

DIFFERENTIAL OPERATORS OF DIRAC TYPES ON COMPLEX AND QUATERNION MANIFOLDS

九州大学大学院数理学研究院 長友康行 (Yasuyuki Nagatomo)
Graduate School of Mathematics, Kyushu University

1. INTRODUCTION

We refer to [4] for this section.

Let $\mathbb{R}^2$ be a Euclidean space of dimension 2. We denote by C_2 the associated Clifford algebra. It is well known that C_2 is isomorphic to the field of quaternions $\mathbb{H}$ as an algebra and has $\mathbb{Z}^2$ -grading:

$$C_2 \cong C_2^0 \oplus C_2^1.$$

This corresponds to the decomposition $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$. To be more precise, let e_1, e_2 be the standard basis of $\mathbb{R}^2$. We set

$$a + be_2e_1 \cong a + \sqrt{-1}b, \quad ae_1 + be_2 \cong a + \sqrt{-1}b,$$

where a and b are real numbers. Then we obtain the desired identification $C_2^0 \cong \mathbb{C}$ and $C_2^1 \cong \mathbb{C}$. The Euclidean structure with the orientation also induces the complex structure on $\mathbb{R}^2$ which is the same as the complex structure given by $C_2^1 \cong \mathbb{C}$.

Next, we consider the Dirac operator $D : \Gamma(C_2) \rightarrow \Gamma(C_2)$, where $\Gamma(C_2)$ is the space of C_2 -valued functions. The Dirac operator D is expressed as:

$$Df = \sum_{i=1}^2 e_i \cdot e_i(f),$$

where $\cdot$ denotes the Clifford multiplication and $e_i(f)$ is the derivative along e_i . If we introduce coordinates on $\mathbb{R}^2$ by $(x, y) \cong xe_1 + ye_2$, then D can be written as:

$$Df = e_1 \cdot \frac{\partial f}{\partial x} + e_2 \cdot \frac{\partial f}{\partial y}.$$

By definition, the Dirac operator respects the $\mathbb{Z}^2$ -grading:

$$D : \Gamma(C_2^0) \rightarrow \Gamma(C_2^1), \quad D : \Gamma(C_2^1) \rightarrow \Gamma(C_2^0).$$

Let $f = u(x, y)1 + v(x, y)e_2e_1$ be a C_2^0 -valued function. Then we have

$$Df = u_x e_1 + v_x e_2 + u_y e_2 - v_y e_1 = (u_x - v_y) e_1 + (v_x + u_y) e_2$$

When we adopt the identification $\mathbb{C}_2^0 \cong \mathbb{C}_2^1 \cong \mathbb{C}$, we obtain

$$Df = \frac{\partial}{\partial \bar{z}}(u + \sqrt{-1}v) = 2 \frac{\partial f}{\partial \bar{z}},$$

where

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right).$$

OPERATORS OF DIRAC TYPES

In brief, the Dirac operator is the Cauchy-Riemann operator. Note that the whole setting is preserved by the action of $\text{Spin}(2)$.

Let $\mathbb{R}^4$ be a Euclidean space of dimension 4. We denote by C_4 the associated Clifford algebra. It is well known that C_4 is isomorphic to $\mathbb{H}(2)$ the 2×2 matrix algebra over quaternions. If we denote by $e_1, \dots, e_4$ the standard basis of $\mathbb{R}^4$, then the identification is realised as:

$$\begin{aligned} e_1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & e_2 &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ e_3 &\mapsto \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, & e_4 &\mapsto \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \end{aligned}$$

Note that $\mathbb{H}$ is not a commutative field. The scalar product on $\mathbb{H}^2$ is provided with multiplication by quaternions on the *right* and the quaternion matrix acts on $\mathbb{H}^2$ from the *left*. Consequently,

$$\begin{aligned} e_1 \begin{pmatrix} 1 \\ i \end{pmatrix} &= \begin{pmatrix} 1 \\ -i \end{pmatrix}, & e_2 \begin{pmatrix} 1 \\ i \end{pmatrix} &= \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} i \\ e_3 \begin{pmatrix} 1 \\ i \end{pmatrix} &= \begin{pmatrix} j \\ -k \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} j, & e_4 \begin{pmatrix} 1 \\ i \end{pmatrix} &= \begin{pmatrix} k \\ j \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} k \end{aligned}$$

Hence we obtain a $\mathbb{Z}^2$ -grading of $\mathbb{H}$ as a module of C_4 :

$$\mathbb{H}^2 \cong \mathbb{H} \begin{pmatrix} 1 \\ i \end{pmatrix} \oplus \mathbb{H} \begin{pmatrix} 1 \\ -i \end{pmatrix} =: V_0 \oplus V_1.$$

The Dirac operator D is defined in a similar way and respects the $\mathbb{Z}^2$ -grading:

$$D : \Gamma(V_0) \rightarrow \Gamma(V_1), \quad D : \Gamma(V_1) \rightarrow \Gamma(V_0).$$

We introduce coordinates on $\mathbb{R}^4$ by $(x_0, x_1, x_2, x_3) \cong \sum_{i=0}^3 x_i e_{i+1}$ and identify $\mathbb{R}^4$ with $\mathbb{H}$ as $(x_0, x_1, x_2, x_3) \cong x_0 + x_1 i + x_2 j + x_3 k = q$.

Let $f = u_0 + u_1 i + u_2 j + u_3 k$ be a V_0 -valued function. (f may be regarded as $\begin{pmatrix} 1 \\ i \end{pmatrix} f$.) Then we have

$$\begin{aligned} Df &= \sum_{i=0}^3 e_{i+1} \cdot \left(\frac{\partial f}{\partial x_i} \right) \cong \sum_{i=0}^3 e_{i+1} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \left(\frac{\partial f}{\partial x_i} \right) \\ &= \begin{pmatrix} 1 \\ -i \end{pmatrix} \left\{ \left(\frac{\partial f}{\partial x_0} \right) + i \left(\frac{\partial f}{\partial x_1} \right) + j \left(\frac{\partial f}{\partial x_2} \right) + k \left(\frac{\partial f}{\partial x_3} \right) \right\} \\ &\cong \frac{\partial}{\partial \bar{q}} f, \end{aligned}$$

where, of course,

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3}.$$

We should also mention that the whole setting is preserved by the action of $\text{Spin}(4)$.

OPERATORS OF DIRAC TYPES

2. GENERALISATION

2.1. Another interpretation. Although the Dirac operator can be defined on any dimensional Euclidean space, Dirac operators in the previous section can be re-interpreted from a different viewpoint.

In the 2-dimensional case, this is obvious. Let J denote the complex structure of $\mathbb{R}^2$:

$$Je_1 = e_2, \quad Je_2 = -e_1$$

Consider the complexification $(\mathbb{R}^2)^\mathbb{C} \cong \mathbb{C}^2$ of $\mathbb{R}^2$. Then J can be extended as the complex linear transformation of $\mathbb{C}^2$. Since $J^2 = -1$, $\mathbb{C}^2$ is decomposed into the eigenspaces of J :

$$\mathbb{C}^2 = \mathbb{C}_{(1,0)} \oplus \mathbb{C}_{(0,1)},$$

where

$$\mathbb{C}_{(1,0)} = \{z \in \mathbb{C}^2 | Jz = \sqrt{-1}z\}, \quad \mathbb{C}_{(0,1)} = \{z \in \mathbb{C}^2 | Jz = -\sqrt{-1}z\},$$

in other words,

$$\frac{1}{2}(u - \sqrt{-1}Ju) \in \mathbb{C}_{(1,0)}, \quad \frac{1}{2}(u + \sqrt{-1}Ju) \in \mathbb{C}_{(0,1)},$$

where $u \in \mathbb{R}^2$. We can easily show that $(\mathbb{R}^2, J)$ is isomorphic to $\mathbb{C}_{(1,0)}$ as a complex vector space. The tangent space $T_x\mathbb{R}^2$ at a point $x \in \mathbb{R}^2$ is naturally identified with the vector space $\mathbb{R}^2$.

Let $\mathbb{C}^{2*}$ be the dual space. According to the decomposition of $\mathbb{C}^2$, we have

$$\mathbb{C}^{2*} = \mathbb{C}dz \oplus \mathbb{C}d\bar{z}, \quad dz = dx + \sqrt{-1}dy, \quad d\bar{z} = dx - \sqrt{-1}dy.$$

Then, for a function f , we have a differential df :

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right).$$

Taking the complex conjugate of $\mathbb{C}^2$, we define a Hermitian inner product h on $\mathbb{C}^2$ as

$$h(z, w) = g(z, \bar{w})$$

where $g(u, v)$ is the bi-linear extension of the inner product on $\mathbb{R}^2$. Then $\mathbb{C}_{(1,0)} \perp \mathbb{C}_{(0,1)}$. Let $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}d\bar{z}$ be the orthogonal projection. We can define a differential operator $\pi \circ d$. For a function f , we write explicitly $\pi \circ df$ down:

$$\pi \circ df = \frac{\partial f}{\partial \bar{z}}d\bar{z} \cong \frac{\partial f}{\partial \bar{z}}$$

and so

$$2\pi \circ df = Df.$$

This consistency is based on the group isomorphism $\text{Spin}(2) \cong \text{U}(1)$. The group $\text{U}(1)$ can be considered as the unit complex numbers:

$$\text{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}.$$

OPERATORS OF DIRAC TYPES

Next, we concern the 4-dimensional case. We follow S.Salamon's description [9]. When we regard $\mathbb{R}^4$ as $\mathbb{H}$, $\mathbb{H}$ has a natural "quaternion"-Hermitian inner product $h_{\mathbb{H}}$:

$$h_{\mathbb{H}}(p, q) = \bar{q}p, \quad \bar{q} = x_0 - ix_1 - jx_2 - kx_3.$$

The set of unit quaternions has a group structure induced from multiplication of $\mathbb{H}$, which is denoted by $\text{Sp}(1)$:

$$\text{Sp}(1) = \{q \in \mathbb{H} \mid |q| = 1\}.$$

If we identify $\mathbb{H}$ with $\mathbb{C}^2$ using the identification $i \cong \sqrt{-1}$, then $h_{\mathbb{H}}$ is decomposed into a Hermitian inner product and a complex volume form:

$$\begin{aligned} h_{\mathbb{H}}(p, q) &= \bar{q}p = \overline{(u + jv)}(z + jw) = (\bar{u} - j\bar{v})(z + jw) \\ &= (z\bar{u} + w\bar{v}) + j(wu - zv) = h\left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) + \omega\left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right)j. \end{aligned}$$

The group $\text{Sp}(1)$ preserves $h_{\mathbb{H}}$ and so, preserves a Hermitian inner product h and a complex volume form ω . This observation yields the isomorphism $\text{Sp}(1) \cong \text{SU}(2)$.

Since unit quaternions act on $\mathbb{H}$ from the both sides and the left action commutes with the right action, $\text{Sp}(1) \times \text{Sp}(1)$ acts on $\mathbb{H}$. This action preserves the inner product and the volume form on $\mathbb{R}^4$ and so, we obtain a group homomorphism $\rho : \text{Sp}(1) \times \text{Sp}(1) \rightarrow \text{SO}(4)$. Since $\rho(1, 1) = \rho(-1, -1) = \text{Id}$, $\text{Ker } \rho \cong \mathbb{Z}^2$. In this way, we have an identification:

$$\text{Sp}(1) \times \text{Sp}(1) / \mathbb{Z}^2 \cong \text{SO}(4), \quad \text{or} \quad \text{Sp}(1) \times \text{Sp}(1) \cong \text{Spin}(4).$$

To define differential operators from the quaternionic viewpoint, we recall the representation theory of $\text{Sp}(1) \cong \text{SU}(2)$. Let $\mathbb{C}^2$ be the standard representation of $\text{SU}(2)$. Then the k -th symmetric tensor product $S^k \mathbb{C}^2$ ($\dim S^k \mathbb{C}^2 = k + 1$) is an irreducible representation of $\text{SU}(2)$ and each finite dimensional irreducible representation is one of them. In particular, we need an irreducible decomposition of $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \wedge^2 \mathbb{C}^2 \oplus S^2 \mathbb{C}^2$. Then $\wedge^2 \mathbb{C}^2 = \mathbb{C}\omega$ and the classification of irreducible representation yields that

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C} \oplus S^2 \mathbb{C}^2.$$

We denote two copies of the standard representation of $\text{Sp}(1)$ by $\mathbb{H}$ and $\mathbb{E}$. The tensor product $\mathbb{H} \otimes_{\mathbb{C}} \mathbb{E}$ (for short, $\mathbb{H} \otimes \mathbb{E}$) is of complex dimension 4. Here we identify $\mathbb{H}$ and $\mathbb{E}$ with $\mathbb{C}^2$ using $i \cong \sqrt{-1}$. Since $ij = -ji$, j is an anti-linear (or conjugate-linear) transformation. We refer to j as the quaternion structure which is preserved by $\text{Sp}(1)$. Then $\sigma = j \otimes j$ acts on $\mathbb{H} \otimes \mathbb{E}$ and $\text{Sp}(1) \times \text{Sp}(1)$ also preserves σ . By definition, $\sigma^2 = j^2 \otimes j^2 = 1$ and σ is still an anti-linear transformation and so, σ is called the real structure. The invariant subset $(\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}$ under σ of $\mathbb{H} \otimes \mathbb{E}$ is a real vector space of dimension 4, and $\text{Sp}(1) \times \text{Sp}(1)$ acts on $(\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}$. Hence we recover $\mathbb{R}^4 \cong (\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}$.

OPERATORS OF DIRAC TYPES

Let f be an $\mathbb{H}$ -valued function. Then $\frac{\partial f}{\partial x_i} = f_{x_i}$ is also an $\mathbb{H}$ -valued function and so,

$$\begin{pmatrix} f_{x_0} \\ f_{x_1} \\ f_{x_2} \\ f_{x_3} \end{pmatrix} \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}^4 \cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{E} \cong (\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E}$$

We define the orthogonal projections π_i by

$$\pi_1 : (\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E} \rightarrow \mathbb{C} \otimes \mathbb{E} \cong \mathbb{E}, \quad \pi_2 : (\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E} \rightarrow S^2\mathbb{H} \otimes \mathbb{E}.$$

We use π_1 to define the differential operator $\pi_1 \circ d : \Gamma(\mathbb{H}) \rightarrow \Gamma(\mathbb{E})$.

To compute explicitly, we use the standard basis h_1, h_2 of $\mathbb{H}$ which is a unitary basis and satisfies $h_2 = jh_1$. We also take the standard basis e_1 and e_2 of $\mathbb{E}$. Then the identification $\mathbb{R}^4 \cong \mathbb{H} \otimes \mathbb{E}$ is realised by:

$$\begin{aligned} (x_0, x_1, x_2, x_3) &\cong x_0(h_2 \otimes e_1 - h_1 \otimes e_2) + x_1\sqrt{-1}(h_2 \otimes e_1 + h_1 \otimes e_2) \\ &\quad + x_2(h_1 \otimes e_1 + h_2 \otimes e_2) + x_3\sqrt{-1}(h_1 \otimes e_1 - h_2 \otimes e_2) \end{aligned}$$

Since

$$x_0 + ix_1 + jx_2 + kx_3 = (x_0 + ix_1) + j(x_2 - ix_3),$$

we put

$$z = x_0 + x_1, \quad w = x_2 - ix_3.$$

If $f = uh_1 + vh_2 \in \Gamma(\mathbb{H})$, where u and v are $\mathbb{C}$ -valued functions, then

$$\begin{aligned} df &= (u_{x_0}h_1 + v_{x_0}h_2) \otimes (h_2 \otimes e_1 - h_1 \otimes e_2) \\ &\quad + (u_{x_1}h_1 + v_{x_1}h_2) \otimes \sqrt{-1}(h_2 \otimes e_1 + h_1 \otimes e_2) \\ &\quad + (u_{x_2}h_1 + v_{x_2}h_2) \otimes (h_1 \otimes e_1 + h_2 \otimes e_2) \\ &\quad + (u_{x_3}h_1 + v_{x_3}h_2) \otimes \sqrt{-1}(h_1 \otimes e_1 - h_2 \otimes e_2) \\ &\xrightarrow{\pi_1} (u_{x_0}e_1 + v_{x_0}e_2) + \sqrt{-1}(u_{x_1}e_1 - v_{x_1}e_2) \\ &\quad + (u_{x_2}e_2 - v_{x_2}e_1) - \sqrt{-1}(u_{x_3}e_2 + v_{x_3}e_1) \\ &= (u_{x_0} + \sqrt{-1}u_{x_1} - v_{x_2} - \sqrt{-1}v_{x_3})e_1 \\ &\quad + (v_{x_0} - \sqrt{-1}v_{x_1} + u_{x_2} - \sqrt{-1}u_{x_3})e_2. \end{aligned}$$

We set

$$\begin{aligned} \partial_z &= \frac{\partial}{\partial x_0} - \sqrt{-1}\frac{\partial}{\partial x_1}, & \partial_{\bar{z}} &= \frac{\partial}{\partial x_0} + \sqrt{-1}\frac{\partial}{\partial x_1}, \\ \partial_w &= \frac{\partial}{\partial x_2} + \sqrt{-1}\frac{\partial}{\partial x_3}, & \partial_{\bar{w}} &= \frac{\partial}{\partial x_2} - \sqrt{-1}\frac{\partial}{\partial x_3}. \end{aligned}$$

Then

$$\begin{aligned} \pi_1 \circ df &= (\partial_{\bar{z}}u - \partial_wv)e_1 + (\partial_zv + \partial_{\bar{w}}u)e_2 \cong (\partial_{\bar{z}}u - \partial_wv) + j(\partial_zv + \partial_{\bar{w}}u) \\ &= \partial_{\bar{z}}u + j\partial_zv + j(\partial_{\bar{w}}u + j\partial_wv) = (\partial_{\bar{z}} + j\partial_{\bar{w}})(u + jv) = \frac{\partial}{\partial \bar{q}}f. \end{aligned}$$

We obtain

$$\pi_1 \circ df = Df.$$

We also have a differential operator $\mathcal{D} : \Gamma(\mathbb{H}) \rightarrow \Gamma(S^2\mathbb{H} \otimes \mathbb{E})$:

$$\mathcal{D} = \pi_2 \circ d,$$

which is called the twistor operator [1].

2.2. Higher dimensional analogue. From the viewpoint of complex number field, it is now clear that we have a generalisation of the differential operator $2\pi \circ d = D$ on $\mathbb{R}^2$. We may replace $\mathbb{C} \cong \mathbb{R}^2$ by $\mathbb{C}^n$ or the structure group $U(1)$ by $U(n)$. Let $(z_1, \dots, z_n)$ be the standard coordinates of $\mathbb{C}^n$. Then, for a $\mathbb{C}$ -valued function f , we define a system of differential operators:

$$\bar{\partial}f = \left(\frac{\partial}{\partial \bar{z}_1} f, \dots, \frac{\partial}{\partial \bar{z}_n} f \right).$$

In an invariant way, we regard $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ with a complex structure J . The complex structure J can be extended to a complex linear transformation on the complexified vector space $\mathbb{C}^{2n} \cong \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. As before, we obtain the eigenspaces of J :

$$\mathbb{C}_{(1,0)} = \{v \in \mathbb{C}^{2n} \mid Jv = \sqrt{-1}v\}, \quad \mathbb{C}_{(0,1)} = \{v \in \mathbb{C}^{2n} \mid Jv = -\sqrt{-1}v\}$$

and an isomorphism $\mathbb{C}^n \cong \mathbb{C}_{(1,0)}$. Let $\mathbb{C}^{n*}$ be the dual space of $\mathbb{C}^n$. Since the dual space has also a natural complex structure, we obtain in a similar way

$$\mathbb{C}^{(1,0)} = \{\phi \in \mathbb{C}^{2n*} \mid J\phi = \sqrt{-1}\phi\}, \quad \mathbb{C}^{(0,1)} = \{\phi \in \mathbb{C}^{2n*} \mid J\phi = -\sqrt{-1}\phi\}$$

When we use coordinates $z_1, \dots, z_n$ on $\mathbb{C}^n$, the basis of $\mathbb{C}^{(1,0)}$ (resp. $\mathbb{C}^{(0,1)}$) consists of

$$dz_1, \dots, dz_n \quad (\text{resp. } d\bar{z}_1, \dots, d\bar{z}_n).$$

Then the differential of f is expressed as:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

We can also define the orthogonal projection $\pi : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{(0,1)}$. It can be shown that

$$\bar{\partial}f = \pi \circ df$$

Although $2\bar{\partial} = D$ is an elliptic operator in the case $n = 1$, higher dimensional analogue $\bar{\partial}$ is not an elliptic operator when $n \geq 2$. But we have an elliptic complex. The complex vector space generated by $dz_{i_1} \wedge \dots \wedge dz_{i_p}$ (resp. $d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_q}$) is denoted by $\Lambda^{p,0}$ (resp. $\Lambda^{0,q}$). We can consider a k -form ϕ of bidegree (p, q) :

$$\phi = \sum_{\substack{p+q=k \\ 0 \leq i_1 < \dots < i_p \leq n \\ 0 \leq j_1 < \dots < j_q \leq n}} \phi_{i_1, \dots, i_p, j_1, \dots, j_q} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q},$$

OPERATORS OF DIRAC TYPES

where $\phi_{i_1, \dots, i_p, j_1, \dots, j_q}$ is a function on $\mathbb{C}^{2n}$. Since

$$d\phi_{i_1, \dots, i_p, j_1, \dots, j_q} = \partial_{z_i} \phi_{i_1, \dots, i_p, j_1, \dots, j_q} dz_i + \partial_{\bar{z}_i} \phi_{i_1, \dots, i_p, j_1, \dots, j_q} d\bar{z}_i,$$

we have a differential of ϕ :

$$\begin{aligned} d\phi = \sum_{i=1}^n \left(\partial_{z_i} \phi_{i_1, \dots, i_p, j_1, \dots, j_q} dz_i \wedge dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{j_q} \right. \\ \left. + \sum_{j=1}^n \partial_{\bar{z}_j} \phi_{i_1, \dots, i_p, j_1, \dots, j_q} d\bar{z}_j \wedge dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{j_q} \right) \end{aligned}$$

Hence, we can also define

$$\bar{\partial}\phi = \sum \bar{\partial}\phi_{i_1, \dots, i_p, j_1, \dots, j_q} \wedge dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{j_q}.$$

Note that

$$\bar{\partial}\phi_{i_1, \dots, i_p, j_1, \dots, j_q} = \sum_{j=1}^n \partial_{\bar{z}_j} \phi_{i_1, \dots, i_p, j_1, \dots, j_q} d\bar{z}_j,$$

which is already defined. Consequently, we obtain a generalisation of a differential $\bar{\partial}$ for a function to a differential for a k -form which is denoted by the same symbol $\bar{\partial}$. By definition, $\bar{\partial} \circ \bar{\partial} = 0$. We use $\bar{\partial}$ to get an elliptic complex:

$$0 \rightarrow \Omega^{p,q} \xrightarrow{\bar{\partial}} \Omega^{p,q+1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{p,n} \rightarrow 0,$$

where

$$\Omega^{p,q} = \Gamma(\wedge^{p,q})$$

In the case of the field of quaternions, $\mathbb{H}^n \cong \mathbb{R}^{4n}$ can be taken as a generalisation in an obvious sense [8]. The group $\text{Sp}(1) \times \text{Sp}(1)$ is replaced by $\text{Sp}(1) \times \text{Sp}(n)$. We also denote by $\mathbb{E}$ the standard representation of $\text{Sp}(n)$ ($\mathbb{E} \cong \mathbb{C}^{2n} \cong \mathbb{H}^n$). Therefore

$$\mathbb{R}^{4n} \cong (\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}.$$

Now we define a differential operator $D : \Gamma(\mathbb{H}) \rightarrow \Gamma(\mathbb{E})$ as

$$D : \Gamma(\mathbb{H}) \xrightarrow{d} \Gamma(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{E}) \cong \Gamma((\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E}) \xrightarrow{\pi_1} \Gamma(\mathbb{E})$$

[9]. We have another differential operator $\mathcal{D} : \Gamma(\mathbb{H}) \rightarrow \Gamma(S^2\mathbb{H} \otimes \mathbb{E})$ as

$$\mathcal{D} : \Gamma(\mathbb{H}) \xrightarrow{d} \Gamma(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{E}) \cong \Gamma((\mathbb{C} \oplus S^2\mathbb{H}) \otimes \mathbb{E}) \xrightarrow{\pi_2} \Gamma(S^2\mathbb{H} \otimes \mathbb{E}).$$

Using the Clebsch-Gordan formula, we obtain

$$\mathbb{H} \otimes S^p\mathbb{H} \cong S^{p-1}\mathbb{H} \oplus S^{p+1}\mathbb{H}.$$

OPERATORS OF DIRAC TYPES

Combined with an algebraic homomorphism : $\mathbb{E} \otimes \wedge^q \mathbb{E} \rightarrow \mathbb{E} \otimes (\otimes^q \mathbb{E}) \rightarrow \wedge^{q+1} \mathbb{E}$, we have two extensions of differential operators D and $\mathcal{D}$:

$$\begin{aligned} D : \Gamma(S^p \mathbb{H} \otimes \wedge^q \mathbb{E}) &\xrightarrow{d} \Gamma(S^p \mathbb{H} \otimes \wedge^q \mathbb{E} \otimes \mathbb{H} \otimes \mathbb{E}) \\ &\cong \Gamma((S^p \mathbb{H} \otimes \mathbb{H}) \otimes (\wedge^q \mathbb{E} \otimes \mathbb{E})) \xrightarrow{\pi_1} \Gamma(S^{p+1} \mathbb{H} \otimes \wedge^{q+1} \mathbb{E}) \\ \mathcal{D} : \Gamma(S^p \mathbb{H} \otimes \wedge^q \mathbb{E}) &\xrightarrow{d} \Gamma(S^p \mathbb{H} \otimes \wedge^q \mathbb{E} \otimes \mathbb{H} \otimes \mathbb{E}) \\ &\cong \Gamma((S^p \mathbb{H} \otimes \mathbb{H}) \otimes (\wedge^q \mathbb{E} \otimes \mathbb{E})) \xrightarrow{\pi_2} \Gamma(S^{p+1} \mathbb{H} \otimes \wedge^{q+1} \mathbb{E}) \end{aligned}$$

Here D is called a quaternion-Dirac operator and $\mathcal{D}$ is called a twistor operator.

We also obtain an elliptic complex:

$$\begin{aligned} 0 \rightarrow C^\infty(\mathbb{R}^{4n}) &\xrightarrow{d} \Gamma(\mathbb{H} \otimes \mathbb{E}) \xrightarrow{\mathcal{D}} \Gamma(S^2 \mathbb{H} \otimes \wedge^2 \mathbb{E}) \xrightarrow{\mathcal{D}} \dots \\ &\xrightarrow{\mathcal{D}} \Gamma(S^p \mathbb{H} \otimes \wedge^p \mathbb{E}) \xrightarrow{\mathcal{D}} \dots \xrightarrow{\mathcal{D}} \Gamma(S^{2n} \mathbb{H} \otimes \wedge^{2n} \mathbb{E}) \rightarrow 0. \end{aligned}$$

3. GENERALISATION II

We have already found *linear* differential operators. Here we consider a *non-linear* problem which relates to our differential operators.

For our purpose, we replace a vector valued function or form by a matrix valued function or form such that

$$A = A_1 dx_1 + \dots + A_n dx_n \quad \text{on } \mathbb{R}^n,$$

where

$$A_i \in \mathbb{C}(r) := \{r \times r \text{ matrices over } \mathbb{C}\}.$$

We introduce a new differentiation ∇ :

$$\nabla = d + A, \quad \nabla_i = \partial_i + A_i, \quad \partial_i = \frac{\partial}{\partial x_i}$$

which acts on $\mathbb{C}^r$ -valued function. Although $d^2 = 0$ which means that $\partial_i \partial_j = \partial_j \partial_i$, we have $\nabla^2 \neq 0$. In fact,

$$[\nabla_i, \nabla_j] = \nabla_i \nabla_j - \nabla_j \nabla_i = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

In this way, we obtain a curvature 2-form or a gauge field F which associates to A which is called the connection form or the gauge potential:

$$F = \sum_{i,j} F_{ij} dx_i \wedge dx_j, \quad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

Note that F is a non-linear function of a given A . Finally, we can extend the operator ∇ to the covariant exterior differentiation d^∇ which acts on $\mathbb{C}^r$ -valued k -forms. For a $\mathbb{C}^r$ -valued k -form

$$\phi_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Gamma(\wedge^k \otimes \mathbb{C}^r) = \Omega^k(\mathbb{C}^r),$$

we define

$$\begin{aligned} d^\nabla (\phi_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}) &= (\nabla \phi_{i_1, \dots, i_k}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ &= \left(\sum_{i=1}^n \nabla_i \phi_{i_1, \dots, i_k} dx_i \right) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^{k+1}(\mathbb{C}^r). \end{aligned}$$

Then it is easily shown that

$$d^\nabla d^\nabla s = F s \quad \text{for } s \in \Gamma(\mathbb{C}^r) = \Omega^0(\mathbb{C}^r), \quad (d^\nabla d^\nabla = F)$$

When f is a function, then we have

$$d^\nabla d^\nabla f s = f d^\nabla d^\nabla s,$$

and so, F can be regarded as

$$F \in \Omega^2(\text{End}(\mathbb{C}^r)).$$

3.1. $\mathbb{C}$ -case. We begin with a vector space $\mathbb{C}^n$ as a *base* space. For a given gauge potential $A = \sum_{i=1}^n (A_{z_i} dz_i + A_{\bar{z}_i} d\bar{z}_i)$ and a $\mathbb{C}^r$ (a *fibre*) -valued (p, q) -form $\phi \in \Omega^{p,q}(\mathbb{C}^r) := \Gamma(\wedge^{p,q} \otimes \mathbb{C}^r)$, we have

$$d^\nabla \phi \in \Omega^{p+1,q}(\mathbb{C}^r) \oplus \Omega^{p,q+1}(\mathbb{C}^r),$$

and so, we can define in an obvious way,

$$\partial^\nabla \phi = \pi_1 d^\nabla \phi \in \Omega^{p+1,q}(\mathbb{C}^r), \quad \bar{\partial}^\nabla \phi = \pi_2 d^\nabla \phi \in \Omega^{p,q+1}(\mathbb{C}^r).$$

Consider a differential equation

$$(*) \quad \bar{\partial}^\nabla s = 0,$$

for a $\mathbb{C}^r$ -valued function s . If such an s exists, then

$$\bar{\partial}^\nabla \bar{\partial}^\nabla s = 0,$$

which means

$$F^{0,2} s = 0.$$

Note that

$$F \in \Omega^2(\text{End}(\mathbb{C}^r)) \cong \Omega^{2,0}(\text{End}(\mathbb{C}^r)) \oplus \Omega^{1,1}(\text{End}(\mathbb{C}^r)) \oplus \Omega^{0,2}(\text{End}(\mathbb{C}^r)).$$

Indeed, the condition

$$F^{0,2} = 0$$

is the integrability condition for the equation $(*)$ [1]. If $F^{0,2} = 0$ is satisfied, then we can find locally enough solutions for $(*)$, which provide a basis of $\mathbb{C}^r$ at each point of $\mathbb{C}^n$. In other words, there exist locally defined $s_1, \dots, s_r$ which satisfy $(*)$ and span the vector space $\mathbb{C}_z^r$ at each point $z \in \mathbb{C}^n$. Hence $\mathbb{C}_z^r$ can be thought as varying holomorphically with z . In this way, we obtain a *holomorphic vector bundle*. Then the *frame* $s_1, \dots, s_r$ is called the *holomorphic gauge*.

As a consequence, a connection form A satisfying $F^{0,2} = 0$ gives a holomorphic vector bundle. Since

$$F_{ij}^{0,2} = \bar{\partial}_i A_{\bar{z}_j} - \bar{\partial}_j A_{\bar{z}_i} + [A_{\bar{z}_i}, A_{\bar{z}_j}],$$

the equation $F^{0,2} = 0$ is a non-linear equation of the first order and we can find the Cauchy-Riemann operator as the linearisation:

$$\bar{\partial}^\nabla B = 0, \quad \text{for } B \in \Omega^{0,1}(\text{End}(\mathbb{C}^r)).$$

Using again $\bar{\partial}^\nabla \bar{\partial}^\nabla = F^{0,2} = 0$, we have an elliptic complex:

$$0 \rightarrow \Omega^{0,0}(\text{End}(\mathbb{C}^r)) \xrightarrow{\bar{\partial}^\nabla} \Omega^{0,1}(\text{End}(\mathbb{C}^r)) \xrightarrow{\bar{\partial}^\nabla} \dots \xrightarrow{\bar{\partial}^\nabla} \Omega^{0,n}(\text{End}(\mathbb{C}^r)) \rightarrow 0.$$

3.2. H-case. Let $\mathbb{R}^{4n} \cong (\mathbb{H} \otimes \mathbb{E})^{\mathbb{R}}$ be the vector space acted by $\text{Sp}(1) \cdot \text{Sp}(n) = \text{Sp}(1) \times \text{Sp}(n)/\mathbb{Z}^2$. Although $\Omega^1 = \Gamma(\mathbb{H} \otimes \mathbb{E})$ is irreducible, $\wedge^2 = \wedge^2(\mathbb{H} \otimes \mathbb{E})$ can be decomposed into irreducible components:

$$(3.2.1) \quad \wedge^2 = (S^2\mathbb{H} \otimes \wedge^2\mathbb{E}) \oplus (\wedge^2\mathbb{H} \otimes S^2\mathbb{E}).$$

We define the orthogonal projection π as:

$$\pi : \wedge^2 \rightarrow S^2\mathbb{H} \otimes \wedge^2\mathbb{E}.$$

We consider a connection form A and its curvature 2-form $F \in \Omega^2(\text{End}\mathbb{C}^r)$. Then the equation

$$(**) \quad \pi \circ F = 0$$

is a non-linear differential equation of the first order for A . By analogy with the $\mathbb{C}$ -case, we call a vector bundle with such a connection form ($\pi \circ F = 0$) *quaternion-holomorphic* vector bundle (for example, see [5]).

For brevity, we focus our attention on the case $n = 1$. Then the decomposition (3.2.1) reduces to:

$$\wedge^2 = \wedge_+ \oplus \wedge_-,$$

where a basis of each space is

$$\wedge_+ : \begin{cases} dx_0 \wedge dx_1 + dx_2 \wedge dx_3 \\ dx_0 \wedge dx_2 - dx_1 \wedge dx_3 \\ dx_0 \wedge dx_3 + dx_1 \wedge dx_2, \end{cases} \quad \wedge_- : \begin{cases} dx_0 \wedge dx_1 - dx_2 \wedge dx_3 \\ dx_0 \wedge dx_2 + dx_1 \wedge dx_3 \\ dx_0 \wedge dx_3 - dx_1 \wedge dx_2. \end{cases}$$

Then the equation (**) is written down as:

$$F^+ = 0$$

which is called the *anti-self-dual* equation [1].

A linearisation of the equation (**) is the composition:

$$\begin{aligned} \Omega^1(\text{End}(\mathbb{C}^r)) &= \Gamma(\text{End}(\mathbb{C}^r) \otimes \mathbb{H} \otimes \mathbb{E}) \xrightarrow{d^\nabla} \Omega^2(\text{End}(\mathbb{C}^r)) \\ &\xrightarrow{\pi} \Gamma(\text{End}(\mathbb{C}^r) \otimes S^2\mathbb{H} \otimes \wedge^2\mathbb{E}), \end{aligned}$$

and so, we obtain the twistor operator $\mathcal{D}^\nabla = \pi \circ d^\nabla$ coupled to the connection A . We also have

$$\mathcal{D}^\nabla \mathcal{D}^\nabla = \mathcal{D}^\nabla \nabla = \pi \circ d^\nabla \nabla = \pi \circ F = 0$$

OPERATORS OF DIRAC TYPES

for a quaternionic holomorphic vector bundle. Consequently, there exists an elliptic complex:

$$0 \rightarrow \Gamma(\text{End}(\mathbb{C}^r)) \xrightarrow{d^\nabla} \Gamma(\mathbb{H} \otimes \mathbb{E} \otimes \text{End}(\mathbb{C}^r)) \xrightarrow{\mathcal{D}^\nabla} \Gamma(S^2\mathbb{H} \otimes \wedge^2\mathbb{E} \otimes \text{End}(\mathbb{C}^r)) \\ \xrightarrow{\mathcal{D}^\nabla} \dots \xrightarrow{\mathcal{D}^\nabla} \Gamma(S^{2n}\mathbb{H} \otimes \wedge^{2n}\mathbb{E} \otimes \text{End}(\mathbb{C}^r)) \rightarrow 0.$$

In particular, in the 4-dimensional case, our elliptic complex is reduced to:

$$0 \rightarrow \Gamma(\text{End}(\mathbb{C}^r)) \xrightarrow{d^\nabla} \Omega^1(\text{End}(\mathbb{C}^r)) \xrightarrow{\mathcal{D}^\nabla} \Omega^+(\text{End}(\mathbb{C}^r)) \rightarrow 0,$$

which is called the Atiyah-Hitchin-Singer complex [1].

4. GENERALISATION III

For the first generalisation, after the identification $\text{Spin}(2) \cong \text{U}(1)$ and $\text{Spin}(4) \cong \text{Sp}(1) \times \text{Sp}(1)$, the groups $\text{U}(1)$ and $\text{Sp}(1) \times \text{Sp}(1)$ are replaced by $\text{U}(n)$ and $\text{Sp}(1) \times \text{Sp}(n)$, respectively. We obtained differential operators of Dirac types according to the structure groups.

In the second process of a generalisation, we replace a function by a $\mathbb{C}^r$ -valued function or a matrix valued function. As a consequence, we had a vector bundle, a connection form and a curvature form. Then we found non-linear differential equations which relate to the "Dirac equations" via linearisation.

Here we concern a manifold with a structure group $\text{U}(n)$ or $\text{Sp}(1) \times \text{Sp}(n)$ and a vector bundle with a connection.

The manifold with a structure group $\text{U}(n)$ is called a **Kähler manifold**. The typical example is the complex projective line $\mathbb{C}P^1 \cong S^2$.

The tangent space of a Kähler manifold has a complex structure and can be regarded as a complex vector space with a Hermitian inner product. The parallel transport makes sense, because a Kähler manifold is also a Riemann manifold. Then, the complex structure and the Hermitian metric are preserved by the parallel transport.

Since our construction in the previous sections is purely local in nature, the Cauchy-Riemann operator $\bar{\partial}$ and a holomorphic vector bundle can be defined on a Kähler manifold.

We pay an attention on a complex line bundle L over a compact Kähler manifold M and the resulting elliptic complex:

$$0 \rightarrow \Omega^{0,0}(M; L) \xrightarrow{\bar{\partial}_0^\nabla} \Omega^{0,1}(M; L) \xrightarrow{\bar{\partial}_1^\nabla} \dots \xrightarrow{\bar{\partial}_{n-1}^\nabla} \Omega^{0,n}(M; L) \rightarrow 0.$$

Then we can consider the cohomology

$$H^q(M; L) := \text{Ker } \bar{\partial}_q^\nabla / \text{Im } \bar{\partial}_{q-1}^\nabla$$

of the elliptic complex.

Theorem 4.1. (Kodaira vanishing theorem) (for example, see [2]) *If the holomorphic line bundle L is negative in some sense, then we have*

$$H^q(M; L) = 0 \quad \text{for } q \leq n-1$$

OPERATORS OF DIRAC TYPES

Finally, a manifold with a structure group $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$ concerns us. (As seen previously, the group $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ does not act on $\mathbb{R}^4$ effectively, but $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$ really acts effectively.) Such a manifold is called a **quaternion-Kähler manifold** [8]. The typical example of a quaternion-Kähler manifold is the quaternion projective line $\mathbb{H}P^1 \cong S^4$.

Let V be a quaternion-holomorphic vector bundle over a compact quaternion Kähler manifold M . (Note that we do not suppose that V is a line bundle). A related elliptic complex is

$$\begin{aligned} 0 \rightarrow \Gamma(M; V) &\xrightarrow{d^\nabla} \Gamma(M; \mathbb{H} \otimes \mathbb{E} \otimes V) \xrightarrow{\mathcal{D}^\nabla} \Gamma(M; S^2\mathbb{H} \otimes \wedge^2\mathbb{E} \otimes V) \\ &\xrightarrow{\mathcal{D}^\nabla} \dots \xrightarrow{\mathcal{D}^\nabla} \Gamma(M; S^p\mathbb{H} \otimes \wedge^p\mathbb{E} \otimes V) \xrightarrow{\mathcal{D}^\nabla} \dots \xrightarrow{\mathcal{D}^\nabla} \Gamma(M; S^{2n}\mathbb{H} \otimes \wedge^{2n}\mathbb{E} \otimes V) \\ &\rightarrow 0. \end{aligned}$$

We again consider the cohomology of the elliptic complex:

$$H^q(M; V) := \mathrm{Ker} \mathcal{D}^\nabla / \mathrm{Im} \mathcal{D}^\nabla.$$

Theorem 4.2. ([3](4-dimensional case), [6] and [7]) *If a quaternion-Kähler manifold has a positive scalar curvature, then we have*

$$H^q(M; V) = 0 \quad \text{for } q \geq n + 1.$$

If a quaternion-Kähler manifold has a negative scalar curvature, then we have

$$H^q(M; V) = 0 \quad \text{for } 1 \leq q \leq n + 1.$$

REFERENCES

- [1] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Self-duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A* **362** (1978), 425–461
- [2] P. Griffiths and J. Harris, “Principles of algebraic geometry” Wiley & Sons, New York (1978)
- [3] N.J. Hitchin, Linear field equations on self-dual spaces, *Proc. R. Soc. A.* **370** (1980), 173–191
- [4] B. Lawson & L. Michelsohn, “Spin geometry”, Princeton University Press, Princeton (1989)
- [5] M. Mamone Capria and S.M. Salamon, Yang-Mills fields on quaternionic spaces, *Non-linearity* **1** (1988), 517–530
- [6] Y. Nagatomo, Vanishing theorem for cohomology groups of c_2 -self-dual bundles on quaternionic Kähler Manifolds, *Differential Geom. Appl.* **5** (1995), 79–95
- [7] Y. Nagatomo and T. Nitta, Vanishing theorem for quaternionic complexes, *Bull. London Math. Soc.* **29** (1997), 359–366
- [8] S.M. Salamon, Quaternionic Kähler Manifolds, *Invent. Math.* **67** (1982), 143–171
- [9] S.M. Salamon, Differential geometry of quaternionic manifolds, *Ann. SC. Ec. Norm. Sup.* **19** (1986), 31–55

GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, ROPPONMATSU, FUKUOKA 810-8560, JAPAN

E-mail address: nagatomo@math.kyushu-u.ac.jp