

SUFFICIENT CONDITIONS FOR MEROMORPHIC STARLIKENESS AND CLOSE-TO-CONVEXITY OF ORDER α

*NAK EUN CHO AND **SHIGEYOSHI OWA

ABSTRACT. The object of the present paper is to drive a property of certain meromorphic functions in the punctured unit disk. Our main theorem contains certain sufficient conditions for starlikeness and close-to-convexity of order α of meromorphic functions.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

which are analytic in the punctured unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma$ is said to be meromorphic starlike of order α if it satisfies

$$-\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathcal{U} = \mathcal{D} - \{0\})$$

for some $\alpha (0 \leq \alpha < 1)$. We denote $\Sigma^*(\alpha)$ the class of all meromorphic starlike functions of order α .

Let $MC(\alpha)$ be the subclass of Σ consisting of functions f which satisfy

$$-\operatorname{Re}\{z^2 f'(z)\} > \alpha \quad (z \in \mathcal{U})$$

for some $\alpha (0 \leq \alpha < 1)$. A function f in $MC(\alpha)$ is meromorphic close-to-convex of order α in $\mathcal{D}$ [1].

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2. Main result

In proving our main theorem, we need the following lemma due to Owa, Nunokawa, Saitoh and Fukui [2].

Lemma 2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that $\operatorname{Re} p(z) > 0$ ($|z| < |z_0|$), $\operatorname{Re} p(z_0) = 0$, and $p(z) \neq 0$. Then we have $p(z) = ia$ ($a \neq 0$) and*

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \frac{k}{2} \left(a + \frac{1}{a} \right),$$

k is a real number with $k \geq 1$.

With the aid of above Lemma 2.1, we drive

Theorem 2.1. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$ and*

$$\operatorname{Re} \left\{ \alpha \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \right\} < 2(2 - \alpha) - \beta \quad (z \in \mathcal{U}),$$

then

$$-\operatorname{Re} \left\{ \frac{z^{2-\alpha} f'(z)}{f^\alpha(z)} \right\} > \frac{1}{1 + 2(2 - \alpha) - 2\beta} \quad (z \in \mathcal{U}),$$

where $\alpha \leq 2$ and $\frac{2(2-\alpha)-1}{2} \leq \beta < 2 - \alpha$.

Proof. We define the function p in U by

$$-\frac{z^{2-\alpha} f'(z)}{f^\alpha(z)} = \gamma + (1 - \gamma)p(z)$$

with $\gamma = \frac{1}{1+2(2-\alpha)-2\beta}$. Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and

$$\alpha \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = 2 - \alpha - \frac{(1 - \gamma)zp'(z)}{\gamma + (1 - \gamma)p(z)}.$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re} p(z) > 0 (|z| < |z_0|), \operatorname{Re} p(z_0) = 0, \text{ and } p(z) \neq 0.$$

Then, applying Lemma 2.1, we have $p(z) = ia$ ($a \neq 0$) and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \frac{k}{2} \left(a + \frac{1}{a} \right) \quad (k \geq 1).$$

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It follows from this that

$$\begin{aligned}\alpha \frac{z_0 f'(z_0)}{f(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)} &= 2 - \alpha - \frac{(1 - \gamma) z_0 p'(z_0)}{\gamma + (1 - \gamma) p(z_0)} \\ &= 2 - \alpha + \frac{k(1 - \gamma)(1 + a^2)}{2(\gamma + i(1 - \gamma)a)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\operatorname{Re} \left\{ \alpha \frac{z_0 f'(z_0)}{f(z_0)} - \frac{z_0 f''(z_0)}{f'(z_0)} \right\} &= 2 - \alpha + \frac{k(1 - \gamma)(1 + a^2)}{2(\gamma^2 + (1 - \gamma)^2 a^2)} \\ &\geq 2 - \alpha + \frac{k(1 - \gamma)}{2\gamma} \\ &\geq 2(2 - \alpha) - \beta.\end{aligned}$$

This contracts our assumption. Thus, we conclude that $\operatorname{Re} p(z) > 0$ for all $z \in \mathcal{U}$, that is, that

$$-\operatorname{Re} \left\{ \frac{z^{2-\alpha} f'(z)}{f^\alpha(z)} \right\} > \gamma = \frac{1}{1 + 2(2 - \alpha) - 2\beta} \quad (z \in \mathcal{U}).$$

Putting $\beta = \frac{2(2-\alpha)-1}{2}$ in Theorem 2.1, we have

Corollary 2.1. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$ and*

$$\operatorname{Re} \left\{ \alpha \frac{z f'(z)}{f(z)} - \frac{z f''(z)}{f'(z)} \right\} < \frac{3}{2} - \alpha \quad (z \in \mathcal{U}),$$

then

$$-\operatorname{Re} \left\{ \frac{z^{2-\alpha} f'(z)}{f^\alpha(z)} \right\} > \frac{1}{2} \quad (z \in \mathcal{U}),$$

where $\alpha \leq 2$.

Taking $\alpha = 1$ in Theorem 2.1, we have

Corollary 2.2. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$ and*

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} - \frac{z f''(z)}{f'(z)} \right\} < 2 - \beta \quad (z \in \mathcal{U}),$$

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then

$$-\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1}{3-2\beta} \quad (z \in \mathcal{U}),$$

that is, $f \in \Sigma^* \left(\frac{1}{3-2\beta} \right)$, where $\frac{1}{2} \leq \beta < 1$.

Further, letting $\alpha = 0$ in Theorem, we have

Corollary 2.3. *If $f \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathcal{D}$ and*

$$-\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} < 4 - \beta \quad (z \in \mathcal{U}),$$

then

$$-\operatorname{Re} \{ z^2 f'(z) \} > 5 - 2\beta \quad (z \in \mathcal{U}),$$

that is, $f \in MC \left(\frac{1}{5-2\beta} \right)$, where $\frac{3}{2} \leq \beta < 2$.

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DEPARTMENT OF APPLIED MATHEMATICS, PUKYONG NATIONAL UNIVERSITY, PUSAN 608-737, KOREA

**DEPARTMENT OF MATHEMATICS, KINKI UNIVERSITY, HIGASHI-OSAKA, OSAKA 577-8502, JAPAN