

5.

Non-monotone Bifurcation on Quadratic Rational Families

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5.1 Moduli space $\mathcal{M}_2(\mathbf{R})$ of real quadratic rational maps

$\text{Rat}_2(\mathbf{R})$ is the space of all real quadratic rational maps $f : \mathbf{R} \cup \{\infty\} \rightarrow \mathbf{R} \cup \{\infty\}$,

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_0x^2 + a_1x + a_2}{b_0x^2 + b_1x + b_2}.$$

Definition 1 $\mathcal{M}_2(\mathbf{R}) = \text{Rat}_2(\mathbf{R})/\text{PSL}_2(\mathbf{R})$ is called the **moduli space** of holomorphic conjugacy class (f) of real quadratic rational maps f .

Remark 1 The definitions of moduli space $\mathcal{M}_2(\mathbf{C})$ for the complex quadratic maps, $\text{Rat}_2(\mathbf{C})$, is identify with $\text{Rat}_2(\mathbf{C})/\text{PSL}_2(\mathbf{C})$.

For each $f \in \text{Rat}_2$, let z_1, z_2, z_3 be fixed points of f , μ_i the multiplier of z_i ($1 \leq i \leq 3$); $\mu_i = f'(z_i)$. Now consider elementary symmetric functions of three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

Milnor introduces coordinates of $\mathcal{M}_2(\mathbf{C})$ as follows [Mil92].

Lemma 1 (lemma 3.1 of [Mil92]) *These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that*

$$\mu_1 \mu_2 \mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0, \quad (1)$$

or in other words

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$, with coordinates σ_1 and σ_2 .

Here after we treat only the real case. σ_i ($1 \leq i \leq 3$) are all real, because three fixed points and multipliers are either all real or one real and a pair of complex conjugate numbers.

Proposition 1 $\mathcal{M}_2(\mathbf{R})$ is isomorphic to $\mathbf{R}^2$ except on the cubic algebraic curve,

$$F(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \quad (2)$$

For each (σ_1, σ_2) on this curve, two real representatives $\langle f_1 \rangle, \langle f_2 \rangle$ are determined. These classes correspond under the complex conjugacy $x \mapsto ix$. Although there is the singular curve (2), yet we regard moduli space $\mathcal{M}_2(\mathbf{R})$ as $\mathbf{R}^2$.

Milnor describes the curve (2) implicitly (compare Figure 15 in [Mil92]). Here we can give a defining equation (2) of this cubic curve.

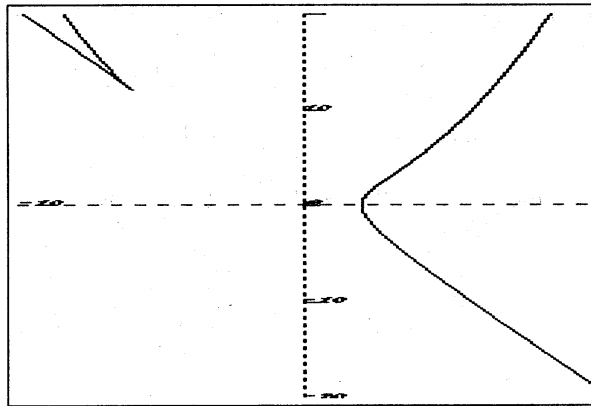


FIG 1 Moduli space with $2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0$.

5.2 A quadratic family with non-monotone bifurcation

M. Bier and T. C. Bountis studied “period-bubbling” bifurcation [BB84]. Their purpose is to demonstrate that monotone bifurcation commonly arise in some of the simplest nonlinear

dynamical systems involving the variation of more than one parameter. As a simple example of non-monotone bifurcation, they treat quadratic rational mapping, $x_{t+1} = Q + Ax_t/(x_t^2 + 1)$, ($A, Q > 0$).

H. E. Nusse and J. A. Yorke gave an example of exponential function family that has non-monotone bifurcation, even though it has negative Schwarzian derivative [NY88]. Their question was arisen of whether having a negative Schwarzian derivative rules out non-monotone bifurcation. They describe in [NY88] that if the above quadratic rational family is written in the following form,

$$\left\{ f_{m,r}(x) = m \frac{rx^2 + x + r}{1 + x^2} \right\},$$

it does not exhibit non-monotone bifurcation as the parameter m is increased. But we can show that this family exhibit non-monotone bifurcation for suitable parameter r .

Since $f_{m,r} \sim f_{m,-r}$, we can regard parameter r as $r \geq 0$.

In general, we obtain next results for a fixed parameter r .

Proposition 2 On $\mathcal{M}_2(\mathbf{R})$, one parameter family $\{f_{m,r}(x)\}_m$ for a fixed $r \neq \frac{1}{2}, 0$ is characterized as the following irreducible algebraic curve of degree 4,

$$\begin{aligned} H_r(\sigma_1, \sigma_2) = & -4096r^6 + (-128\sigma_1^2 + 512\sigma_1 + 512\sigma_2 + 1536)r^4 \\ & + (-\sigma_1^4 + 8\sigma_1^3 + (8\sigma_2 + 8)\sigma_1^2 + (-32\sigma_2 - 96)\sigma_1 - 16\sigma_2^2 - 96\sigma_2 - 144)r^2 \\ & - 2\sigma_1^3 + (-\sigma_2 + 1)\sigma_1^2 + (8\sigma_2 - 12)\sigma_1 + 4\sigma_2^2 - 12\sigma_2 + 36 = 0. \end{aligned} \quad (3)$$

For $r = \frac{1}{2}$, following irreducible algebraic curve of degree 3.

$$H_{\frac{1}{2}}(\sigma_1, \sigma_2) = -\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0.$$

For $r = 0$,

$$H_0(\sigma_1, \sigma_2) = F(\sigma_1, \sigma_2).$$

Proof. Three fixed points z_1, z_2, z_3 of f are the roots of the equation

$$x^3 - mrx^2 + (1 - m)x - mr = 0.$$

From the relation between coefficients and solutions, following equations hold.

$$\begin{cases} z_1 + z_2 + z_3 = mr \\ z_1 z_2 + z_2 z_3 + z_3 z_1 = 1 - m \\ z_1 z_2 z_3 = mr \end{cases}$$

Let μ_i ($i = 1, 2, 3$) be multiplier of each fixed point z_i ($i = 1, 2, 3$) given by,

$$\mu_i = m \frac{z_i^2 - 1}{(z_i^2 + 1)^2}.$$

By using “Gröbner basis” of Risa/Asir, symbolic and algebraic computation system, we can obtain $\sigma_1(= \mu_1 + \mu_2 + \mu_3)$ and $\sigma_2(= \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)$ as functions of m and r :

$$\begin{cases} 4m^2r^2 - m^2 + (\sigma_1 + 2)m - 4 = 0 \\ -4m^4r^4 + (m^4 - 12m^3 - 8m^2)r^2 + 2m^3 + (\sigma_2 - 5)m^2 + 4m - 4 = 0. \end{cases} \quad (4)$$

Using “Gröbner basis” again, we can remove m from (4), and we have (3).

In the case of $r = \frac{1}{2}$, $-\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0$.

In the case of r equal to 0, algebraic curve of (3) coincides with the curve of (2). ■

Remark 2 The equation of σ_1 in (4) is obtained from the Program 2. Takeshi Shimoyama(Fujitsu Laboratories) guided me in usage of Risa/Asir and he suggested this program.

Program 2

```

if (vtype(gr)!=3) load("gr")$$
extern Ord$

def moduliS1()
{
    S1=nm(m*((z1^2-1)/(z1^2+1)^2
            +(z2^2-1)/(z2^2+1)^2+(z3^2-1)/(z3^2+1)^2)-s1);
    X=z1+z2+z3-m*r;
    Y=z1*z2+z2*z3+z3*z1-1+m;
    Z=z1*z2*z3-m*r;
    Ord=2;
    G=gr([S1,X,Y,Z],[z1,z2,z3,m,r,s1]);
    for (I=length(G)-1;I>=0;I--){
        E=G[I];
        if (vars(E)==[r,m,s1])
            break;
    }
    return E;
}
end$

```

To say superfluously, the required equation (3) is obtained from following command of Risa/Asir.

```

gr([4*m^2*r^2-m^2+(s1+2)*m-4,
    -4*m^4*r^4+(m^4-12*m^3-8*m^2)*r^2+2*m^3+(s2-5)*m^2+4*m-4],[m,r]);

```

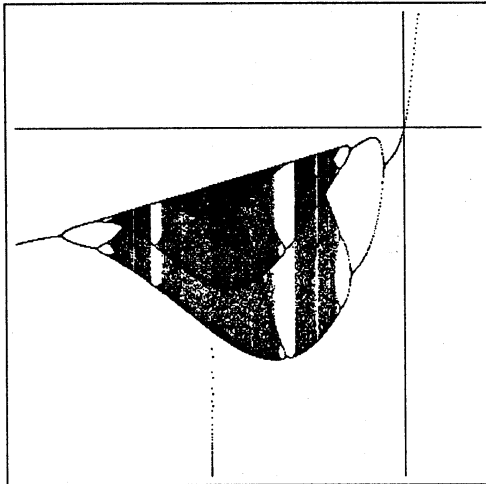


Figure 2 Non-monotone bifurcation;
 $-25.0 \leq m \leq 5.0$, $-3.0 \leq x \leq 1.0$,
 Parameter $r = 0.54$.

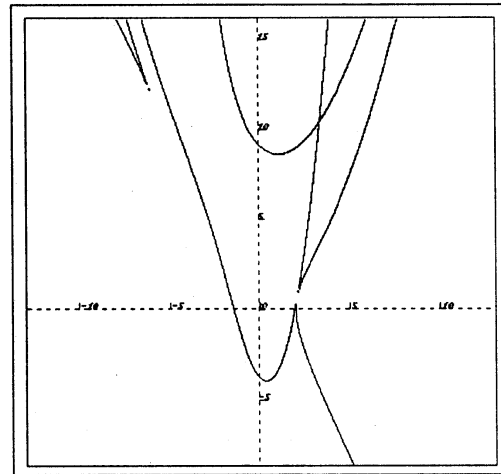


Figure 3 Algebraic curve of degree 4 and cubic curve in the moduli space. In the case of $r = 0.54$.

Example 1 Non-monotone bifurcation can occur at $r = 0.54$, See Figure 2. And its characteristic curve is Figure 3.

We can analyze the non-monotone bifurcation by overwriting the algebraic curve of degree 4 on the $\mathcal{M}_2(\mathbb{R})$.

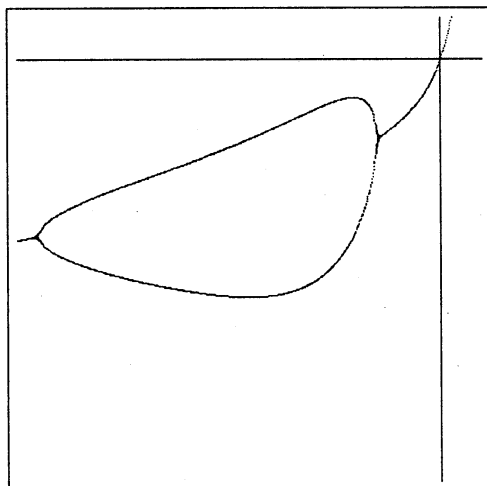


Figure 4 Period-bubbling bifurcation:
 $-10 \leq m \leq 1$, $-2 \leq x \leq 0.2$,
 Parameter $r = 0.58$.

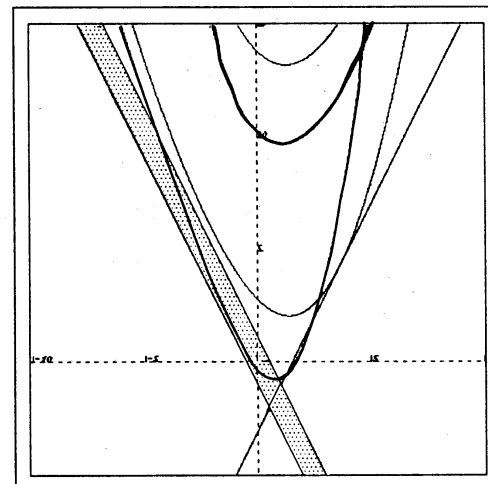


Figure 5 Algebraic curves of degree 4 in the "classified" moduli space. Thick curve corresponds with $r = 0.58$, thin curve corresponds with $r = 0.7$.

Example 2 One parameter family $\{f_{m,0.58}\}$ has non-monotone (period-bubbling) bifurcation. See Figure 4.

In Figure 5, the thick line indicates this family, and the gray belt is the region on which each map has attracting period 2 cycle. When algebraic curve of degree 4 through this gray belt, period-doubling bifurcation occurs. In this case, the curve intersects the gray belt (period-doubling occurs) and intersects again the period 1 region (period-halving occurs). Hence period-bubbling bifurcation occurs, as in Figure 4.

Theorem 1 *For a fixed parameter r , there are following three possibilities;*

1. *various bifurcations occur if $0 < r \leq \frac{1}{2}$,*
2. *non-monotone bifurcations occur if $\frac{1}{2} < r < \frac{3\sqrt{3}}{8}$, or*
3. *any bifurcation can't occur if $\frac{3\sqrt{3}}{8} \leq r$.*

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