

On the deformation of A-branes in String theory

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1 A brief sketch

In this paper, we discuss the deformation theory of A-branes in String theory, from the point of view of CR structures and give an outline of our approach. The full paper will appear in another paper. Let W be a Kaehler manifold and let ω_W be its Kaehler form. Let M be a real hypersurface in W . We assume that our M admits an A-brane structure. Namely, there is a real line bundle L on M , and a connection ∇ on L , satisfying;

- [1] The curvature of the connection, F , is an element of $\Gamma(M, \wedge^2 \mathcal{F}^*)$,
- [2] $J := \omega_W^{-1} F$ determines a complex structure on $\mathcal{F}$, where $\mathcal{F} := \frac{TM}{\mathcal{L}}$, and $\mathcal{L}$ is a characteristic foliation $\mathcal{L}$, defined by: for $p \in M$, $\mathcal{L}_p = \{Y_p, Y'_p \in T_p W, \omega_W(Y_p, Y'_p) = 0, Y'_p \in T_p M\}$.

In this paper, by using the notion of almost CR structures, we reformulate the notion of A-branes. Our J determines an almost CR structure (M, T''_J) on M . For this almost CR structure, we prove that $C \otimes \mathcal{L} + T''_J$ is integrable on M . And show the deformation complex of A-branes (the Kapustin-Orlov complex)(see (2.7)). This is a natural generalization of the case $M = W$ (Kapustin-Orlov consider the case; A-branes wrap the whole W , and obtain the standard $\bar{\partial}$ -complex as a deformation complex).

Here we treat A-branes of the type hypersurfaces. Now for a given A-brane, we introduce the notion of family of A-branes, $\{(M, L, \nabla_t)\}_{t \in T}$. In this paper, we introduce the deformation complex of A-branes, and construct the Kodaira-Spencer map for the given family of A-branes. On the parameter space, a complex structure is given. But, we are relying on the Hamilton deformation, so we can't discuss in the complex analytic category (so we have to use that $\{(M, L, \nabla_t)\}_{t \in T}$ depends on t, C^∞ -ly). And because of this fact, we have to discuss in the category, $\text{mod } (t^2, \bar{t})$.

The author would like to thank Prof.A.Kapustin for allowing me to use the name, the Kapustin-Orlov complex and valuable suggestions during the preparation of this paper (the author learned that Kapustin and his student Yi Li, independently, obtained the integrability of $C \otimes \mathcal{L} + T''_J$).

2 The Kapustin-Orlov complex

In [Kap-Or], Kapustin-Orlov formulate the D-branes of A-type (in their language, A-branes), mathematically. We consider the deformation theory of A-branes in the case real hypersurfaces. For this, we recall the notion of A-branes. Let W be a Kaehler manifold. Let ω_W be its Kaehler metric. Let M be a real submanifold of W . Then, for this M , we have a characteristic foliation $\mathcal{L}$. This is defined by: for $p \in M$,

$$\mathcal{L}_p = \{Y_p, Y_p \in T_p W, \omega_W(Y_p, Y'_p) = 0, Y'_p \in T_p M\}.$$

By this definition, $\mathcal{L}$ is a subbundle of $TW|_M$ and the rank of $\mathcal{L}$ is $2n - \dim_R M$, because of ω_W being non-degenerate (here n is the complex dimension of W).

Definition 2.1. *If for $p \in M$, $\mathcal{L}_p \subset T_p M$, then M is called coisotropic.*

Henceforth we assume that our real submanifold is coisotropic. So, on M , we have a quotient bundle

$$\mathcal{F} := \frac{TM}{\mathcal{L}}.$$

Definition 2.2. (*A-branes*). *Let M be a coisotropic submanifold. Then M admits the A-brane if and only if there is a real line bundle L and a connection ∇ of L , (L, ∇) which satisfies*

- [1] *The curvature of the connection, F , is an element of $\Gamma(M, \wedge^2 \mathcal{F}^*)$,*
- [2] *$J := \omega_W^{-1} F$ determines a "Tac" structure on M (this means that: $J^2 = -1$ and this J is integrable modulo characteristic foliation).*

Now for the submanifold M , a CR structure $(M, {}^0T'')$ is introduced by:

$${}^0T'' = C \otimes TM \cap T''W|_M,$$

where $C \otimes TM$ means the complexified tangent bundle of M . Let $D = \{Y : Y \in TM, Y = X + \bar{X}, X \in {}^0T''\}$. Then, naturally,

$$D \cong \mathcal{F}.$$

By this identification, J is defined on D , satisfying: $J^2 = -1$. Hence J determines an almost CR structure on M . We study this structure. J is defined on D . We extend this J on $C \otimes D$, naturally. Set

$$\begin{aligned} T'_J &= \{X : X \in C \otimes D, JX = \sqrt{-1}X\}, \\ T''_J &= \{X' : X' \in C \otimes D, JX' = -\sqrt{-1}X'\}. \end{aligned}$$

Then, as mentioned in [Kap-Or], we have

Proposition 2.1.

$$C \otimes D = T'_J + T''_J, T'_J \cap T''_J = 0, \quad (2.1)$$

$$[\Gamma(M, T'_J), \Gamma(M, T'_J)] \subset \Gamma(M, T'_J) \text{ mod } \mathcal{L}. \quad (2.2)$$

Proof. (0.1) is obvious. We see (0.2). By the definition, $dF = 0$, $d\omega_W = 0$, and

$$\omega_W(X, JX') = F(X, X'), X, X' \in C \otimes D.$$

With these, we compute : for $X_1, X_2 \in \Gamma(M, T'_J)$, $X \in \Gamma(M, C \otimes TM)$,

$$dF(X_1, X_2, X) = 0, \quad (2.3)$$

$$d\omega_W(X_1, X_2, X) = 0. \quad (2.4)$$

We compute (0.3). Then,

$$\begin{aligned} & X_1 F(X_2, X) - X_2 F(X_1, X) + X F(X_1, X_2) \\ & - F([X_1, X_2], X) + F([X_1, X], X_2) - F([X_2, X], X_1) = 0. \end{aligned}$$

We rewrite this by using : $\omega_W(X, JX') = F(X, X')$, $X, X' \in C \otimes D$.

$$\begin{aligned} & X_1 \omega_W(JX_2, X) - X_2 \omega_W(JX_1, X) + X \omega_W(JX_1, X_2) \\ & - \omega_W(J[X_1, X_2], X) + \omega_W([X_1, X], JX_2) - \omega_W([X_2, X], JX_1) = 0. \end{aligned}$$

By $JX_i = \sqrt{-1}X_i$, $i = 1, 2$, this becomes

$$\begin{aligned} & X_1 \omega_W(\sqrt{-1}X_2, X) - X_2 \omega_W(\sqrt{-1}X_1, X) + X \omega_W(\sqrt{-1}X_1, X_2) \\ & - \omega_W(J[X_1, X_2], X) + \omega_W([X_1, X], \sqrt{-1}X_2) - \omega_W([X_2, X], \sqrt{-1}X_1) = 0. \end{aligned}$$

While, by (0.4),

$$\begin{aligned} & X_1 \omega_W(X_2, X) - X_2 \omega_W(X_1, X) + X \omega_W(X_1, X_2) \\ & - \omega_W([X_1, X_2], X) + \omega_W([X_1, X], X_2) - \omega_W([X_2, X], X_1) = 0. \end{aligned}$$

Hence, we have

$$\omega_W(J[X_1, X_2], X) = \omega_W(\sqrt{-1}[X_1, X_2], X) \text{ for any } X \in C \otimes D.$$

This means that: $[X_1, X_2] \in T'_J$ modulo $\mathcal{L}$. □

The following proposition is also mentioned in [Kap-Or].

Proposition 2.2.

$$\omega_W(X_1, X_2) = 0 \text{ for } X_1 \in T'_J, X_2 \in T''_J.$$

So, J -structure is different from the CR structure, naturally, induced from W . Here for the convenience, we give a proof.

Proof. We use $\omega_W(X, JY) = F(X, Y)$, for any $X, Y \in C \otimes TM$. For $X_1 \in T'_J, X_2 \in T''_J$,

$$\omega_W(X_1, JX_2) = F(X_1, X_2).$$

By $JX_2 = -\sqrt{-1}X_2$,

$$\omega_W(X_1, -\sqrt{-1}X_2) = F(X_1, X_2),$$

so,

$$\omega_W(X_1, X_2) = \sqrt{-1}F(X_1, X_2).$$

On the other hand,

$$\omega_W(X_2, JX_1) = F(X_2, X_1).$$

So, by $JX_1 = \sqrt{-1}X_1$,

$$\omega(X_2, X_1) = -\sqrt{-1}F(X_2, X_1).$$

Hence

$$\omega(X_1, X_2) = -\sqrt{-1}F(X_1, X_2).$$

This means that $\omega_W(X_1, X_2) = 0$. $\square$

As is mentioned in [Kap-Or], the following corollary follows from this proposition.

Corollary 2.3.

$$\dim_C T'_J = \text{even}.$$

Now we set a C^∞ vector bundle decomposition

$$C \otimes TM = C \otimes \mathcal{L} + T''_J + T'_J.$$

Here $C \otimes \mathcal{L}$ means the complexified $\mathcal{L}$. While in our case, (M, T''_J) may not be a CR structure (only integrable modulo $\mathcal{L}$). But,

Proposition 2.4. $\mathcal{L}$ preserves J , namely,

$$[\Gamma(M, T'_J), \mathcal{L}] \subset \Gamma(M, T'_J) \text{ modulo } \mathcal{L}.$$

Proof. By the same ways as in Proposition 2, we see this proposition.

For $X \in T'_J, Y \in T''_J, \zeta \in \mathcal{L}$, as F, ω_W are closed,

$$\begin{aligned} dF(X, Y, \zeta) &= 0, \\ d\omega_W(X, Y, \zeta) &= 0. \end{aligned}$$

By the first equation,

$$\begin{aligned} XF(Y, \zeta) - YF(X, \zeta) + \zeta F(X, Y) \\ - F([X, Y], \zeta) + F([X, \zeta], Y) - F([Y, \zeta], X) = 0. \end{aligned}$$

As $\mathcal{L}$ is a characteristic foliation, this becomes

$$\zeta F(X, Y) + F([X, \zeta], Y) - F([Y, \zeta], JX) = 0.$$

With $\omega_W(X', JY') = F(X', Y')$ for $X', Y' \in C \otimes D$,

$$\zeta \omega_W(JX, Y) + \omega_W([X, \zeta], JY) - \omega_W([Y, \zeta], JY) = 0.$$

While, by Proposition 2,

$$\begin{aligned} \omega_W(JX, Y) &= \omega_W(\sqrt{-1}X, Y) \\ &= 0. \end{aligned}$$

Hence

$$\omega_W([X, \zeta], -\sqrt{-1}Y) - \omega([Y, \zeta], \sqrt{-1}X) = 0. \quad (2.5)$$

While by the second equation,

$$\begin{aligned} & X\omega_W(Y, \zeta) - Y\omega_W(X, \zeta) + \zeta\omega_W(X, Y) \\ & - \omega_W([X, Y], \zeta) + \omega_W([X, \zeta], Y) - \omega_W([Y, \zeta], X) = 0. \end{aligned}$$

So, by the same way, this becomes

$$\omega_W([X, \zeta], Y) - \omega([Y, \zeta], X) = 0. \quad (2.6)$$

With (0.5), (0.6), we have

$$\omega_W([X, \zeta], Y) = 0, \text{ for } X \in T'_J, Y \in T''_J$$

This means that: the T''_J part of $[X, \zeta]$ vanishes because of ω_W being nondegenerate with Proposition 2.2. Hence

$$[X, \zeta] \in \Gamma(M, T'_J) \text{ modulo } \mathcal{L}.$$

□

Now we can state our theorem.

Theorem 2.5. *We set $T'' := C \otimes \mathcal{L} + T''_J$. Then,*

$$[\Gamma(M, T''), \Gamma(M, T'')] \subset \Gamma(M, T'').$$

By this theorem, we have the deformation complex of A-branes (Kapustin-Orlov complex). Namely, for $u \in \Gamma(M, C)$, we set $\bar{\partial}u$ of $\Gamma(M, (T'')^*)$ by;

$$\bar{\partial}u(X) = Xu, \text{ for } X \in T''.$$

By the same way as for ordinary differential forms, we can introduce $\bar{\partial}^p$ from $\Gamma(M, \wedge^p(T'')^*)$ to $\Gamma(M, \wedge^{p+1}(T'')^*)$.

$$\bar{\partial}^p : \Gamma(M, \wedge^p(T'')^*) \rightarrow \Gamma(M, \wedge^{p+1}(T'')^*).$$

Then, by the integrability theorem(Theorem 2.5),

$$\bar{\partial}^{p+1}\bar{\partial}^p = 0.$$

So, we have a deformation complex of A-branes(Kapustin-Orlov complex).

$$0 \rightarrow \Gamma(M, C) \xrightarrow{\bar{\partial}} \Gamma(M, (T'')^*) \xrightarrow{\bar{\partial}^1} \Gamma(M, \wedge^2(T'')^*) \rightarrow \dots \quad (2.7)$$

Furthermore, by this theorem, we can introduce a sheaf, $\mathcal{O}_{T''}$, composed of $\bar{\partial}$ -closed elements, which are holomorphic in the direction T''_J , and constant in the direction $\mathcal{L}$.

3 A family of deformations of A-branes

We introduce the notion of a family of deformations of A-branes,

Definition 3.1. *A set of A-branes $\{(M, L, \nabla_t), i_t\}_{t \in T}$, where T is an analytic space with the origin o , is a family of deformations of A-branes if*

- (1) *connections ∇_t depends on t , C^∞ -ly, and $\nabla_o = \nabla$,*
- (2) *embeddings i_t depends on t , C^∞ -ly, and $i_o = i$.*

Unlike CR structures, we rely on C^∞ category. Because, in the case symplectic structures, the Hamiltonian deformations play an essential part. We study a family of deformations of A-branes in the case real hypersurfaces. For the embedding i_t , we have the characteristic vector field, ξ_t . By using this vector field, the condition of $\{(M, L, \nabla_t), i_t\}$ being the A-brane is rewritten as follows.

- [1]_t The curvature of the connection ∇_t , F_t , is an element of $\Gamma(M, \wedge^2 \mathcal{F}_t^*)$,
- [2]_t Let $J_t := (i_t^* \omega_W)^{-1} F_t$. Then, $J_t^2 = -1$ on $\mathcal{F}_t$, where

$$\mathcal{F}_t := \frac{TM}{\mathcal{L}_t},$$

and $\mathcal{L}_t$ is generated by ξ_t . While the inclusion map induces a bundle isomorphism map ρ_t ; from D to $\frac{TM}{\mathcal{L}_t}$, induced by the inclusion map ; D to TM . The structure defined by J_t induces an almost CR structure on D by;

$$J'_t := \rho_t^{-1} J_t \rho_t.$$

Henceforth, we use the same notation J_t for J'_t and we regard J_t as an almost CR structure on D . Therefore [1]_t, [2]_t are written as

- [1]_t' The curvature of the connection ∇_t , F_t , satisfies $F_t(\xi_t, Y) = 0$ for $Y \in D$,
- [2]_t' Let $J_t := (i_t^* \omega_W)^{-1} F_t$. Then, $J_t^2 = -1$ on D .

We see why we call this complex a deformation complex of A-branes.

Definition 3.2. *The quartets of A-branes, $\{(M, L, \nabla), i\}$, $\{(M, L', \nabla'), i'\}$ are equivalent if there is a gauge transform of the line bundle L (we write this bundle map by q), and there is a Hamiltonian diffeomorphism map of W , defined by a C^∞ function g (we write it by V_g), satisfying;*

- (1) *the composition of maps V_g and i , $V_g i = i'$,*
- (2) *$V_g^* q^* \nabla = \nabla'$*

Next we introduce an equivalence relation for a family of deformations of A-branes.

Definition 3.3. *The family of deformations of A-branes, $\{(M, L, \nabla_t), i_t\}_{t \in T}$, $\{(M, L', \nabla'_s), i'_s\}_{s \in S}$ are equivalent if there is a local biholomorphic map h from T to S satisfying : $h(o) = o$, there is a gauge transform of the line bundle L (we write this bundle map by q), and there is a Hamiltonian diffeomorphism map of W , defined by a C^∞ function g_t (we write it by V_{g_t}), satisfying;*

- (1) *the composition of maps V_{g_t} and i_t , $V_{g_t} i_t = i'_{h(t)}$,*
- (2) *$V_{g_t}^* q_t^* \nabla_t = \nabla'_{h(t)}$*

4 The infinitesimal case

For a family of deformations of A-branes, $\{(M, L, \nabla_t), i_t\}_{t \in T}$, we can introduce the Kodaira-Spencer map, like the case the deformation theory of complex structures.

Theorem 4.1.

$$\frac{\partial}{\partial t} \{(M, L, \nabla_t), i_t\}_{t \in T} |_{t=0}$$

determines an element of $\text{Ker } \bar{\partial}^{(1)} / \text{Im } \bar{\partial}$ (the first cohomology of the differential complex (2.7)).

Definition 4.1. Let W be a Kaehler manifold and $\{(M, L, \nabla)\}$ be an A-brane in W . Let $\{\nabla_t\}_{t \in T}$ be a family of connections of L , satisfying $L_0 = L$. Let ξ_t of a section of $\Gamma(M, TW|_M)$, satisfying that $\xi_0 = 0$ and ξ_t can be extended to a neighborhood of M , and let i_t be the embedding map, induced by ξ_t . If the following holds, then $\{(M, L_t, \nabla_t)\}_{t \in T}$ is called an infinitesimal deformation of A-branes.

[1]'_t The curvature of the connection ∇_t , F_t , satisfies $F_t(\xi_t, Y) \equiv 0$ for $Y \in D$, mod $(t^2, \bar{t})$

[2]'_t Let $J_t := (i_t^* \omega_W)^{-1} F_t$. Then, $J_t^2 \equiv -1$ mod $(t^2, \bar{t})$ on D .

With this correspondence, we have

Theorem 4.2. For $\phi \in \Gamma(M, (C \otimes \mathcal{L} + T_f^*)^*)$, satisfying $\bar{\partial}^{(1)} \phi = 0$, on M , we can set a family of deformations of A-branes, infinitesimally.

In a forthcoming paper, the proof is given.

References

- [AGL1] T. Akahori, P. M. Garfield, and J. M. Lee, *Deformation theory of five-dimensional CR structures and the Rumin complex*, **50** (2002), 517-549, Michigan Mathematical Journal.
- [AGL2] T. Akahori, P. M. Garfield, and J. M. Lee, *The bigraded Rumin complex on CR manifolds*, preprint.
- [AG] T. Akahori, P. M. Garfield, *On the ordinary double point from the point of view of CR structures*, to appear in Michigan Mathematical Journal.
- [Kap-Or] A. Kapustin, D. Orlov, *Remarks on A-branes, mirror symmetry, and the Fukaya category*, **48** (2003), 84-99, Journal of Geometry and Physics.
- [OP] Y.-G. Oh and J.-S. Park, *Deformations of coisotropic submanifolds and strongly homotopy Lie algebroid*, preprint, math.SG/0305292