{[u(s, \phi)(\xi) - x, F(\xi, u_\xi(s, \phi)) - y]_+ + \theta(\xi, r)\} d\xi \\
 & + \alpha_0 \int_\tau^t \|u(s, \phi)(\xi) - x\| d\xi,
 \end{aligned}$$

which yields (3.4).

Q.E.D.

Next, we recall the definition of weak solutions in the sense of Kartsatos and Parrott [6,7] and consider the existence of weak solutions of $(FDE; \phi)_0$.

Definition 2. A function $u(t) \in C([-r, T]; X)$ is said to be a weak solution of $(FDE; \phi)_0$ if $u(t) = \phi(t)$ for $t \in [-r, 0]$ and

$$\begin{aligned}
 (DE) \quad & v'(t) + A(t)v(t) \ni F(t, u_t), \quad t \in [0, T] \\
 & v(0) = \phi(0)
 \end{aligned}$$

has a solution $v(t)$ in the sense of Evans [5] such that $v(t) = u(t)$ for $t \in [0, T]$.

Remark. By definition and [5, Theorem 3], there exists at most one weak solution of $(FDE; \phi)_0$. Indeed, if $u_1(t)$ and $u_2(t)$ are two weak solutions, they satisfy the integral inequality

$$\|u_1(t) - u_2(t)\| \leq \int_0^t \|F(\tau, u_1) - F(\tau, u_2)\| d\tau. \text{ (See [5, (8.3)].)}$$

Thus, by (A.3) and the Gronwall inequality, $u_1(t) = u_2(t)$ for $t \in [0, T]$.

Theorem 5. Suppose that $\{A(t); t \in [0, T]\}$ satisfy (A.1) with $\alpha_0 = 0$ and (A.2) and $F: [0, T] \times C \rightarrow X$ satisfy (A.3) and (A.4). If $\phi \in \text{Lip}$ and $\phi(0) \in \hat{D}$, then $(\text{FDE}; \phi)_0$ has a unique weak solution.

Proof. It suffices to show a generalized solution $u(0, \phi)(t)$ of $(\text{FDE}; \phi)_0$ is a weak solution. Note that $t \rightarrow F(t, u_t(0, \phi))$ is of bounded variation by (A.3) and (A.4) because $u(0, \phi)(t)$ is Lipschitz continuous. Then (DE) has a solution $v(t)$ in the sense of Evans, i.e., there exist sequence $\{t_k^n\}$ and $\{u_k^n\}$ such that

$$\text{i) } \frac{u_k^n - u_{k-1}^n}{h_k^n} + A(t_k^n)u_k^n \ni F(t_k^n, u_{t_k^n}(0, \phi)), \text{ where } h_k^n = t_k^n - t_{k-1}^n,$$

ii) the step functions $v^n(t)$ ($\equiv u_k^n$ on $(t_{k-1}^n, t_k^n]$) converge uniformly on $[0, T]$ to $v(t)$.

Note here that

$$M \equiv \max \left\{ \sup \|u_k^n\|, \sup \left\| \frac{u_k^n - u_{k-1}^n}{h_k^n} - F(t_k^n, u_{t_k^n}(0, \phi)) \right\| \right\} < \infty.$$

(See [5, Proof of Theorem 2].)

Let $v_k^n \in A(t_k^n)u_k^n$. By (3.5) we see that

$$\begin{aligned} & \|u(0, \phi)(t) - u_k^n\| = \|u(0, \phi)(\tau) - u_k^n\| \\ & \leq \int_{\tau}^t \{ [u(0, \phi)(\xi) - u_k^n, F(\xi, u_{\xi}(0, \phi)) - v_k^n]_+ + \theta_1(\xi, t_k^n) \} d\xi \\ & \quad + \alpha_0 \int_{\tau}^t \|u(0, \phi)(\xi) - u_k^n\| d\xi, \end{aligned}$$

where $\theta_1(\xi, r) = M_1 \|h(\xi) - h(r)\|$ and $M_1 = L_1(M)(1 + M)$.

Since $h_k^n[u(0, \phi)(\xi) - u_k^n, F(\xi, u_\xi(0, \phi)) - v_k^n]_+$

$$\leq \|u(0, \phi)(\xi) - u_{k-1}^n\| - \|u(0, \phi)(\xi) - u_k^n\| \\ + h_k^n \|F(\xi, u_\xi(0, \phi)) - F(t_k^n, u_{t_k^n}(0, \phi))\|,$$

it follows by the standard argument that

$$\int_{t_j^n}^{t_i^n} (\|u(0, \phi)(t) - v^n(\eta)\| - \|u(0, \phi)(\tau) - v^n(\eta)\|) d\eta \\ \leq \int_{\tau}^t (\|u(0, \phi)(\xi) - v^n(t_j^n)\| - \|u(0, \phi)(\xi) - v^n(t_i^n)\|) d\xi \\ + \int_{t_j^n}^{t_i^n} \int_{\tau}^t \{\alpha_0 \|u(0, \phi)(\xi) - v^n(\eta)\| + \theta_1^n(\xi, \eta) \\ + \|F(\xi, u_\xi(0, \phi)) - F^n(\eta)\|\} d\xi d\eta,$$

where θ_1^n and F^n are functions defined by

$$\theta_1^n(\xi, \eta) = \theta_1(\xi, t_k^n) \quad \text{for } \eta \in (t_{k-1}^n, t_k^n]$$

and

$$F^n(\eta) = F(t_k^n, u_{t_k^n}(0, \phi)) \quad \text{for } \eta \in (t_{k-1}^n, t_k^n], \text{ respectively.}$$

Letting $t_i^n \rightarrow t'$, $t_j^n \rightarrow \tau'$ as $n \rightarrow \infty$ and applying [8, Proposition 2.5] we obtain that $u(0, \phi)(t) = v(t)$ for $t \in [0, T]$. Q.E.D.

Finally, we consider the existence of strong solutions of $(FDE; \phi)_S$.

Corollary 1. Let $\phi \in \text{Lip}$ with $\phi(0) \in \hat{D}$. Assume that $\{A(t); t \in [0, T]\}$ and $F: [0, T] \times C \rightarrow X$ satisfy conditions (A.1)-(A.4). If X is reflexive, or, more generally, X satisfies the Radon-Nikodym property, then $(FDE; \phi)_S$ has a unique strong solution.

Proof. By virtue of Theorem 2, there exists a generalized solution $u(s, \phi)(t)$ and by the Remark after Lemma 1, $u(s, \phi)(t)$ is Lipschitz continuous and hence $u(s, \phi)(t)$ is differentiable a.e. $t \in [s, T]$. Now, let $h > 0$ and t_0 be any point at which $u(s, \phi)(\cdot)$ is differentiable. Putting $\tau = r = t_0$ and $t = t_0 + h$ in (3.5), we see that

$$\begin{aligned} & \|u(s, \phi)(t_0 + h) - x\| - \|u(s, \phi)(t_0) - x\| \\ & \leq \int_{t_0}^{t_0+h} \{ [u(s, \phi)(\xi) - x, F(\xi, u_\xi(s, \phi)) - y]_+ + \theta(\xi, t_0) \} d\xi \\ & \quad + \alpha_0 \int_{t_0}^{t_0+h} \|u(s, \phi)(\xi) - x\| d\xi \text{ for } [x, y] \in A(t_0). \end{aligned}$$

Dividing the above inequality by h and letting $h \downarrow 0$, it follows $[u(s, \phi)(t_0) - x, u'(s, \phi)(t_0)]_+$

$$\leq [u(s, \phi)(t_0) - x, F(t_0, u_{t_0}(s, \phi)) - y]_+ + \alpha_0 \|u(s, \phi)(t_0) - x\|,$$

i.e., for $[x, y] \in A(t_0)$

$$\begin{aligned} (3.6) \quad & [u(s, \phi)(t_0) - x, -u'(s, \phi)(t_0) + F(t_0, u_{t_0}(s, \phi)) \\ & + \alpha_0 u(s, \phi)(t_0) - (\alpha_0 x + y)]_+ \geq 0. \end{aligned}$$

By condition (A.1), it is easy to see that $A(t_0) + \alpha_0$ is m -accretive. Therefore, by (3.6), we see that

$$u'(s, \phi)(t_0) + A(t_0)(u(s, \phi)(t_0)) \ni F(t_0, u_{t_0}(s, \phi)).$$

Q.E.D.

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