

# Defining ideal of the Segre locus in arbitrary characteristic

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$X \subset \mathbb{P}^N$  : a non-deg proj variety over  $k = \bar{k}$  with  $p = \text{char}(k) \geq 0$ .

**Def (Segre locus).**  $\pi_z : \mathbb{P}^N \setminus \{z\} \rightarrow \mathbb{P}^{N-1}$  : the proj from a pt  $z \in \mathbb{P}^N$ .

$\mathfrak{S}^{\text{out}}(X) := \{z \in \mathbb{P}^N \setminus X \mid \pi_{z|X} : X \rightarrow \pi_z(X) \text{ is **not** birational}\}^-$ ,

$\mathfrak{S}^{\text{inn}}(X) := \{z \in X \mid \pi_{z|X} : X \setminus \{z\} \rightarrow \pi_z(X) \text{ is **not** birational}\}^-$ .

**Theorem A (B. Segre, 1936).** Assume  $p = 0$ . Then,

$\mathfrak{S}^{\text{out}}(X) = (\text{finitely many **linear subspaces** in } \mathbb{P}^N)$ .

- Recently,  $\mathfrak{S}^{\text{out}}(X)$ ,  $\mathfrak{S}^{\text{inn}}(X)$ , and related topics have been studied by A. Calabri-C. Ciliberto, E. Ballico, A. Noma, ...

**Q.** Does the **linearity** of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$  hold in **char**  $p \geq 0$ ?

**Theorem 1 (F).** Assume  $p \geq \deg(X)$  or  $p = 0$ . Then,

$\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X) = (\text{finitely many **linear subspaces** in } \mathbb{P}^N)$ .

**Example 2 ( $p < \deg(X)$ , Fukasawa-F).**  $\ell$  : a prime number,  
 $X := (x_0^\ell - x_1 x_3^{\ell-1} = x_1^\ell - x_2 x_3^{\ell-1} = 0) \subset \mathbb{P}^3$  (a rational curve of  $\deg \ell^2$ ).

1. Either  $p \neq \ell$  or  $p = 0 \implies \mathfrak{S}^{\text{out}}(X) = \{(1, 0, 0, 0)\}$ ,

Moreover,  $\ell \geq 3 \implies \mathfrak{S}^{\text{inn}}(X) = \{(0, 0, 0, 1)\}$ .

2.  $p = \ell \implies \mathfrak{S}^{\text{out}}(X) = (x_3 = x_0^\ell x_2 - x_1^{\ell+1} = 0)$ , **(non-linear)**  
Moreover,  $p = \ell \geq 3 \implies \mathfrak{S}^{\text{inn}}(X) = X$ .

3.  $\ell = 2$  &  $p \geq 0 \implies \mathfrak{S}^{\text{inn}}(X) = \emptyset$ .

**Rem.** Linearity of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X) \implies$  linearity of  $\mathfrak{S}^{\text{out}}(X)$ .

( $\because \forall$  irred comp  $Z \subset \mathfrak{S}^{\text{out}}(X)$ ,  $Z = (\text{an irred comp of } \mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X))$ .)

**Rem.** The following is an essential fact in  $p = 0$ :

For a variety  $Y \subset \mathbb{P}^N$ , and for a linear subspace  $L \subset \mathbb{P}^N$ ,

( $\diamond$ )  $\dim \pi_L(Y \setminus L) = \dim(Y) - \dim(\mathbb{T}_y Y \cap L) - 1$  ( $y \in Y$ : general pt).

In  $p > 0$ , equality ( $\diamond$ ) does *not* hold in general, since the **separability of the map**  $\pi_L$  is *not* guaranteed.

## Our strategy to prove Thm 1 & Ex 2.

Assume  $p \geq 0$  &  $\text{codim}(X, \mathbb{P}^N) = 2$ .

( $\star$ ) Give a **method to compute the defining ideal** of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$ .

$\Downarrow$

Assume  $p \geq \deg(X)$  &  $\text{codim}(X, \mathbb{P}^N) = 2$ .  
Show the **linearity** of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$ .

$\Downarrow$

By **induction** on  $\text{codim}(X, \mathbb{P}^n)$ , show the **linearity** of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$ . **(Thm 1)**

$\Downarrow$

For  $X$  in Ex 2, calc  
the ideal of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$ .  
 $\leadsto$  The **non-linearity** is shown. **(Ex 2)**

**Defining ideal of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$ .**

$X$  : not a cone &  $\text{codim}(X, \mathbb{P}^N) = 2$ . For  $e \in \mathbb{N}$ ,

$\text{Loc}_e(X) := \{z \in \mathbb{P}^N \mid \deg(\pi_z(X \setminus \{z\}))^- \leq e\}$ .

**Prop 3.** For every irred comp

$Z$  of  $\mathfrak{S}^{\text{out}}(X) \cup \mathfrak{S}^{\text{inn}}(X)$ ,

$\exists e < \deg(X)$  s.t.

$Z = (\text{an **irred comp** of } \text{Loc}_e(X))$ .

**Thm 4.**  $r := h^0(\mathbb{P}^N, \mathcal{I}_X(e))$ . Then,

$\text{Loc}_e(X) = \left( \begin{array}{l} \text{the zero set of all} \\ r \times r \text{ minors of } \Lambda(e) \end{array} \right)$ ,

where  $\Lambda(e)$  is constructed as follows.

**Construction of the matrix  $\Lambda(e)$ .**

$\mathbb{P}^N = \{(x_0, x_1, \dots, x_N)\}$ ,  
 $i = (i_0, i_1, \dots, i_N) \in \mathbb{Z}_{\geq 0}^{N+1}$ .

For  $0 \leq k \leq N$ ,  $\check{k}$

$\omega_k := (0, \dots, 0, \check{1}, 0, \dots, 0)$ .

For each  $i$ ,

$D_i := \frac{1}{i_0! i_1! \cdots i_N!} \left( \frac{\partial}{\partial x_0} \right)^{i_0} \left( \frac{\partial}{\partial x_1} \right)^{i_1} \cdots \left( \frac{\partial}{\partial x_N} \right)^{i_N}$

**(iterative higher derivation).**

**Def 5.**  $\{h_1, \dots, h_r\}$  : a basis of  $H^0(\mathbb{P}^N, \mathcal{I}_X(e))$ . For an integer  $s \leq e$ ,

$\Lambda_{e-s} := \begin{bmatrix} D_{(e-s)\omega_0} h_1 & D_{(e-s-1)\omega_0+\omega_1} h_1 & \cdots & D_i h_1 & \cdots & D_{(e-s)\omega_N} h_1 \\ D_{(e-s)\omega_0} h_2 & D_{(e-s-1)\omega_0+\omega_1} h_2 & \cdots & D_i h_2 & \cdots & D_{(e-s)\omega_N} h_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_{(e-s)\omega_0} h_r & D_{(e-s-1)\omega_0+\omega_1} h_r & \cdots & D_i h_r & \cdots & D_{(e-s)\omega_N} h_r \end{bmatrix}_{(|i| = e-s)}.$

$\Lambda(e) := \begin{cases} \Lambda_{e-1} & (p > e \text{ or } p = 0), \\ [\Lambda_{e-1} \ \Lambda_{e-p} \ \Lambda_{e-2p} \ \cdots \ \Lambda_{e-\lfloor e/p \rfloor \cdot p}] & (p \leq e). \end{cases}$