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Semiclassical perturbations of single-degree-of-freedom Hamiltonian systems II: Nonintegrability

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ABSTRACT

Continuing from Paper I [Ohsawa and Yagasaki, J. Math. Phys. 65, 102706 (2024)], we study semiclassical perturbations of single-degree-of-freedom analytic Hamiltonian systems and provide a sufficient condition for its meromorphic nonintegrability such that the first integrals depend on the small parameter meromorphically. Our approach is based on a generalization due to Ayoul and Zung of the Morales-Ramis theory, which enables us to show the meromorphic nonintegrability of dynamical systems by using the differential Galois theory. We remark that standard systems of Hagedorn and Heller for the semiclassical Gaussian wave packet dynamics are analytically integrable as well as the corresponding classical systems. We illustrate our theory for a bounded potential.

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I. INTRODUCTION

Continuing from the companion paper,²³ which we refer to as Paper I²³ here, we study semiclassical perturbations of single-degree-of-freedom Hamiltonian systems of the form

$$\begin{aligned}\dot{q} &= p, & \dot{p} &= -V'(q) - \frac{1}{4}\varepsilon(Q_1^2 + Q_2^2)V'''(q), \\ \dot{Q}_j &= P_j, & \dot{P}_j &= -V''(q)Q_j, \quad j = 1, 2, \quad (q, p, Q_1, Q_2, P_1, P_2) \in \mathbb{R}^6,\end{aligned}\tag{1.1}$$

where the overdot and prime represent differentiation with respect to t and q , respectively, ε is the semiclassical parameter such that $0 < \varepsilon \ll 1$, and $V(q)$ is a scalar function. In addition, we assume the following on the potential $V(q)$:

(A1) $V(q)$ is bounded from below and its second derivative $V''(q)$ is bounded.

The corresponding classical Hamiltonian system is given by

$$\dot{q} = p, \quad \dot{p} = -V'(q), \quad (q, p) \in \mathbb{R} \times \mathbb{R},\tag{1.2}$$

with the Hamiltonian

$$H_0(q, p) = \frac{1}{2}p^2 + V(q).$$

The semiclassical system (1.1) is also Hamiltonian on $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ with the symplectic form

$$\Omega^\varepsilon := dq \wedge dp + \frac{1}{2}\varepsilon \sum_{j=1}^2 (dQ_j \wedge dP_j) \quad (1.3)$$

and the Hamiltonian

$$H^\varepsilon(q, p, Q, P) = H_0(q, p) + \varepsilon H_1(q, p, Q, P), \quad (1.4)$$

where $Q = (Q_1, Q_2)$ and $P = (P_1, P_2)$ and

$$H_1(q, p, Q, P) = \frac{1}{4}(P_1^2 + P_2^2 + (Q_1^2 + Q_2^2)V''(q)).$$

The semiclassical system (1.1) naturally appears as an approximation of the corresponding quantum system. Consider the initial value problem of the Schrödinger equation,

$$\begin{aligned} i\varepsilon \frac{\partial}{\partial t} \psi(t, x) &= \hat{H}\psi(t, x) \quad \text{with } \hat{H} := \frac{1}{2}\hat{p}^2 + V(x), \\ \psi(0, x) &= \phi_0(q(0), p(0), Q(0), P(0), S(0); x), \end{aligned} \quad (1.5)$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\hat{p} := -i\varepsilon \partial / \partial x$ is the momentum operator, and ϕ_0 is the Gaussian wave function

$$\phi_0(q, p, Q, P, S; x) := \frac{Q^{-1/2}}{(\pi\varepsilon)^{1/4}} \exp \left\{ \frac{i}{\varepsilon} \left(\frac{1}{2} PQ^{-1}(x - q)^2 + p(x - q) + S \right) \right\}$$

with $Q = Q_1 + iQ_2$ and $P = P_1 + iP_2$. Under a weak version of assumption (A1) in which $V(q)$ is only assumed to be C^4 , it was shown in Ref. 22 that the (q, p) -components of the solution to (1.1) with $(q(0), p(0)) = (\langle \hat{x} \rangle(0), \langle \hat{p} \rangle(0))$ approximate the expectation value $(\langle \hat{x} \rangle(t), \langle \hat{p} \rangle(t))$ for the initial value problem (1.5) in the sense that $q(t) - \langle \hat{x} \rangle(t)$ and $p(t) - \langle \hat{p} \rangle(t)$ are both $O(\varepsilon^{3/2})$. Here $\hat{x}$ represents the position operator and

$$\langle \hat{x} \rangle(t) := \langle \psi(t), \hat{x} \psi(t) \rangle, \quad \langle \hat{p} \rangle(t) := \langle \psi(t), \hat{p} \psi(t) \rangle,$$

where we have used the shorthand $\psi(t) := \psi(t, \cdot) \in L^2(\mathbb{R})$ for the exact solution to (1.5). The case of (1.1) with $\varepsilon = 0$,

$$\dot{q} = p, \quad \dot{p} = -V'(q), \quad \dot{Q}_j = P_j, \quad \dot{P}_j = -V''(q)Q_j, \quad j = 1, 2, \quad (1.6)$$

which is referred to as the unperturbed system below, is the same as proposed earlier by Hagedorn^{9,10} and Heller¹¹ for studying the expectation value dynamics of (1.5). Note that the system (1.6) is not Hamiltonian since the symplectic form (1.3) is degenerate at $\varepsilon = 0$.

Suppose that the classical system (1.2) possesses a hyperbolic saddle at $(q, p) = (q_0, 0) \in \mathbb{R} \times \mathbb{R}$ with a homoclinic orbit. Then the semiclassical system (1.1) also has a hyperbolic saddle at

$$(q, p, Q, P) = (q_0, 0, 0, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$$

which has three-dimensional stable and unstable manifolds, denoted by W_ε^s and W_ε^u , respectively. In Paper I,²³ we gave a sufficient condition for W_ε^s and W_ε^u to split, slightly extending a Melnikov-type approach developed in Ref. 27. The occurrence of separatrix splitting explains a mechanism for the existence of trajectories of (1.5) to cross the separatrix on the classical phase plane, the (q, p) -plane, in the expectation value dynamics. We remark that such separatrix splitting does not occur in the unperturbed system (1.6) as well as in the classical system (1.2). See Paper I²³ for more details. The separatrix splitting also suggests the nonintegrability of (1.1).

In this paper we provide a sufficient condition for the meromorphic nonintegrability of (1.1) such that the first integrals depend on the small parameter $\varepsilon > 0$ meromorphically, under the assumptions that the classical system (1.2) has a one-parameter analytic family of periodic orbits. Such nonintegrability does not immediately imply that the system (1.1) is meromorphic nonintegrable for specific values of $\varepsilon > 0$, but it is still interesting and was studied much earlier by Poincaré^{24,25} and Kozlov^{12,13} for Hamiltonian systems including the restricted three-body problem and recently by the author and his co-worker^{20,21,28–32} for more general systems including forced oscillators and dissipative planar systems. See also Ref. 16 for another related work. Our approach is based on a generalization due to Ayoul and Zung³ of the Morales-Ramis theory,^{15,18} which enables us to show the meromorphic nonintegrability of dynamical systems (see Definition 3.1 below for its precise meaning) by using the differential Galois theory,^{6,26} as in Refs. 21, and 28–32. Here we need the generalized Morales-Ramis theory because the semiclassical system (1.1) is not Hamiltonian for $\varepsilon = 0$ although it is for $\varepsilon > 0$.

On the other hand, the classical system (1.2) is analytically integrable since it is a single-degree-of-freedom analytic Hamiltonian system, and the (Q_j, P_j) -components of (1.6) are solved by quadrature for $j = 1, 2$, as seen in Sec. II. The latter is a consequence of the Morales-Ramis theory^{15,18} (see also Remark 2.2 and Theorem 3.3 below) since they have the same form as the variational equations of the analytically

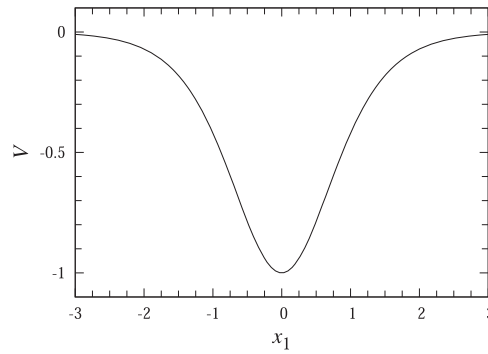


FIG. 1. Potential $V(x)$ given by (1.7).

integrable Hamiltonian system (1.2). Morales-Ruiz and his co-workers^{1,17} used this fact and analyzed semiclassical models derived by the WKB approximation^{5,7,19} based on the Feynman path integrals⁸ for Hamiltonian systems. We illustrate our result for the potential

$$V(q) = -\text{sech}^2 q, \quad (1.7)$$

for which the quantum dynamics of a particle in the field was treated in such a standard textbook on quantum mechanics as Ref. 14. Moreover, the nonintegrability of an autonomous perturbation of the corresponding classical system was discussed in Ref. 29. See Fig. 1 for the graph of the potential (1.7).

The outline of this paper is as follows. In Sec. II, we state our main theorem after making the additional assumptions on the classical system (1.2) precise. We briefly review necessary information on the differential Galois theory and generalized Morales-Ramis theory in Sec. III and give a proof of the main result in Sec. IV. Finally, we present an example for the potential (1.7).

II. MAIN RESULT

In this section we state our main result. Additionally, we make the following assumptions on (1.2):

- (A2) There exists a one-parameter family of periodic orbits $(q^\alpha(t), p^\alpha(t))$, $\alpha \in (\alpha_1, \alpha_2)$, with period $T^\alpha > 0$ for some $\alpha_1 < \alpha_2$ (see Fig. 2);
- (A3) $q^\alpha(t), p^\alpha(t)$ are analytic with respect to $\alpha \in (\alpha_1, \alpha_2)$.

Note that in assumption (A2) $x^\alpha(t)$ is automatically analytic with respect to t since the vector field of (1.2) is analytic. Henceforth we assume that $p^\alpha(0) \neq 0$ without loss of generality by shifting the time if not.

We next show that there exists a three-parameter family of periodic orbits in the unperturbed system (1.6), corresponding to the one-parameter family of periodic orbits $(q^\alpha(t), p^\alpha(t))$ in (1.2), as follows.

Lemma 2.1. *The system (1.6) has a three-parameter family of periodic orbits given by*

$$(q, p, Q_1, Q_2, P_1, P_2) = (q^\alpha(t), p^\alpha(t), \beta_1 p^\alpha(t), \beta_2 p^\alpha(t), \beta_1 \dot{p}^\alpha(t), \beta_2 \dot{p}^\alpha(t)), \quad (2.1)$$

where $\beta = (\beta_1, \beta_2) \in \mathbb{R}$.

Proof. We first notice that the (Q_j, P_j) -components of (1.6) represent a nonautonomous linear system for $j = 1, 2$ when $q = q^\alpha(t)$ is given. We compute

$$\ddot{p}^\alpha(t) = -\frac{d}{dt} V'(q^\alpha(t)) = -V''(q^\alpha(t)) p^\alpha(t).$$

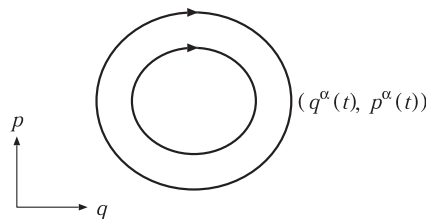
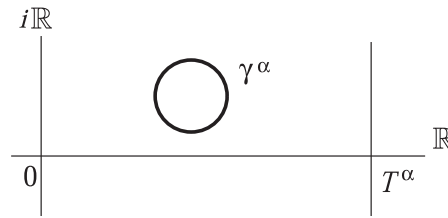


FIG. 2. Assumption (A2).

FIG. 3. Closed path γ_θ .

Hence, $(Q_j, P_j) = (p^\alpha(t), \dot{p}^\alpha(t))$ is a solution of the linear system and we obtain the desired result. $\square$

Remark 2.2. Using variation of constants, we obtain another linearly independent solution $(Q_j, P_j) = (\chi^\alpha(t), \dot{\chi}^\alpha(t))$ to the (Q_j, P_j) -components of (1.6) with $q = q^\alpha(t)$, where

$$\chi^\alpha(t) = p^\alpha(t) \int_0^t \frac{ds}{p^\alpha(s)^2}.$$

See the Proof of Proposition 4.2 below and also Lemma 2.1 of Paper I.²³ Thus, the unperturbed system (1.6) is solved by quadrature as stated in Sec. I.

We extend the domain of the independent variable t to a region including $\mathbb{R}$ in $\mathbb{C}$ and do so for the dependent variables $q, p, Q_j, P_j, j = 1, 2$. Let γ^α be a closed path in a domain containing $(0, T^\alpha)$ in $\mathbb{C}$ such that $\gamma^\alpha \cap (i\mathbb{R} \cup (T^\alpha + i\mathbb{R})) = \emptyset$. See Fig. 3. We define the complex integral

$$\mathcal{J}^\alpha = \int_{\gamma^\alpha} p^\alpha(t)^2 \dot{p}^\alpha(t) V'''(q^\alpha(t)) dt. \quad (2.2)$$

We now state our main theorem.

Theorem 2.3. Suppose that there exists a closed loop γ^α in a domain including $(0, T^\alpha)$ in $\mathbb{C}$ such that $\gamma^\alpha \cap (i\mathbb{R} \cup (T^\alpha + i\mathbb{R})) = \emptyset$ and the integral $\mathcal{J}^\alpha$ is not zero. Then the semiclassical system (1.1) is not meromorphically integrable near the periodic orbit (2.1) with $\beta \neq 0$ on any domain in $\mathbb{C}/T^\alpha\mathbb{Z}$ containing $\mathbb{R}/T^\alpha\mathbb{Z}$ and γ^α , such that the additional first integrals depend on ε meromorphically in a neighborhood of $\varepsilon = 0$.

We give a proof of Theorem 2.3 in Sec. IV.

Remark 2.4. If the integral $\mathcal{J}^\alpha$ is not zero but $\gamma^\alpha \cap i\mathbb{R}$ or $\gamma^\alpha \cap (T^\alpha + i\mathbb{R})$ is nonempty, then we can apply Theorem 2.3 by shifting the variable t . This approach is used in Sec. V.

III. PRELIMINARIES

In this section we explain preliminary results, the differential Galois theory^{6,26} and the generalized Morales-Ramis theory,^{3,15,18} which are necessary for the Proof of Theorem 2.3 in Sec. IV.

A. Differential Galois theory

We first state necessary information on differential Galois theory for linear differential equations, which is often referred to as the Picard–Vessiot theory. See the textbooks^{6,26} for more details on the theory.

Consider a linear system of differential equations

$$\dot{y} = A(t)y, \quad A(t) \in \text{gl}(n, \mathbb{K}), \quad (3.1)$$

where $\mathbb{K}$ is a differential field and $\text{gl}(n, \mathbb{K})$ denotes the ring of $n \times n$ matrices with entries in $\mathbb{K}$. Here a *differential field* is a field endowed with a derivation ∂ , which is an additive endomorphism satisfying the Leibniz rule and represented by the overdot in (3.1). The set $\mathbb{C}_\mathbb{K}$ of elements of $\mathbb{K}$ for which ∂ vanishes is a subfield of $\mathbb{K}$ and called the *field of constants* of $\mathbb{K}$. In our application of the theory in this paper, the field of constants is $\mathbb{C}$.

A *differential field extension* $\mathbb{L} \supset \mathbb{K}$ is a field extension such that $\mathbb{L}$ is also a differential field and the derivations on $\mathbb{L}$ and $\mathbb{K}$ coincide on $\mathbb{K}$. A differential field extension $\mathbb{L} \supset \mathbb{K}$ satisfying the following two conditions is called a *Picard–Vessiot extension* for (3.1):

(PV1) The field $\mathbb{L}$ is generated by $\mathbb{K}$ and elements of a fundamental matrix of (3.1);

(PV2) The fields of constants for $\mathbb{L}$ and $\mathbb{K}$ coincide.

The system (3.1) admits a Picard–Vessiot extension which is unique up to isomorphism.

We now fix a Picard–Vessiot extension $\mathbb{L} \supset \mathbb{K}$ and fundamental matrix Φ with entries in $\mathbb{L}$ for (3.1). Let σ be a $\mathbb{K}$ -automorphism of $\mathbb{L}$, which is a field automorphism of $\mathbb{L}$ that commutes with the derivation of $\mathbb{L}$ and leaves $\mathbb{K}$ pointwise fixed. Obviously, $\sigma(\Phi)$ is also a fundamental matrix of (3.1) and consequently there is a matrix M_σ with constant entries such that $\sigma(\Phi) = \Phi M_\sigma$. This relation gives a faithful representation of the group of $\mathbb{K}$ -automorphisms of $\mathbb{L}$ on the general linear group as

$$R : \text{Aut}_{\mathbb{K}}(\mathbb{L}) \rightarrow \text{GL}(n, \mathbb{C}_{\mathbb{L}}), \quad \sigma \mapsto M_\sigma,$$

where $\text{GL}(n, \mathbb{C}_{\mathbb{L}})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{C}_{\mathbb{L}}$. The image of R is a linear algebraic subgroup of $\text{GL}(n, \mathbb{C}_{\mathbb{L}})$, which is called the *differential Galois group* of (3.1) and often denoted by $\text{Gal}(\mathbb{L}/\mathbb{K})$. This representation is not unique and depends on the choice of the fundamental matrix Φ , but a different fundamental matrix only gives rise to a conjugated representation. Thus, the differential Galois group is unique up to conjugation as an algebraic subgroup of the general linear group.

Let $\mathcal{G} \subset \text{GL}(n, \mathbb{C}_{\mathbb{L}})$ be an algebraic group. Then it contains a unique maximal connected algebraic subgroup $\mathcal{G}^0$, which is called the *connected component of the identity or identity component*. The identity component $\mathcal{G}^0 \subset \mathcal{G}$ is the smallest subgroup of finite index, i.e., the quotient group $\mathcal{G}/\mathcal{G}^0$ is finite.

Let $\mathbb{K}$ be the field of meromorphic functions on a Riemann surface Γ , and consider the linear system (3.1). Let $t_0 \in \Gamma$ be a nonsingular point for (3.1). We prolong the fundamental matrix $\Phi(t)$ analytically along any loop γ based at t_0 and containing no singular point, and obtain another fundamental matrix $\gamma * \Phi(t)$. So there exists a constant nonsingular matrix $M_{[\gamma]}$ such that

$$\gamma * \Phi(t) = \Phi(t) M_{[\gamma]}.$$

The matrix $M_{[\gamma]}$ depends on the homotopy class $[\gamma]$ of the loop γ and is called the *monodromy matrix* of $[\gamma]$.

Let $\mathbb{L}$ be a Picard–Vessiot extension of (3.1) and let $\text{Gal}(\mathbb{L}/\mathbb{K})$ be the differential Galois group. Since analytic continuation commutes with differentiation, we have $M_{[\gamma]} \in \text{Gal}(\mathbb{L}/\mathbb{K})$.

B. Generalized Morales-Ramis theory

We next briefly review the version of Ayoul and Zung³ of the Morales-Ramis theory^{15,18} in our necessary setting. Consider n -dimensional systems of the form

$$\dot{x} = f(x), \quad x \in \mathbb{C}^n, \quad (3.2)$$

where $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is meromorphic. We adopt the following definition of integrability due to Bogoyavlenskij⁴ for (3.2).

Definition 3.1 (Bogoyavlenskij). For $n \in \mathbb{N}$ an n -dimensional dynamical system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \text{ or } \mathbb{C}^n,$$

is called $(\ell, n - \ell)$ -integrable or simply integrable if there exist ℓ vector fields $f_1(x) (= f(x))$, $f_2(x), \dots, f_\ell(x)$ and $n - \ell$ scalar-valued functions $F_1(x), \dots, F_{n-\ell}(x)$ such that the following two conditions hold:

- (i) $f_1(x), \dots, f_\ell(x)$ are linearly independent almost everywhere (a.e.) and commute with each other, i.e., $[f_j, f_k](x) := Df_k(x)f_j(x) - Df_j(x)f_k(x) \equiv 0$ for $j, k = 1, \dots, \ell$, where $[\cdot, \cdot]$ denotes the Lie bracket;
- (ii) The derivatives $DF_1(x), \dots, DF_{n-\ell}(x)$ are linearly independent a.e. and $F_1(x), \dots, F_{n-\ell}(x)$ are first integrals of $f_1, \dots, f_\ell$, i.e., $DF_k(x) \cdot f_j(x) \equiv 0$ for $j = 1, \dots, \ell$ and $k = 1, \dots, n - \ell$, where “ $\cdot$ ” represents the dot product.

We say that the system is meromorphically (resp. analytically) integrable if the first integrals and commutative vector fields are meromorphic (resp. analytic).

Definition 3.1 is considered as a generalization of Liouville-integrability for Hamiltonian systems.^{2,15} An n -degree-of-freedom Hamiltonian system with $n \geq 1$ is called *integrable* in the sense of Liouville if there exist n first integrals which are linearly independent a.e. and in involution (i.e., their Poisson brackets are zero). An n -degree-of-freedom Liouville-integrable Hamiltonian system has not only n functionally independent first integrals but also n linearly independent commutative (Hamiltonian) vector fields generated by the first integrals. We have the following proposition.

Proposition 3.2. Suppose that the system (3.2) is Hamiltonian and integrable in the Liouville sense. Then it is also integrable in the Bogoyavlenskij sense.

It immediately follows from Proposition 3.2 that if the system (3.2) is Hamiltonian and not Bogoyavlenskij-integrable, then it is not Liouville-integrable.

Let $x = \phi(t)$ be its nonstationary particular solution to (3.2). The variational equation (VE) of (3.2) along $x = \phi(t)$ is given by

$$\dot{\xi} = Df(\phi(t))\xi, \quad \xi \in \mathbb{C}^n. \quad (3.3)$$

Let $\mathcal{C}$ be a curve given by $x = \phi(t)$. We take the meromorphic function field on $\mathcal{C}$ as the coefficient field $\mathbb{K}$ of (3.3). We have the following result.^{3,15,18}

Theorem 3.3. *Let $\mathcal{G}$ be the differential Galois group of (3.3). If the system (3.2) is meromorphically integrable near $\mathcal{C}$ in the Bogoyavlenskij sense, then the identity component $\mathcal{G}^0$ of $\mathcal{G}$ is commutative.*

IV. PROOF OF THEOREM 2.3

We now prove Theorem 2.3 using an approach similar to one developed in Ref. 31, based on the version of Ayoul and Zung³ of the Morales-Ramis theory.^{15,18}

Let $p_0(t)$ and $q_0(t)$ denote solutions to the classical system (1.2), and let $q = q_0 + \varepsilon q_1$ and $p = p_0 + \varepsilon p_1$. We rewrite (1.1) as

$$\begin{aligned} \dot{q}_0 &= p_0, & \dot{p}_0 &= -V'(q_0), & \dot{q}_1 &= p_1, \\ \dot{p}_1 &= -V''(q_0)q_1 - \frac{1}{4}(Q_1^2 + Q_2^2)V'''(q_0) \\ &\quad - \varepsilon \left(\frac{1}{2}V'''(q_0)q_1^2 + \frac{1}{4}(Q_1^2 + Q_2^2)V''''(q_0)q_1 \right), \\ \dot{Q}_j &= P_j, & \dot{P}_j &= -V''(q_0)Q_j - \varepsilon V'''(q_0)Q_1Q_j, & j &= 1, 2, \end{aligned} \quad (4.1)$$

up to $O(\varepsilon)$. When $\varepsilon = 0$, the system (4.1) becomes

$$\begin{aligned} \dot{q}_0 &= p_0, & \dot{p}_0 &= -V'(q_0), & \dot{q}_1 &= p_1, \\ \dot{p}_1 &= -V''(q_0)q_1 - \frac{1}{4}(Q_1^2 + Q_2^2)V'''(q_0), \\ \dot{Q}_j &= P_j, & \dot{P}_j &= -V''(q_0)Q_j, & j &= 1, 2. \end{aligned} \quad (4.2)$$

Lemma 4.1. *The system (4.2) has a three-parameter family of periodic solutions given by*

$$\begin{aligned} &(q_0, p_0, q_1, p_1, Q_1, Q_2, P_1, P_2) \\ &= \left(q^\alpha(t), p^\alpha(t), -\frac{1}{4}(\beta_1^2 + \beta_2^2)\dot{p}^\alpha(t), \right. \\ &\quad \left. -\frac{1}{4}(\beta_1^2 + \beta_2^2)\ddot{p}^\alpha(t), \beta_1 p^\alpha(t), \beta_2 p^\alpha(t), \beta_1 \dot{p}^\alpha(t), \beta_2 \dot{p}^\alpha(t) \right), \end{aligned} \quad (4.3)$$

where $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$.

Proof. It follows from Lemma 2.1 that the $(q_0, p_0, Q_1, Q_2, P_1, P_2)$ -components of (4.3) satisfy the corresponding components of (4.2). We also compute

$$\begin{aligned} \ddot{p}^\alpha(t) &= -\frac{d^2}{dt^2} V'(q^\alpha(t)) = -\frac{d}{dt} V''(q^\alpha(t))p^\alpha(t) \\ &= -V'''(q^\alpha(t))\dot{p}^\alpha(t) - V''''(q_0(t))p^\alpha(t)^2. \end{aligned} \quad (4.4)$$

Substituting $q_0 = q^\alpha(t)$ and $Q_j = \beta_j p^\alpha(t)$, $j = 1, 2$, into the right hand side of the p_1 -component of (4.2), we have

$$\dot{p}_1 = -V''(q^\alpha(t))q_1 - \frac{1}{4}(\beta_1^2 + \beta_2^2)p^\alpha(t)^2 V'''(q^\alpha(t))$$

and compare it with (4.4) to obtain the desired result. $\square$

Taking ε as a state variable, we extend (4.1) as

$$\begin{aligned}\dot{q}_0 &= p_0, & \dot{p}_0 &= -V'(q_0), & \dot{q}_1 &= p_1, \\ \dot{p}_1 &= -V''(q_0)q_1 - \frac{1}{4}(Q_1^2 + Q_2^2)V'''(q_0) \\ &\quad - \varepsilon\left(\frac{1}{2}V'''(q_0)q_1^2 + \frac{1}{4}(Q_1^2 + Q_2^2)V''''(q_0)q_1\right), \\ \dot{Q}_j &= P_j, & \dot{P}_j &= -V''(q_0)Q_j - \varepsilon V'''(q_0)q_1Q_j, & j &= 1, 2, \\ \dot{\varepsilon} &= 0\end{aligned}\quad (4.5)$$

up to $O(\varepsilon)$. The VE of (4.5) along the periodic orbits given by (4.3) and $\varepsilon = 0$ is given by

$$\begin{aligned}\delta\dot{q}_0 &= \delta p_0, & \delta\dot{p}_0 &= -V''(q^\alpha(t))\delta q_1, & \delta\dot{q}_1 &= \delta p_1, \\ \delta\dot{p}_1 &= \frac{1}{4}(\beta_1^2 + \beta_2^2)(\dot{p}^\alpha(t)V'''(q^\alpha(t)) - p^\alpha(t)^2V''''(q^\alpha(t)))\delta q_0 \\ &\quad - V''(q^\alpha(t))\delta q_1 - \frac{1}{2}\beta_1 p^\alpha(t)V''''(q^\alpha(t))\delta Q_1 - \frac{1}{2}\beta_2 p^\alpha(t)V''''(q^\alpha(t))\delta Q_2 \\ &\quad - \frac{1}{32}(\beta_1^2 + \beta_2^2)^2(V'''(q^\alpha(t))\dot{p}^\alpha(t)^2 - 2p^\alpha(t)^2\dot{p}^\alpha(t)V''''(q^\alpha(t)))\delta\varepsilon, \\ \delta\dot{Q}_j &= \delta P_j, \\ \delta\dot{P}_j &= -\beta_j p^\alpha(t)V''''(q^\alpha(t))\delta q_0 - V''(q^\alpha(t))\delta Q_j \\ &\quad + \frac{1}{4}\beta_j(\beta_1^2 + \beta_2^2)p^\alpha(t)\dot{p}^\alpha(t)V''''(q^\alpha(t))\delta\varepsilon, & j &= 1, 2, \\ \delta\dot{\varepsilon} &= 0.\end{aligned}\quad (4.6)$$

Letting $\delta q_0, \delta p_0, \delta q_1, \delta p_1 = 0$ in (4.6), we have

$$\begin{aligned}\delta\dot{Q}_j &= \delta P_j, \\ \delta\dot{P}_j &= -V''(q^\alpha(t))\delta Q_j + \frac{1}{4}\beta_j(\beta_1^2 + \beta_2^2)p^\alpha(t)\dot{p}^\alpha(t)V''''(q^\alpha(t))\delta\varepsilon, & j &= 1, 2, \\ \delta\dot{\varepsilon} &= 0.\end{aligned}\quad (4.7)$$

Proposition 4.2. The linear system (4.7) has a fundamental matrix given by

$$\Phi(t) = \begin{pmatrix} p^\alpha(t) & \chi^\alpha(t) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1\eta^\alpha(t) \\ \dot{p}^\alpha(t) & \dot{\chi}^\alpha(t) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1\dot{\eta}^\alpha(t) \\ 0 & 0 & p^\alpha(t) & \chi^\alpha(t) & \frac{1}{4}\tilde{\beta}_2\eta^\alpha(t) \\ 0 & 0 & \dot{p}^\alpha(t) & \dot{\chi}^\alpha(t) & \frac{1}{4}\tilde{\beta}_2\dot{\eta}^\alpha(t) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\tilde{\beta}_j = \beta_j(\beta_1^2 + \beta_2^2)$ and

$$\begin{aligned}\eta^\alpha(t) &= \chi^\alpha(t) \int_0^t p^\alpha(s)^2 \dot{p}^\alpha(s) V''''(q^\alpha(s)) ds \\ &\quad - p^\alpha(t) \int_0^t \chi^\alpha(s) p^\alpha(s) \dot{p}^\alpha(s) V''''(q^\alpha(s)) ds.\end{aligned}$$

Proof. From Lemma 2.1 we see that $(\delta Q_j, \delta P_j) = (p^\alpha(t), \dot{p}^\alpha(t))$, $j = 1, 2$, and $\delta\varepsilon = 0$ give a solution to (4.7). On the other hand, we compute

$$\ddot{\chi}^\alpha(t) = \frac{d}{dt} \left(\dot{p}^\alpha(t) \int_0^t \frac{ds}{p^\alpha(s)^2} + \frac{1}{p^\alpha(t)} \right) = \ddot{p}^\alpha(t) \int_0^t \frac{ds}{p^\alpha(s)^2}.$$

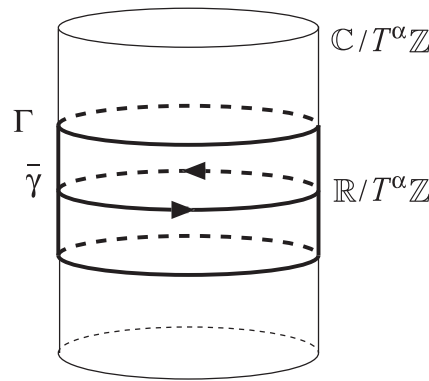


FIG. 4. Riemann surface Γ . The monodromy matrix $M_{\tilde{\gamma}}$ is computed along the loop $\tilde{\gamma}$.

By $\ddot{p}^\alpha(t) = -V''(q^\alpha(t))p^\alpha(t)$, we see that $(\delta Q_j, \delta P_j) = (\chi^\alpha(t), \dot{\chi}^\alpha(t))$, $j = 1, 2$, and $\delta\varepsilon = 0$ give another solution to (4.7).

Let $\delta\varepsilon = 1$. The solution to the $(\delta Q_j, \delta P_j)$ -components of (4.7) with $(\delta Q_j(0), \delta P_j(0)) = (0, 0)$ is given by

$$\begin{aligned} \begin{pmatrix} \delta Q_j \\ \delta P_j \end{pmatrix} &= \begin{pmatrix} p^\alpha(t) & \chi^\alpha(t) \\ \dot{p}^\alpha(t) & \dot{\chi}^\alpha(t) \end{pmatrix} \int_0^t \begin{pmatrix} \dot{\chi}^\alpha(s) & -\chi^\alpha(s) \\ -\dot{p}^\alpha(s) & p^\alpha(s) \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{4}\tilde{\beta}_j p^\alpha(s) \dot{p}^\alpha(s) V'''(q^\alpha(s)) \end{pmatrix} ds \\ &= \frac{1}{4}\tilde{\beta}_j \begin{pmatrix} p^\alpha(t) & \chi^\alpha(t) \\ \dot{p}^\alpha(t) & \dot{\chi}^\alpha(t) \end{pmatrix} \int_0^t \begin{pmatrix} -\chi^\alpha(s) p^\alpha(s) \dot{p}^\alpha(s) V'''(q^\alpha(s)) \\ p^\alpha(s)^2 \dot{p}^\alpha(s) V'''(q^\alpha(s)) \end{pmatrix} ds \end{aligned}$$

since $\chi^\alpha(0) = 0$, $\dot{\chi}(0) = 1/p^\alpha(0)$ and the trace of the coefficient matrix is zero in the $(\delta Q_j, \delta P_j)$ -components of (4.7), so that

$$\det \begin{pmatrix} p^\alpha(t) & \chi^\alpha(t) \\ \dot{p}^\alpha(t) & \dot{\chi}^\alpha(t) \end{pmatrix} = \det \begin{pmatrix} p^\alpha(0) & \chi^\alpha(0) \\ \dot{p}^\alpha(0) & \dot{\chi}^\alpha(0) \end{pmatrix} = 1. \quad (4.8)$$

Thus, we obtain the desired result. $\square$

We regard (4.6) and (4.7) as linear differential equations on a Riemann surface $\Gamma \subset \mathbb{C}/T^\alpha\mathbb{Z}$ containing γ^α and $\mathbb{R}/T^\alpha\mathbb{Z}$. See Fig. 4.

Proposition 4.3. Suppose that there exists a closed loop γ^α in the domain Γ such that the integral $\mathcal{I}^\alpha$ is not zero. Then the identity component $\mathcal{G}^0$ of the differential Galois group $\mathcal{G}$ for (4.7) is not commutative if $\beta \neq 0$.

Proof. We adopt the fundamental matrix $\Phi(t)$ given in Proposition 4.2. We first compute the monodromy matrix M_{γ^α} along γ^α , which is a closed loop on the Riemann surface Γ .

Lemma 4.4. The monodromy matrix M_{γ^α} along the loop γ^α on Γ is given by

$$M_{\gamma^\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{4}\tilde{\beta}_1 \mathcal{I}^\alpha \\ 0 & 1 & 0 & 0 & \frac{1}{4}\tilde{\beta}_1 \mathcal{I}^\alpha \\ 0 & 0 & 1 & 0 & -\frac{1}{4}\tilde{\beta}_2 \mathcal{I}^\alpha \\ 0 & 0 & 0 & 1 & \frac{1}{4}\tilde{\beta}_2 \mathcal{I}^\alpha \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\mathcal{I}^\alpha = \int_{\gamma^\alpha} \chi^\alpha(t) p^\alpha(t) \dot{p}^\alpha(t) V'''(q^\alpha(t)) dt.$$

Proof. Let $t = t_0$ denote the base point of γ^α . The monodromy matrix M_{γ^α} satisfies

$$\Phi(t_0)M_{\gamma^\alpha} = \begin{pmatrix} p^\alpha(t_0) & \chi^\alpha(t_0) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1(\chi^\alpha(t_0)\tilde{\mathcal{J}}^\alpha - p^\alpha(t_0)\mathcal{J}^\alpha + \eta^\alpha(t_0)) \\ \dot{p}^\alpha(t_0) & \dot{\chi}^\alpha(t_0) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1(\dot{\chi}^\alpha(t_0)\tilde{\mathcal{J}}^\alpha - \dot{p}^\alpha(t_0)\mathcal{J}^\alpha + \dot{\eta}^\alpha(t_0)) \\ 0 & 0 & p^\alpha(t_0) & \chi^\alpha(t_0) & \frac{1}{4}\tilde{\beta}_2(\chi^\alpha(t_0)\tilde{\mathcal{J}}^\alpha - p^\alpha(t_0)\mathcal{J}^\alpha + \eta^\alpha(t_0)) \\ 0 & 0 & \dot{p}^\alpha(t_0) & \dot{\chi}^\alpha(t_0) & \frac{1}{4}\tilde{\beta}_2(\dot{\chi}^\alpha(t_0)\tilde{\mathcal{J}}^\alpha - \dot{p}^\alpha(t_0)\mathcal{J}^\alpha + \dot{\eta}^\alpha(t_0)) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since by (4.8)

$$\Phi(t_0)^{-1} = \begin{pmatrix} \dot{\chi}^\alpha(t_0) & -\dot{\chi}^\alpha(t_0) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1(\chi^\alpha(t_0)\dot{\eta}^\alpha(t_0) - \dot{\chi}^\alpha(t_0)\eta^\alpha(t_0)) \\ -\dot{p}^\alpha(t_0) & p^\alpha(t_0) & 0 & 0 & -\frac{1}{4}\tilde{\beta}_1(p^\alpha(t_0)\dot{\eta}^\alpha(t_0) - \dot{p}^\alpha(t_0)\eta^\alpha(t_0)) \\ 0 & 0 & \dot{\chi}^\alpha(t_0) & -\dot{\chi}^\alpha(t_0) & \frac{1}{4}\tilde{\beta}_2(\chi^\alpha(t_0)\dot{\eta}^\alpha(t_0) - \dot{\chi}^\alpha(t_0)\eta^\alpha(t_0)) \\ 0 & 0 & -\dot{p}^\alpha(t_0) & p^\alpha(t_0) & -\frac{1}{4}\tilde{\beta}_2(p^\alpha(t_0)\dot{\eta}^\alpha(t_0) - \dot{p}^\alpha(t_0)\eta^\alpha(t_0)) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4.9)$$

we obtain the desired result. $\square$

Let $\tilde{\gamma} = \{T^\alpha s \mid s \in [0, 1]\}$, which is also a closed loop on the Riemann surface Γ (see Fig. 4).

Lemma 4.5. *The monodromy matrix $M_{\tilde{\gamma}}$ along the loop $\tilde{\gamma}$ on Γ is given by*

$$M_{\tilde{\gamma}} = \begin{pmatrix} 1 & \chi^\alpha(T^\alpha)/p^\alpha(0) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1\tilde{\mathcal{J}}_0^\alpha \\ 0 & 1 & 0 & 0 & \frac{1}{4}\tilde{\beta}_1\mathcal{J}_0^\alpha \\ 0 & 0 & 1 & \chi^\alpha(T^\alpha)/p^\alpha(0) & \frac{1}{4}\tilde{\beta}_2\tilde{\mathcal{J}}_0^\alpha \\ 0 & 0 & 0 & 1 & \frac{1}{4}\tilde{\beta}_2\mathcal{J}_0^\alpha \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{J}_0^\alpha &= \int_0^{T^\alpha} p^\alpha(t)^2 \dot{p}^\alpha(t) V'''(q^\alpha(t)) dt, \\ \tilde{\mathcal{J}}_0^\alpha &= \int_0^{T^\alpha} \chi^\alpha(t) p^\alpha(t) \dot{p}^\alpha(t) V'''(q^\alpha(t)) dt. \end{aligned}$$

Proof. Since $t = 0$ is the base point of $\tilde{\gamma}$, the monodromy matrix $M_{\tilde{\gamma}}$ satisfies

$$\Phi(0)M_{\tilde{\gamma}} = \begin{pmatrix} p^\alpha(0) & \chi^\alpha(T^\alpha) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1(\chi^\alpha(T^\alpha)\tilde{\mathcal{J}}_0^\alpha - p^\alpha(0)\mathcal{J}_0^\alpha) \\ \dot{p}^\alpha(0) & \dot{\chi}^\alpha(T^\alpha) & 0 & 0 & \frac{1}{4}\tilde{\beta}_1(\dot{\chi}^\alpha(T^\alpha)\tilde{\mathcal{J}}_0^\alpha - \dot{p}^\alpha(0)\mathcal{J}_0^\alpha) \\ 0 & 0 & p^\alpha(0) & \chi^\alpha(T^\alpha) & \frac{1}{4}\tilde{\beta}_2(\chi^\alpha(T^\alpha)\tilde{\mathcal{J}}_0^\alpha - p^\alpha(0)\mathcal{J}_0^\alpha) \\ 0 & 0 & \dot{p}^\alpha(0) & \dot{\chi}^\alpha(T^\alpha) & \frac{1}{4}\tilde{\beta}_2(\dot{\chi}^\alpha(T^\alpha)\tilde{\mathcal{J}}_0^\alpha - \dot{p}^\alpha(0)\mathcal{J}_0^\alpha) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\chi^\alpha(0) = 0$, $\dot{\chi}^\alpha(0) = 1/p^\alpha(0)$, we have $\eta^\alpha(0) = 0$ and $\dot{\eta}(0) = 0$, so that Eq. (4.9) becomes

$$\Phi(0)^{-1} = \begin{pmatrix} 1/p^\alpha(0) & 0 & 0 & 0 & 0 \\ -\dot{p}^\alpha(0) & p^\alpha(0) & 0 & 0 & 0 \\ 0 & 0 & 1/p^\alpha(0) & 0 & 0 \\ 0 & 0 & -\dot{p}^\alpha(0) & p^\alpha(0) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

at $t = 0$. Using the above expressions and (4.8), we obtain the desired result. $\square$

Assume that $\beta_1 \neq 0$. Using Lemmas 4.4 and 4.5 and noting (4.8) and $\chi^\alpha(T^\alpha) \neq 0$, we obtain

$$\mathcal{G}^0 \supset \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & C_1 \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| C_1 \in \mathbb{C} \right\} \cup \left\{ \begin{pmatrix} 1 & C_2 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & C_2 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| C_2 \in \mathbb{C} \right\},$$

where each element denoted by “*” may be zero. Hence, $\mathcal{G}^0$ is not commutative since

$$\begin{pmatrix} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & C_1 \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & C_2 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & C_2 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & C_2 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & C_2 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & C_1 \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -C_1 C_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly we show that $\mathcal{G}^0$ is not commutative when $\beta_2 \neq 0$. Thus, we complete the proof of Proposition 4.3. $\square$

Proof of Theorem 2.3. Suppose that the hypotheses of the theorem hold. The conclusion of Proposition 4.3 implies that the identity component $\mathcal{G}^0$ of the differential Galois group $\mathcal{G}$ for (4.6) is not commutative since the identity component $\mathcal{G}^0$ of the differential Galois group $\mathcal{G}$ for (4.7) is commutative otherwise. Hence, by Proposition 3.2 and Theorem 3.3, the system (4.1) is not meromorphically integrable near the periodic orbit (4.3) on any domain Γ in $\mathbb{C}/T^\alpha\mathbb{Z}$ containing $\mathbb{R}/T^\alpha\mathbb{Z}$ and γ^α , such that the additional first integrals depend on ε meromorphically in a neighborhood of $\varepsilon = 0$. Here we remark that meromorphic first integrals generate meromorphic commutative vector fields near $\varepsilon = 0$ if they exist although they may not be linearly independent at $\varepsilon = 0$. Thus, we complete the proof. $\square$

Remark 4.6. As shown in the proof of Theorem 2.3, the semiclassical system (1.1) is nonintegrable in a more general sense, i.e., the Bogoyavlenskij sense, such that the first integrals and commutative vector fields depend on ε meromorphically near $\varepsilon = 0$.

V. EXAMPLE

We now illustrate the above theory for the potential (1.7). The phase portraits of (1.2) are shown in Fig. 5: There exist a center at (0, 0) and a one-parameter family of periodic orbits

$$(q^\alpha(t), p^\alpha(t)) = \left(\operatorname{arcsinh} \left(\frac{\sqrt{2-\alpha^2}}{\alpha} \sin \alpha t \right), \frac{\alpha \sqrt{2-\alpha^2} \cos \alpha t}{\sqrt{\alpha^2 + (2-\alpha^2) \sin^2 \alpha t}} \right), \quad \alpha \in (0, \sqrt{2}), \quad (5.1)$$

around it. See Appendix A of Ref. 29 for the derivation of (5.1). The period of $(q^\alpha(t), p^\alpha(t))$ are given by $T^\alpha = 2\pi/\alpha$.

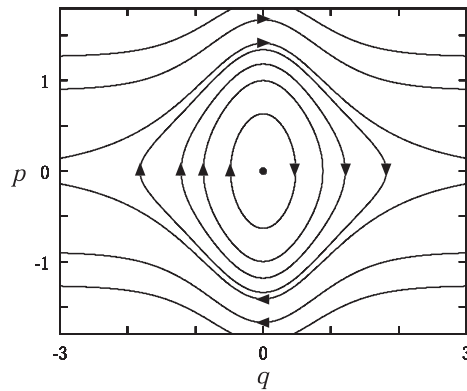


FIG. 5. Phase portraits of (1.2) with the potential (1.7).

We see that $p^\alpha(t)$ has a singularity at

$$t = iK^\alpha, \quad K^\alpha = \frac{1}{\alpha} \operatorname{arcsinh}\left(\frac{\alpha}{\sqrt{2-\alpha^2}}\right).$$

Since

$$\begin{aligned} \alpha^2 + (2 - \alpha^2) \sin^2 \alpha t &= 2\sqrt{2} i \alpha^2 \Delta t + \alpha^2 (\alpha^2 + 2) \Delta t^2 - \frac{4\sqrt{2} i}{3} \alpha^4 \Delta t^3 \\ &\quad - \frac{\alpha^4 (\alpha^2 + 2)}{3} \Delta t^4 + \frac{4\sqrt{2} i}{15} \alpha^6 \Delta t^5 + \dots \end{aligned}$$

and

$$\begin{aligned} \cos \alpha t &= \sqrt{\frac{2}{2-\alpha^2}} - \frac{i \alpha^2}{\sqrt{2-\alpha^2}} \Delta t - \frac{\alpha^2}{\sqrt{2(2-\alpha^2)}} \Delta t^2 \\ &\quad + \frac{i \alpha^4}{6\sqrt{2-\alpha^2}} \Delta t^3 + \frac{\alpha^4}{12\sqrt{2(2-\alpha^2)}} \Delta t^4 + \dots \end{aligned}$$

around $t = iK^\alpha$, we have

$$\begin{aligned} p^\alpha(t) &= \frac{1-i}{2^{3/4}} \Delta t^{-1/2} + \frac{(1+i)(2-3\alpha^2)}{8 \cdot 2^{1/4}} \Delta t^{1/2} + \frac{(1+i)(15\alpha^4 - 20\alpha^2 - 36)}{192 \cdot 2^{3/4}} \Delta t^{3/2} \\ &\quad + \frac{(1-i)(21\alpha^6 - 42\alpha^4 + 28\alpha^2 - 120)}{1536 \cdot 2^{1/4}} \Delta t^{5/2} \\ &\quad - \frac{(1+i)(5\alpha^4 - 12\alpha^2 + 20)(45\alpha^4 - 12\alpha^2 - 140)}{40960 \cdot 2^{3/4}} \Delta t^{7/2} + \dots, \end{aligned}$$

where $\Delta t = t - iK^\alpha$. Taking a small circle centered at $t = iK^\alpha$ as γ^α , we compute (2.2) as

$$\begin{aligned} \mathcal{I}^\alpha &= 4 \int_{\gamma^\alpha} p^\alpha(t)^2 \dot{p}^\alpha(t)^2 (1 - 3\operatorname{sech}^2 q^\alpha(t)) dt \\ &= - \frac{(15\alpha^4 + 20\alpha^2 + 28)(2 - \alpha^2)}{128\sqrt{2}} i \neq 0 \end{aligned}$$

for $\alpha \in (0, \sqrt{2})$, where we have used the relations,

$$\begin{aligned} V'''(q^\alpha(t)) &= 8\operatorname{sech}^2 q^\alpha(t) \tanh q^\alpha(t) - 24\operatorname{sech}^4 q^\alpha(t) \tanh q^\alpha(t) \\ &= 4\dot{p}^\alpha(t)(1 - 3\operatorname{sech}^2 q^\alpha(t)) \end{aligned}$$

and

$$\begin{aligned}\operatorname{sech}^2 q^\alpha(t) &= \frac{\alpha^2}{\alpha^2 + (2 - \alpha^2)\sin^2 at} \\ &= -\frac{i}{2\sqrt{2}\Delta t} + \frac{\alpha^2 + 2}{8} + \frac{3\alpha^4 - 4\alpha^2 + 12}{48\sqrt{2}}i\Delta t - \frac{(\alpha^2 + 2)(\alpha^2 - 2)^2}{64}\Delta t^2 \\ &\quad - \frac{45\alpha^8 - 120\alpha^6 + 56\alpha^4 - 480\alpha^2 + 720}{5760\sqrt{2}}i\Delta t^3 + \dots\end{aligned}$$

and the method of residues. Applying Theorem 2.3, we obtain the following.

Proposition 5.1. *The semiclassical system (1.1) with the potential (1.7) is not meromorphically integrable near the periodic orbits*

$$(q, p, Q_1, Q_2, P_1, P_2) = (q^\alpha(t), p^\alpha(t), \beta_1 p^\alpha(t), \beta_2 p^\alpha(t), \beta_1 \dot{p}^\alpha(t), \beta_2 \dot{p}^\alpha(t))$$

for $\alpha \in (0, \sqrt{2})$ and $\beta \neq 0$ on any domain in $\mathbb{C}/T^\alpha\mathbb{Z}$ containing $\mathbb{R}/T^\alpha\mathbb{Z}$ and $t = iK^\alpha$ such that the additional first integrals depend on ε meromorphically in a neighborhood of $\varepsilon = 0$.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Kazuyuki Yagasaki: Conceptualization (lead); Formal analysis (lead); Funding acquisition (lead); Investigation (lead); Methodology (lead); Project administration (lead); Resources (lead); Supervision (lead); Validation (lead); Writing – original draft (lead); Writing – review & editing (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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