

Stratification of proper real analytic mappings
and
structural stability of a family of real analytic sets.

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0. Introduction.

Abstractly what we call the problem of stability can be expressed as follows: We consider an equivalence relation $\sim$ between objects (or morphisms) of some category. Let $\{E_a\}$, $a \in A$, be a family of objects (or morphisms) of this category, the parameter space A being a topological space. We say that E_a is stable in the family $\{E_a\}$ with respect to the relation $\sim$ if there exists a neighborhood U of a in A such that for any point $c \in U$, we have $E_c \sim E_a$. The set K of points $b \in A$ such that E_b is not stable is called the bifurcation set of the family $\{E_a\}$. Then the problem of structural stability is : Is the bifurcation set K nowhere dense in A ?

Now, we consider the following situation: Let A and B be complex or real analytic spaces or analytic sets. Consider a system of analytic equations

$$F_j(x,y) = 0 \quad , \quad x \in A, \quad y \in B$$

Then we have a family $\{E_a\}$, $a \in A$, of real analytic sets defined by

$$E_a = \{y \in B \mid F_j(a,y) = 0\} \quad .$$

We discuss in the present paper the structural stability of this family in the topological sense : E_a is topologically stable if there is a neighborhood U of a in A such that for any point $c \in U$, E_c is homeomorphic to E_a . Furthermore we are interested in the topological structure of the bifurcation set K of this family.

If we replace the term "analytic" by "algebraic" in the above situation, it is known that the bifurcation set K is constructible (or semi-algebraic) subset of A and nowhere dense in A . So , in this case, we obtain a positive response for the topological stability problem. The main technique is to stratify the projection map $p:G \rightarrow A$, where $G = \{(x,y) \in A \times B \mid F_j(x,y) = 0\}$ and P is defined by

$$p(x,y) = x.$$

For a method of stratification of $p:G \rightarrow A$, see [1].

If A, B and F_j are complex analytic, it is also known that if $p:G \rightarrow A$ is proper, that is, if the counter image of a compact set is compact, the fact mentioned above holds exact: K is an analytic subset of A and nowhere dense in A . If $p:G \rightarrow A$ is not proper, we can not say any thing about the bifurcation sets K .

In real analytic case, Thom says [6], "une caractérisation intrinsèque de ces ensembles très vraisemblablement stratifiés, n'a pas encore été explicitée dans la littérature." The purpose of the present paper is to give a similar response:

THEOREM 1. In real analytic case, if $p:G \rightarrow A$ is proper, then the bifurcation set K is a subanalytic subset of A and nowhere dense in A .

The main tools of the proof are the notion of subanalytic sets obtained by H.Hironaka [9] and the stratification of proper real analytic morphisms:

THEOREM 2. A proper real analytic morphism is a stratified map.

Next, we consider the following situation: Let A and B be real analytic or suanalytic sets. and let $f: A \times B \rightarrow \mathbb{R}$ be a real analytic function. Suppose that B is compact. Then we have a family $\{f_a\}$, $a \in A$, of real analytic functions defined by

$$f_a: B \rightarrow \mathbb{R} \quad : \quad f_a(b) = f(a, b).$$

We consider the structural stability of this family in the following sense: f_a is topologically stable if there exists a neighborhood U of a in A such that for any point $c \in U$, f_c is topologically equivalent to f_a : i.e. there exist homeomorphisms $h_1: B \rightarrow B$ and $h_2: \mathbb{R} \rightarrow \mathbb{R}$ such that the following diagramm commutes

$$\begin{array}{ccc} B & \xrightarrow{h_1} & B \\ f_a \downarrow & & \downarrow f_c \\ \mathbb{R} & \xrightarrow{h_2} & \mathbb{R} \end{array}$$

Then we have

THEOREM 3. In the above situation, the bifurcation set K is suanalytic and nowhere dense in A .

This theorem follows from Thom's second isotopy lemma (proposition 2.5.) and the following theorem.

THEOREM 4. Let A, B and f be as above. Let $F: A \times B \rightarrow A \times R$
be the map defined by $F(a,b) = (a, f(a,b))$. Then A admits a Whitney
stratification $S(A)$ such that for any stratum X of $S(A)$, the map
 $F|_{X \times B}: X \times B \rightarrow X \times A$ is a Thom mapping over the projection map
 $p: X \times A \rightarrow X$.

In the present paper, we only give the proof of Theorem 1
 and theorem 2. For the proof of theorem 3 and 4 see [2].

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1. Whitney stratification.

In which we introduce the notion of stratifications which is due to H. Whitney [7], [8]. Here we recall only the definitions and some properties which we need. For the proof of these properties and more details, we are referred to R. Thom [5] and J. Mather [4].

Let X and Y be differentiable submanifolds of R^n . Let y be a point of Y and let $r = \dim X$. In what follows, $T_p(M)$ denotes the tangent space to a manifold M at a point p of M .

DEFINITION 1.1. We say that the pair (X, Y) satisfies condition (a) at $y \in Y$ if the following holds: Given any sequence x_i of points in X such that $x_i \rightarrow y$ and the tangent space $T_{x_i}(X)$ converges to some r -plane τ , we have $T_y(Y) \subset \tau$.

Here and in what follows "convergence" means convergence in the standard topology on the Grassmannian manifold of r -planes in R^n .

For any two distinct points $x, y \in R^n$, the secant $\widehat{xy}$ denotes the line in R^n which is parallel to the line joining x and y and passes through the origin.

Let X, Y be smooth submanifolds of R^n . Let $y \in Y$. Let $r = \dim X$.

DEFINITION 1.2. We say that the pair (X, Y) satisfies condition (b) at y if the following holds. Given any sequences x_i of points in X and y_i of points in Y such that $x_i \neq y_i$, $x_i \rightarrow y$ and $y_i \rightarrow y$ and such that $T_{x_i}(X)$ converges to some r -plane τ and the secants $\widehat{x_i y_i}$ converge to some line $\ell \subset \mathbb{R}^n$, we have $\ell \subset \tau$.

We say the pair (X, Y) satisfies condition (a) (resp. (b)) if it satisfies condition (a) (resp. (b)) at every point of Y .

REMARK. (Mather [4].) If (X, Y) satisfies condition (b) at y , then it satisfies condition (a) at y .

DEFINITION 1.3. A W-complex is a set $S = \{X_\alpha\}$ of connected smooth manifolds in $\mathbb{R}^n$, called strata of S , satisfying the following conditions:

- (i) The strata X_α are pair-wise disjoint.
- (ii) (X, Y) satisfies condition (b) for any pair (X, Y) of strata of $S = \{X_\alpha\}$.
- (iii) The family $S = \{X_\alpha\}$ is locally finite: each point of $\mathbb{R}^n$ has a neighborhood which meets at most finitely many strata.

DEFINITION 1.4. A stratified set is a subset E of $\mathbb{R}^n$ provided a W-complex $S(E) = \{X_\alpha\}$ with $E = \bigcup X_\alpha$. We call $S(E)$ a Whitney stratification of E .

REMARK. (Mather [4].). The local finiteness of strata and the condition (b) imply the condition of frontier: For each stratum X of $S(E)$, its frontier $(\bar{X} - X) \cap E$ is a union of strata.

NOTATION. Let X, Y be two strata of $S(E)$ with $Y \cap \bar{X} \neq \emptyset$. Then by the above remark, we have $Y \subset \bar{X} - X$. We represent this situation by the symbol $Y < X$ and we say Y is incident to X .

2. Stratified mappings and Thom's isotopy lemmas.

Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$.

DEFINITION. We say that a continuous mapping $f: E \rightarrow F$ is a W-morphism or a stratified mapping if there exist stratifications $S(E)$ of E and $S(F)$ of F and the following conditions hold:

(i) f is extendable to a differentiable mapping of a neighborhood of E into $\mathbb{R}^m$.

(ii) For any stratum X of $S(E)$, the image $f(X)$ is contained in a stratum Y of $S(F)$ and the restricted mapping $f|_X: X \rightarrow Y$ is a submersion.

A W-morphism $f: E \rightarrow F$ will be said to be exact if for any stratum X of $S(E)$, $f(X)$ is a stratum of $S(F)$.

REMARK. A proper W-morphism is an exact W-morphism. (See Mather's existence theorem for tubular neighborhoods [4].).

PROPOSITION 2.2. (Thom's first isotopy lemma). If $f:E \rightarrow F$ is a proper stratified mapping, then for each stratum Y of $S(F)$, the restricted mapping $f|_{f^{-1}(Y)}:f^{-1}(Y) \rightarrow Y$ is a locally trivial fibre bundle.

For the proof, see Mather [4], Thom [5] or Fukuda [1].

DEFINITION 2.3. (Thom's condition a_f). Let X and Y be smooth submanifolds of R^n and let N be a smooth manifold. Let $f:U \rightarrow N$ be a differentiable mapping defined on a neighborhood U of $X \cup Y$ in R^n . Suppose that $f|_X$ and $f|_Y$ are of constant rank. Then we say the pair (X,Y) satisfies condition a_f at a point $y \in Y$ if the following holds: Given any sequence x_i of points in X converging to y such that the sequence of planes $\ker(f|_X)_{x_i}$ converges to a plane τ in the appropriate Grassmannian manifold, we have

$$\ker(f|_Y)_y \subset \tau,$$

where $\ker(f|_X)_x$ denotes the kernel of the differential

$$(df|_X)_x: T_x(X) \rightarrow T_{f(x)}(N)$$

of $f|_X: X \rightarrow N$.

We say that the pair (X,Y) satisfies condition a_f if it satisfies condition a_f at every point of Y .

DEFINITION 2.4. (Thom mapping). Let $f:E \rightarrow F$ and $g:F \rightarrow V$ be stratified mappings. Suppose that V is a connected smooth manifold and it is considered as a stratified set with its trivial stratification $S(V) = \{V\}$. Then we say that f is a Thom mapping over g if for each point p of V and ^{for} any pair (X,Y) of strata of $S(E)$, the pair $(X \cap (g f)^{-1}(p), Y \cap (g f)^{-1}(p))$ satisfies condition a_f .

Let $f:E \rightarrow F$ be a Thom mapping over $g:F \rightarrow V$. For a point p of V , set $E_p = (g f)^{-1}(p)$ and $F_p = g^{-1}(p)$.

PROPOSITION 2.5. (Thom's second isotopy lemma). Let $f:E \rightarrow F$ be a proper Thom mapping over a proper stratified mapping $g:F \rightarrow V$. Then for any two points p and q of V , the restricted mappings $f|_{E_p}:E_p \rightarrow F_p$ and $f|_{E_q}:E_q \rightarrow F_q$ are of same topological type: there exist homeomorphisms $h_1:E_p \rightarrow E_q$ and $h_2:F_p \rightarrow F_q$ such that the following diagram commutes:

$$\begin{array}{ccc}
 E_p & \xrightarrow{h_1} & E_q \\
 f \downarrow & & \downarrow f \\
 F_p & \xrightarrow{h_2} & F_q
 \end{array}$$

For the proof of this proposition, see Mather [4] or Fukuda [1].

3. Subanalytic subsets.

In which we introduce the notion of "subanalyticity" that is due to H. Hironaka [3]. All the properties are stated without proof. For the proof, more details or examples, see [3].

DEFINITION 3.1. Let Ω be an open set in $\mathbb{R}^n$. An analytic set $A \subset \Omega$ is a set such that for any point a of Ω , there is a neighborhood U of a in Ω and analytic functions $f_1, \dots, f_k$ in U such that

$$A \cap U = \{x \in U \mid f_1(x) = \dots = f_k(x) = 0\}.$$

DEFINITION 3.2. (Analytic mappings). Let $A_i, i=1,2$, be analytic sets in open sets $\Omega_i \subset \mathbb{R}^{n_i}$. A continuous mapping $f: A_1 \rightarrow A_2$ is said to be analytic at a point $a \in A_1$ if there exist a neighborhood U of a in Ω_1 and an analytic mapping $F: U \rightarrow \mathbb{R}^{n_2}$ with

$$F|_{A_1 \cap U} = f|_{A_1 \cap U}.$$

An analytic mapping is, at least in the present paper, a continuous mapping of an analytic set A_1 into another analytic set which is analytic at every point of A_1 .

DEFINITION 3.3. (Subanalytic subsets). Let $X \subset \Omega$ be an analytic subset of an open set Ω in $\mathbb{R}^n$. A subanalytic subset $A \subset X$ is a set such that for any point a of X there exist an open neighborhood U of a in X and a finite system of analytic sets Y_{ij} and proper real analytic mappings $f_{ij}: Y_{ij} \rightarrow X$, $1 \leq i \leq p$ and $j = 1, 2$, such that

$$A \cap U = \bigcup_{i=1}^p (f_{i1}(Y_{i1}) - f_{i2}(Y_{i2})).$$

PROPOSITION 3.4. Let A, B, C be subanalytic subsets of an analytic set X . Then so are $A \cup B$, $A \cap B$ and $A - B$.

PROPOSITION 3.5. Let $f: X \rightarrow Y$ be a proper real-analytic mapping.

- (i) If B is a subanalytic subset of Y , then so is $f^{-1}(B)$ in X .
- (ii) If A is a subanalytic subset of X , then so is $f(A)$ in Y .

DEFINITION 3.6. Let A be a subanalytic subset of $X \subset \mathbb{R}^n$. A point $a \in A$ is called a regular point of A of dimension k if there is a neighborhood U of a , $U \subset \Omega$, such that $A \cap U$ is an analytic submanifold of dimension k of U . A point $a \in A$ is called singular if it is not regular.

PROPOSITION 3.7. Let A be a subanalytic subset of an analytic set X . Then we have:

- (i) The closure $\bar{A}$ of A in X is subanalytic in X .
- (ii) Every connected component of A is subanalytic in X and A has locally finite connectedness in X , ie., every point of X has a neighborhood which meets only a finite number of connected components of A .
- (iii) The set of singular points of A is subanalytic in X . The set of regular points of A of dimension p is subanalytic in X .
- (iv) ~~Regular~~ points are dense in A .

DEFINITION 3.8. Thanks to the proposition 3.7 (iv), we can define, as usually, the local dimension of a subanalytic set A at a point $a \in A$. And so we can define the dimension of A as the max. of the local dimensions of A .

NOTATION 3.9. Let X and Y be real analytic submanifolds of $\mathbb{R}^n$. $S_b(X, Y)$ will denote the set of points $y \in Y$ such that the pair (X, Y) does not satisfy condition (b) at y .

PROPOSITION 3.10. Let X and Y be real analytic submanifolds of R^n . Assume that $X \cap Y = \emptyset$ and $\bar{X} \supset Y$ and that X and Y are both subanalytic in an openset of R^n . Then there exists a subanalytic subset B of Y such that

(i) B is closed in Y and $\dim B < \dim Y$.

(ii) $B \supset S_b(X, Y)$.

4. Stratification of a subanalytic subset.

In which we give a proof of Hironaka's following theorem:

PROPOSITION 4.1. (Hironaka [3]). Let A be a subanalytic subset of an analytic set $X \subset \Omega \subset R^n$. Then A admits a Whitney stratification whose strata are subanalytic in X .

DEFINITION 4.2. We say that a W -complex $S = \{Y_\alpha\}$ in R^n is compatible with a submanifold X of R^n if for any stratum Y of S we have $S_b(X, Y) = \emptyset$.

It is clear that in order to prove the proposition 4.1, it is sufficient to prove the following:

PROPOSITION 4.3. Let A be a subanalytic subset of an analytic set $X \subset \Omega \subset \mathbb{R}^n$. Let $X_1, \dots, X_k$ be submanifolds of $\mathbb{R}^n$ which are subanalytic in X . Assume that $A \cap X_i = \emptyset$ for each i . Then A admits a Whitney stratification which is compatible with $X_1, \dots, X_k$ and such that each stratum is subanalytic in X .

Proof. We prove the proposition by induction on dimension of A . If $\dim A = 0$, then the proposition is evident. So we assume the proposition holds for every subanalytic set A with $\dim A < m$ and we shall prove it for a subanalytic set A with $\dim A = m$.

Let A_{sp} denote the set of the regular points of A of dimension m . Then by proposition 3.7, A_{sp} and $A - A_{sp}$ are both subanalytic in X and we have $\dim (A - A_{sp}) < \dim A = m$. Since A_{sp} is subanalytic in X and a submanifold of $\mathbb{R}^n$ and since $A_{sp} \cap X_i \subset A \cap X_i = \emptyset$, there exists, by proposition 3.10, a subanalytic subset B of A_{sp} such that

- (i) B is closed in A_{sp} and $\dim B < \dim A_{sp} = m$.
- (ii) $B \supset S_b(X_i, A_{sp})$ for each $i=1, \dots, k$.

Set $C = B \cup (A - A_{sp})$, $A^0 = A_{sp} - C$ and set $S(A^0) =$ the set of the connected components of A^0 . By proposition 3.7 (ii), $S(A^0)$ is locally finite in Ω , hence $S(A^0)$ is a W -complex which is compatible with $X_1, \dots, X_k$ and such that every stratum is subanalytic in X and disjoint with C .

Since $\dim C < m$, by the hypothesis of our induction, C admits a Whitney stratification $S(C)$ which is compatible with $X_1, \dots, X_k$ and with all of strata of $S(A^0)$.

Thus we have a Whitney stratification $S(A) = S(A^0) \cup S(C)$ that is wanted. Q.E.D.

5. Stratification of a proper real analytic mapping.

In which we prove the following

THEOREM 5.1. Let $f: X \rightarrow Y$ be a proper real analytic mapping of a real analytic set X into another one Y . Let $A \subset X$ and $B \subset Y$ be subanalytic subsets. Suppose that $f(A) \subset B$ and that $f|_A: A \rightarrow B$ is proper. Then $f|_A: A \rightarrow B$ is a stratified mapping with stratifications $S(A)$ of A and $S(B)$ of B such that any stratum of $S(A)$ (resp. of $S(B)$) is subanalytic in X (resp. in Y).

DEFINITION 5.2. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ and let X (resp. Y) be a submanifolds of $\mathbb{R}^n$ (resp. of $\mathbb{R}^m$). Let $f: A \rightarrow B$ be a stratified mapping with stratifications $S(A)$ of A and $S(B)$ of B . Then we say that the stratified mapping $f: A \rightarrow B$ is compatible with X (resp. with Y) if so is $S(A)$ (resp. $S(B)$).

The theorem 5.1. is a immediate consequence of the following proposition.

PROPOSITION 5.3. Let $f: X \rightarrow Y$ and $A \subset X \subset \mathbb{R}^n$, $B \subset Y \subset \mathbb{R}^m$
be as in theorem 5.1. Let $X_1, \dots, X_k$ (resp. $Y_1, \dots, Y_\ell$) be submani-
folds of $\mathbb{R}^n$ (resp. of $\mathbb{R}^m$) which are subanalytic in X (resp. in Y).
Assume that $A \cap X_i = B \cap Y_j = \emptyset$ for each i and j . Then there exist
Whitney stratifications $S(A)$ of A and $S(B)$ of B such that
 (i) $f|_{A: A \rightarrow B}$ is a stratified mapping provided $S(A)$
and $S(B)$ and it is compatible with $X_1, \dots, X_k$ and $Y_1, \dots, Y_\ell$.
 (ii) The strata of $S(A)$ (resp. of $S(B)$) are subanaly-
tic in X (resp. in Y).

To prove the proposition, we need

LEMMA 5.4. (Bertini-Sard). Let $f: X \rightarrow Y$ be a proper real
analytic map, where both X and Y are smooth. Then there exist a
subset S of Y such that

- (i) S is closed and subanalytic in Y and
 $\dim S < \dim Y$.
 (ii) for every connected component U of $Y-S$, either
 $f^{-1}(U) = \emptyset$ or f induces a submersion from $f^{-1}(U)$ to U .

For the proof see [3].

PROOF OF PROPOSITION 5.3.

We prove the proposition by induction on $\dim B$. The verification for the case $\dim B = 0$ is immediate from proposition 4.3. So we assume that the proposition holds for subanalytic set B with $\dim B < p$ and we shall prove it for a subanalytic set B of dimension p .

Let B_{sp} denote the set of the regular points of B of dimension p . By proposition 3.10, there is a closed subanalytic subset B_1 of B_{sp} such that $\dim B_1 < \dim B_{sp}$ and $B_1 \subset S_b(Y_j, B_{sp})$ for each $j=1, \dots, \ell$. Set $B_0 = B_{sp} - B_1$ and $A_0 = A \cap f^{-1}(B_{sp})$. Then A_0 is subanalytic in X and so is B_0 in Y . By proposition 4.3, A_0 admits a Whitney stratification $S(A_0)$ which is compatible with $X_1, \dots, X_k$ and such that each stratum is subanalytic in X .

Now for each stratum W of $S(A_0)$, consider the restricted map $f|_W: W \rightarrow B_0$ and set $\Sigma_W = \{x \in W \mid \text{the rank of } f|_W \text{ at } x < \dim B_{sp} = p\}$. Then Σ_W is subanalytic in X . Since $S(A_0)$ is locally finite, $\Sigma = \bigcup \Sigma_W$ is subanalytic in X and closed in A_0 .

Then $f(\Sigma)$ and its closure $\overline{f(\Sigma)}$ are subanalytic in Y and $\dim \overline{f(\Sigma)} < \dim B$. Set $B_{00} = B_0 - \overline{f(\Sigma)}$ and $A_{00} = A \cap f^{-1}(B_{00})$. Set

$$S(A_{00}) = \{W \cap A_{00} \mid W \in S(A_0)\}$$

$$S(B_{00}) = \text{the set of the connected components of } B_{00}.$$

With these stratifications $S(A_{00})$ and $S(B_{00})$, $f: A_{00} \rightarrow B_{00}$ is a stratified mapping which is compatible with $X_1, \dots, X_k$ and $Y_1, \dots, Y_\ell$.

$B - B_{00}$ is closed in B and $\dim(B - B_{00}) = \dim B - p$. So by the hypothesis of our induction, there exist stratifications $S(B - B_{00})$ and $S(A - A_{00})$ with which $f: A - A_{00} \rightarrow (B - B_{00})$ is a stratified mapping such that it is compatible with $X_1, \dots, X_k, Y_1, \dots, Y_\ell$ and with all strata of $S(A_{00})$ and $S(B_{00})$.

Thus we have stratifications $S(A) = S(A_{00}) \cup S(A - A_{00})$ and $S(B) = S(B_{00}) \cup S(B - B_{00})$ which satisfy the conditions in the proposition. Q.E.D.

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