

On Generalized Radix Representations

By S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner

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1. Radix representation and two generalizations

Positive base: Let $g \geq 2$ be an integer. Then every $n \in \mathbb{Z}$ can be represented in the form

$$n = \pm \sum_{j=0}^{\ell} n_j g^j, \quad 0 \leq n_j < g.$$

Negative base: V. Grünwald (1885): Let $g \leq -2$ be an integer. Then every $n \in \mathbb{Z}$ can be represented in the form

$$n = \sum_{j=0}^{\ell} n_j g^j, \quad 0 \leq n_j < g.$$

You can find details and more material about the here studied kind of questions in the following papers:

- [1] S. Akiyama and A. Pethő, *On canonical number systems*, Theor. Comp. Sci., 270 (2002), 921–933.
- [2] S. Akiyama, H. Brunotte and A. Pethő, *Cubic CNS polynomials, notes on a conjecture of W.J. Gilbert*, J. Math. Anal. and Appl., 281 (2003), 402–415.
- [3] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner, *On a generalization of the radix representation - a survey*, Fields Institute Communications, to appear.
- [4] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner, *Generalized radix representations and dynamical systems I*, Acta Math. Hungar., submitted

1.1. β representation

A. Rényi (1957): Let $\beta > 1$ be a real number and $\mathcal{A} = \{0, 1, \dots, \lfloor \beta \rfloor\}$ be the set of digits. Then each $\gamma \in [0, \infty)$ can be represented by

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} \dots \quad (1)$$

with $a_i \in \mathcal{A}$. This β -representation is usually not unique.

Assuming however that

$$0 \leq \gamma - \sum_{i=n}^m a_i \beta^i < \beta^n \quad (2)$$

hold for all $n \leq m$ the β -representation become unique. For $\gamma \in [0, 1)$ this greedy expansion can be given by the β -transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor$$

This concept is the topics of a lot of research:

- Description of the representations of 1, Erdős, Joó, Horváth.
- Characterization of univoque numbers, i.e. those β for which 1 has a unique representation, Daróczy, Kátai, Komornik, Loretti, Allouche, Cosnard.
- Connections with fractals: Rauzy, Thurston, Akiyama.
- Characterization of those β which leads to finite or eventually finite representations, Bertrand, Schmidt, Frougny, Solomyak, Hollander.
- Connection with radix representations based on linear recursive sequences, Zeckendorf, Fraenkel, Grabner, Tichy, Pethő.

1.3. CNS polynomials

Observation: If $\mathbb{Z}_K$ is monogenic then $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathbb{Z}_K$. This means $\mathbb{Z}_K \cong \mathbb{Z}[x]/P(x)\mathbb{Z}[x]$, where $P(x)$ is the minimal polynomial of α .

Moreover, $\{\alpha, \mathcal{N}\}$ is a CNS in $\mathbb{Z}_{Q(\alpha)}$ means nothing else that every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element (a representative) such that its coefficients is bounded by $|p_0| - 1$.

A monic polynomial $P(x) = x^d + p_{d-1}x^{d-1} + \dots + p_0$ is called CNS polynomial if every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element

$$a_0 + a_1x + \dots + a_kx^k \quad (3)$$

such that $0 \leq a_i < |p_0|$.

1.2. CNS representation

Number rings: Knuth, Kátai, J. Szabó, B. Kovács, Gilbert (1960-1981):

Let $\mathbb{Z}_K$ be the ring of integers of the algebraic number field K .

$\{\alpha, \mathcal{N}\}; \quad \alpha \in \mathbb{Z}_K, \quad \mathcal{N} = \{0, \dots, |\text{Norm}(\alpha)| - 1\}$

is called a *canonical number system* if every $\nu \in \mathbb{Z}_K$ can be represented in the form

$$\nu = \sum_{j=0}^l n_j \alpha^j, \quad n_j \in \mathcal{N}.$$

2. Comparison of the properties of greedy expansions and of CNS-polynomials

Let β be the root of $B(X) = X^d - b_1X^{d-1} - \dots - b_d \in \mathbb{Z}[X]$.

Let $\text{Fin}(\beta)$ be the set of positive real numbers having finite greedy expansion with respect to β . We say that $\beta > 1$ has property (F) if

$$\text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty).$$

Property (F)

β is a Pisot number:
 $\beta > 1$, but its
conjugates are < 1

If $b_1 \geq \dots \geq b_d \geq 1$,
Frougny and
Solomyak (1992)

CNS-polynomials

the absolute value of
all zeroes of $P(X)$ are
larger than 1.

If $p_{d-1} \leq \dots \leq p_0$,
 $p_0 \geq 2$, B. Kovács
(1981)

Characterization results if

$b_1 > |b_2| + \dots + |b_d|$,
 $b_d \neq 0$, Hollander
(1996)

$p_0 > |p_1| + \dots + |p_{d-1}|$,
Akiyama, Pethő, (2002),
Scheicher, Thuswaldner

3. Shift Radix Systems

Let $r = (r_1, \dots, r_d) \in \mathbb{R}^d$. To r we associate the mapping

$\tau_r : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$: if $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$ then let

$$\tau_r(a) = (a_2, \dots, a_d, -\lfloor ra \rfloor),$$

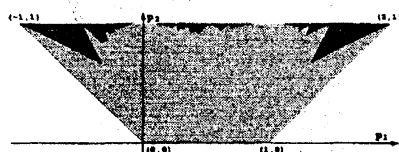
where $ra = r_1 a_1 + \dots + r_d a_d$, i.e. the inner product of r and a .

Let r be fixed. We will show: r gives rise to a Pisot number β with property (F) as well as to a CNS-polynomial P iff

$$\text{for all } a \in \mathbb{Z}^d \exists k > 0 \text{ with } \tau_r^k(a) = 0. \quad (4)$$

If (4) holds, we will call τ_r a *shift radix system* (SRS for short). Hence SRS is a common generalization of greedy expansions with property (F) and CNS-polynomials.

It is clear that $\mathcal{D}_1 = [-1, 1]$ and $\mathcal{D}_1^0 = [0, 1]$. To illustrate the difficulty of the characterization problem of $\mathcal{D}_d^0$ we show an approximation of $\mathcal{D}_2^0$.



An approximation of $\mathcal{D}_2^0$.

4. Relation between SRS and CNS-polynomials

This is a more delicate question.

Let $P(x) = p_d x^d + p_{d-1} x^{d-1} + \dots + p_0 \in \mathbb{Z}[x]$ with $p_d = 1$. Every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element of form

$$A_0 + A_1 x + \dots + A_{d-1} x^{d-1}, \quad A_i \in \mathbb{Z}. \quad (5)$$

3.1. Relation between SRS and β -expansions

Two basic definitions. Let

$$\mathcal{D}_d^0 := \{r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d \exists k > 0 : \tau_r^k(a) = 0\} \text{ and}$$

$$\mathcal{D}_d := \{r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d : \{\tau_r^k(a)\}_{k \geq 0} \text{ is ultimately periodic}\}.$$

Now we can formulate the connection between SRS and greedy expansions.

Theorem 1 (Hollander, 1996). Let $\beta > 1$ be a Pisot number with minimal polynomial $X^d - b_1 X^{d-1} - \dots - b_{d-1} X - b_d$. Set

$$r_1 := 1,$$

$$r_j := b_j \beta^{-1} + b_{j+1} \beta^{-2} + \dots + b_d \beta^{j-d-1},$$

($2 \leq j \leq d$). Then β has property (F) if and only if $(r_d, \dots, r_2) \in \mathcal{D}_{d-1}^0$.

Let $\mathbb{Z}'[x] = \{A(x) \in \mathbb{Z}[x] : \deg A < d\}$ and

$$T(A) = \sum_{i=0}^{d-1} (A_{i+1} - q p_{i+1}) x^i,$$

where $A_d = 0$ and $q = [A_0/p_0]$.

Then $T : \mathbb{Z}'[x] \rightarrow \mathbb{Z}'[x]$ and

$$A = a_0 + xT(A), \text{ with } a_0 = A_0 - qp_0.$$

This backward division process can become:

- divergent $A(X) = -1$ for $P(X) = X^2 + 4X + 2$

$$T_{X^2+4X+2}^k(-1) = -1, X + 4, -2X - 8, 4X + 16, \dots \text{ or}$$

- ultimately periodic $A(X) = -1$ for $P(X) = X^2 - 2X + 2$

$$T_{X^2-2X+2}^k(-1) = -1, X - 2, X - 1, X - 1, \dots \text{ or}$$

- can terminate after finitely many steps $A(X) = -1$ for $P(X) = X^2 + 2X + 2$
 $-1 = 1 + x^2 + x^3 + x^4.$

Let

$$\Pi(P) = \{A : T_P^\ell(A) = A \text{ for some } \ell > 0\}$$

denote the set of periodic points of the mapping T_P .

We always have $0 \in \Pi(P)$. With help of this set we define

$$\begin{aligned} C_d^0 &= \{(p_0, p_1, \dots, p_{d-1}) \in \mathbb{Z}^d : \\ &\quad \Pi(X^d + p_{d-1}X^{d-1} + \dots + p_0) = \{0\}\} \quad \text{and} \\ C_d &= \{(p_0, p_1, \dots, p_{d-1}) \in \mathbb{Z}^d : \\ &\quad T_{X^d + p_{d-1}X^{d-1} + \dots + p_0} \text{ has only finite orbits}\}. \end{aligned}$$

Clearly, we have $C_d^0 \subset C_d$.

The elements of C_d^0 will be called CNS polynomials.

It is convenient to replace T_P by the conjugate mapping

$\tilde{T}_P : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ defined as

$$\tilde{T}_P(A) = (A_1 - qp_1, \dots, A_{d-1} - qp_{d-1}, -qp_d),$$

where $A = (A_0, \dots, A_{d-1})$ and $q = \lfloor A_0/p_0 \rfloor$.

4.1. Affect of a new representation

H. Brunotte (2000) and K. Scheicher and J. Thuswaldner (2001) observed that the basis transformation

$$\begin{aligned} \{1, x, \dots, x^{d-1}\} &\rightarrow \{w_1, \dots, w_d\}, \\ w_j &= \sum_{i=d-j+1}^d p_i x^{i+j-d-1} \end{aligned}$$

of R implies a nicer and much better applicable transformation than $\tilde{T}_P$ is. Indeed, if

$$A(x) = \sum_{j=1}^d A_j w_j, \quad \text{then}$$

$$\tilde{T}_P(A) = -tw_d + \sum_{j=1}^{d-1} A_{j+1} w_j, \quad \text{where}$$

$$t = \left\lfloor \frac{p_1 A_d + \dots + p_d A_1}{p_0} \right\rfloor.$$

Hence, $\tilde{T}_P$ implies the mapping $\tau_P : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$

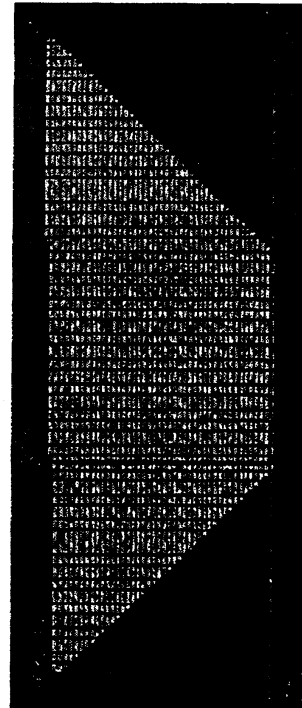
$$\tau_P(A) = \left(A_2, \dots, A_d, - \left\lfloor \frac{p_1 A_d + \dots + p_d A_1}{p_0} \right\rfloor \right)$$

where $A = (A_1, \dots, A_d)$. The mapping τ_P will be called *Brunotte's mapping*.

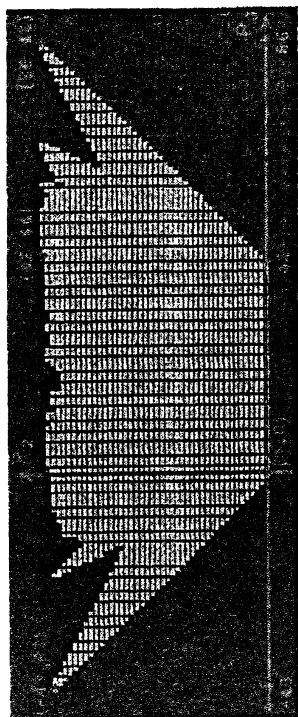
Theorem 2. Let $P(X) := X^d + p_{d-1}X^{d-1} + \dots + p_1X + p_0 \in \mathbb{Z}[X]$. Then $P(X)$ is a CNS polynomial (or belongs to C_d^0) if and only if $r = (\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}) \in \mathcal{D}_d^0$.

4.2. C_d^0 for small d 's

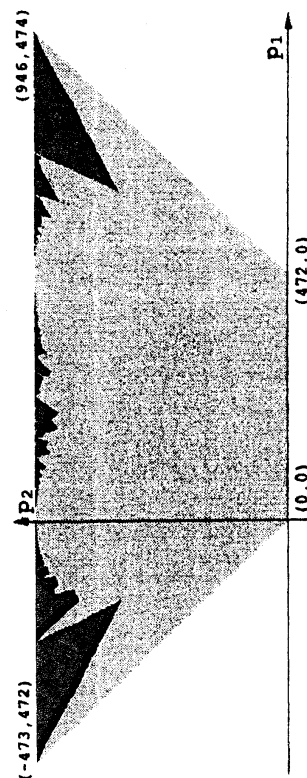
- $C_1^0 = \{p_0 : p_0 \geq 2\}$, V. Grünwald
- $C_2^0 = \{(p_0, p_1) : -1 \leq p_1 \leq p_0, p_0 \geq 2\}$, Kátai, Szabó, B. Kovács, Gilbert.
- Conjecture of Gilbert, 1981: $(p_0, p_1, p_2) \in C_3^0$ if and only if
 - (i) $p_0 \geq 2$,
 - (ii) $p_2 \geq 0$,
 - (iii) $p_1 + p_2 \geq -1$,
 - (iv) $p_1 - p_2 \leq p_0 - 2$,
 - (v) $p_2 \leq \begin{cases} p_0 - 2, & \text{if } p_1 \leq 0, \\ p_0 - 1, & \text{if } 1 \leq p_1 \leq p_0 - 1, \\ p_0, & \text{if } p_1 \geq p_0. \end{cases}$



Visualization of Gilbert's conjecture, $p_0 = 44$.



C_3^0 for $p_0 = 44$.



C_3^0 for $p_0 = 474$.

5. Basic properties of SRS

For a matrix M denote the spectral norm by $\|M\|$. For a vector v , $\|v\|$ shall denote the Euclidean norm.

For $r = (r_1, \dots, r_d) \in \mathcal{D}_d$ let

$$R := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -r_1 & -r_2 & \dots & \dots & -r_d \end{pmatrix}. \quad (6)$$

Lemma 1. Let $d \in \mathbb{N}$. If $r = (r_1, \dots, r_d) \in \mathcal{D}_d$ then the spectral radius of R is less than or equal to 1.

In the opposite direction we get the following result.

Lemma 2. Let $r \in \mathbb{R}^d$ such that the spectral radius ρ of the matrix R given above is less than 1. Then $r \in \mathcal{D}_d$.

It is not too hard to prove the following statement

Theorem 3. The sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$ are Lebesgue measurable.

5.1. Convexity property of τ_r

Theorem 4. Let $r_1, \dots, r_k \in \mathbb{R}^d$ and $a \in \mathbb{Z}^d$ be such that $\tau_{r_1}(a) = \dots = \tau_{r_k}(a)$. Let s be any convex linear combination of $r_1, \dots, r_k$. Then we have $\tau_s(a) = \tau_{r_1}(a) = \dots = \tau_{r_k}(a)$.

Corollary 1. Let $r_1, \dots, r_k \in \mathbb{R}$ have the same period, i.e. $\tau_{r_1}^\ell(a) = \dots = \tau_{r_k}^\ell(a)$, $\ell = 0, \dots, q$ and $a = \tau_{r_1}^q(a)$. Then if s is lying in the convex hull of $r_1, \dots, r_k$ the mapping τ_s is periodic and has the same period as τ_{r_1} .

For example, it is easy to check that for the plane vectors $r_1 = \begin{pmatrix} 381 & 253 \\ 254 & 254 \end{pmatrix}$, $r_2 = \begin{pmatrix} 421 & 253 \\ 254 & 254 \end{pmatrix}$ and $r_3 = \begin{pmatrix} 344 & 176 \\ 254 & 254 \end{pmatrix}$ the corresponding mappings have the same period $(-2, 1); 3, -2, 1, 1, -2$, hence the corresponding mapping for any point lying in the rectangle r_1, r_2, r_3 have this period.

5.2. Brunotte's algorithm

To decide $r \in \mathcal{C}_d^0$ Brunotte gave an algorithm, which was realized independently by Scheicher and Thuswaldner. We give here a generalization for $\mathcal{D}_d^0$.

Theorem 5. Suppose that there exists a set $E \subset \mathbb{Z}^d$ satisfying

- (i) E contains $2d$ elements of the form $(0, \dots, 0, \pm 1, 0, \dots, 0)$.
- (ii) $\tau_r(E) \cup \tau_r^*(E) \subset E$, where $\tau_r^*(x) = -\tau_r(-x)$.
- (iii) For each $a \in E$ there is some $k > 0$ such that $\tau_r^k(a) = 0$.

Then $r \in \mathcal{D}_d^0$.

An example: Let $r = \begin{pmatrix} 2 & -1 \\ 5 & -5 \end{pmatrix}$.

Starting from $E_0 = \{(\pm 1, 0), (0, \pm 1)\}$ and using that

$$\tau_r(1, 0) = (0, 0), \tau_r(-1, 0) = (0, 1), \tau_r(0, 1) = (1, 1),$$

$$\tau_r(0, -1) = (-1, 0) \text{ we get}$$

$$E_1 = \tau_r(E_0) \cup \tau_r^*(E_0) = E_0 \cup \{(0, 0), (1, 1), (-1, -1)\}.$$

Now

$$\tau_r(1, 1) = (1, 0), \tau_r(-1, -1) = (-1, 1), \text{ hence we may take}$$

$$E_2 = \tau_r(E_1) \cup \tau_r^*(E_1) = E_1 \cup \{(1, -1), (-1, 1)\}.$$

Finally because

$$\tau_r(-1, 1) = (1, 1), \tau_r(1, -1) = (-1, 0), \text{ we get that}$$

$$E = E_2 \text{ proves } r \in \mathcal{D}_2^0.$$

6. Lifting theorem

Let $d \in \mathbb{N}$ and

$$(a_{1+j}, \dots, a_{d+j}) \in \mathbb{Z}^d, \quad (0 \leq j \leq L-1), \quad (7)$$

with $a_{L+1} = a_1, \dots, a_{L+d} = a_d$.

For which $r = (r_1, \dots, r_d) \in \mathbb{R}^d$ these vectors form a period π of $\mathcal{D}_d$? By the definition of τ_r this is the case if and only if the inequalities

$$0 \leq r_1 a_{1+j} + \dots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (8)$$

hold simultaneously for all $0 \leq j \leq L-1$. They define a (possibly degenerated) polyhedron, which will be denoted by $\mathcal{P}(\pi)$.

Since $r \in \mathcal{D}_d^0$ if and only if τ_r has 0 as its only period we conclude that

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi)$$

where the union is extended over all families of vectors π of the shape (7). We call the family of (non-empty) polyhedra corresponding to this choice the *family of cutout polyhedra* of $\mathcal{D}_d^0$.

Let π be a period of $\mathcal{C}_d$ or $\mathcal{D}_d$ which corresponds to a non-degenerate cutout polyhedron. Then we call π a *non-degenerate period*. We will show that we can "lift" a non-degenerate period to higher dimensions.

Definition 1. Let

$$\pi : (a_1, \dots, a_d); a_{d+1}, \dots, a_L \quad (9)$$

be a non-degenerate period of length L of $\mathcal{C}_d$ or $\mathcal{D}_d$. Then we call

$$l(\pi) : (a_1, a_2, \dots, a_{d+1}); a_{d+2}, \dots, a_L \quad (10)$$

the lift of π to $d+1$.

Note that π and $l(\pi)$ have the same period length L .

Theorem 6 (Lifting Theorem). Let $d \geq 1$ be an integer.

(i) Let $p_0 \geq 2$ and let π be a non-degenerate period for $\mathcal{C}_d$. Then π is also a non-degenerate period for $\mathcal{D}_d$. More precisely, there exist $p_1, \dots, p_{d-1} \in \mathbb{Z}$ such that $(p_0, \dots, p_{d-1}) \in \text{int}(\mathcal{P}'(\pi))$ and

$$\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}\right) \in \text{int}(\mathcal{P}(\pi)).$$

(ii) Let π be a non-degenerate period of $\mathcal{D}_d$. Then its lift $\lambda := l(\pi)$ is a non-degenerate period of $\mathcal{C}_{d+1}$ for each sufficiently large p_0 . More precisely, for all $(r_1, \dots, r_d) \in \text{int}(\mathcal{P}(\pi))$ there exists $\varepsilon > 0$ such that for all $(p_0, \dots, p_d) \in \mathbb{Z}^{d+1}$ with

$$\max_{1 \leq k \leq d} \left| \frac{p_{d+1-k}}{p_0} - r_k \right| < \varepsilon$$

we have $(p_0, \dots, p_d) \in \text{int}(\mathcal{P}'(\lambda))$.

Theorem 7. Fix $n \in \mathbb{N}$, $n > 3$, and set $r = (x_n, y_n) \in \mathbb{R}^2$ with

$$x_n := 1 - \frac{1}{2n^2} + z_n \quad \text{and} \quad y_n := -\frac{2n+1}{2n(n+1)} + u_n,$$

where $|z_n|, |u_n| < 1/n^4$. Then ζ_n is a non-degenerate period of τ_r .

Since we can select n arbitrarily large and the length of the period ζ_n is $4n+1$ the previous theorem implies that there exist non-degenerate periods of arbitrarily large length for $\mathcal{D}_2$.

7. Long periods

Consider the following family of edges.

$$\begin{aligned} \alpha_k : (-k-1, -n+k) &\rightarrow (-n+k, k+1) \\ &\quad (0 \leq k \leq n-1), \\ \beta_k : (-n+k, k+1) &\rightarrow (k+1, n+1-k) \\ &\quad (0 \leq k \leq n-1), \\ \gamma_0 : (1, n+1) &\rightarrow (n+1, 1), \\ \gamma_k : (k+1, n+1-k) &\rightarrow (n+1-k, -k) \\ &\quad (1 \leq k \leq n-1), \\ \gamma_n : (n+1, 1) &\rightarrow (1, -n), \\ \delta_k : (n+1-k, -k) &\rightarrow (-k, -n-1+k) \\ &\quad (1 \leq k \leq n-1). \end{aligned}$$

With these edges we form the cycle

$$\zeta_n : \alpha_0 \beta_0 \gamma_0 \gamma_n \alpha_{n-1} \beta_{n-1} \gamma_{n-1} \delta_{n-1} \alpha_{n-2} \dots \alpha_1 \beta_1 \gamma_1 \delta_1.$$

Note that δ_1 ends up in $(-1, -n)$. In this node α_0 starts. Thus ζ_n is indeed a cycle. We wonder whether there exists $r := (x_n, y_n) \in \mathcal{D}_2$ such that τ_r has ζ_n as a non-degenerate period. This is done in the following result.

By a direct application of the Lifting Theorem we obtain.

Theorem 8. Let $d \geq 2$ be an integer, fix $n \in \mathbb{N}$, $n > 3$. Then there exist some $r \in \mathbb{R}^d$ such that $l^{d-2}(\zeta_n)$ is a non-degenerate period of τ_r . Since we can select n arbitrarily large and the length of the period $l^{d-2}(\zeta_n)$ is $4n+1$ this implies that there exist non-degenerate periods of arbitrarily large length of $\mathcal{D}_d$ and $\mathcal{C}_{d+1}$.

Corollary 2. Fix $n \in \mathbb{N}$, $n > 3$, and set $d \geq 2$ and

$r = (0, \dots, 0, x_n, y_n) \in \mathbb{R}^d$ with x_n, y_n as in Theorem 7. Then $l^{d-2}(\zeta_n)$ is a period of τ_r .

8. Critical points

Definition 2. Let $x \in \mathcal{D}_d$.

- If there exists an open neighborhood of x which contains only finitely many cutout polyhedra then we call x a **regular point**.
- If each open neighborhood of x has nonempty intersection with infinitely many cutout polyhedra then we call x a **weak critical point** for $\mathcal{D}_d$.
- If for each open neighborhood U of x the set $U \setminus \mathcal{D}_d^0$ can not be covered by finitely many cutout polyhedra then x is called a **critical point**.

We will show the existence of critical points for each $d \geq 2$. This shows that there is no way to characterize either of the sets $\mathcal{D}_d^0$ by finitely many cutouts if $d \geq 2$.

Lemma 3. Let x be a weak critical point for $\mathcal{D}_d$. Then $x \in \partial \mathcal{D}_d$.

Lemma 4. Let $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ be sequences with $x_n < 1$, $y_n < 0$, $\lim x_n = 1$, $\lim y_n = 0$ and $1 - x_n = o(y_n)$. Let $\{a_m\}_{m \geq 1}$ be a sequence of integers such that $|a_i| < K$ for some constant K . Then there exists $N \in \mathbb{N}$ such that

$$0 \leq a_{i-1}x_n + a_i y_n + a_{i+1} < 1 \quad (11)$$

can not hold for all i if $n \geq N$ unless $a_i = 0$ for all i large enough. Thus nonzero periods whose elements are bounded by K can not occur in $\mathcal{D}_d$ for $\tau_{(0, \dots, 0, x_n, y_n)}$ if n is large enough.

Theorem 9. Let $d \geq 2$. Then $K_d := (0, \dots, 1, 0) \in \mathbb{R}^d$ is a critical point of $\mathcal{D}_d$.

Problem 1. Characterize the critical points of $\mathcal{D}_d$. Can one show that for a given d there exist only finitely many critical points?