

1. DEFINITIONS

$\mathcal{P} \subset \mathbb{R}^N$: an integral polytope of dimension n
 $\mathcal{A}_{\mathcal{P}} = \{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\} \cap \mathbb{Z}^{N+1}$

- We say that $\mathcal{P}$ is **normal** if $\mathcal{P}$ satisfies

$$\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} = \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}.$$

- We say that $\mathcal{P}$ is **very ample** if $\mathcal{P}$ satisfies

$$|(\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}| < \infty.$$

the elements of $(\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$: the **holes** of $\mathcal{P}$
 Note : “normal $\Rightarrow$ very ample”

For simplicity, we assume

$$N = n \text{ and } \mathbb{Z}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}^{n+1}.$$

Then

$\mathcal{P}$ is normal (resp. very ample) $\iff$

$\mathcal{P}$ satisfies (*) for **all** $m \geq 1$ (resp. for **sufficiently large** m) :

$$\begin{aligned} & \forall \alpha \in m\mathcal{P} \cap \mathbb{Z}^n, \\ (*) \quad & \exists \alpha_1, \dots, \exists \alpha_m \in \mathcal{P} \cap \mathbb{Z}^n \text{ s.t.} \\ & \alpha = \alpha_1 + \dots + \alpha_m. \end{aligned}$$

- $\clubsuit$ $\mathcal{P}$ is called **k-normal** if $\mathcal{P}$ satisfies (*) for all $m \geq k$.

2. BACKGROUNDS

From Commutative Algebra

K : field

$\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}, \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}^{n+1}$: affine semigroups

- $K[\mathcal{P}] = K[\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}], \overline{K[\mathcal{P}]} = K[\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}^{n+1}]$:
affine semigroup graded K -algebras

$\implies \overline{K[\mathcal{P}]}$ is the **normalization** of $K[\mathcal{P}]$.

- $K[\mathcal{P}]$ and $\overline{K[\mathcal{P}]}$ are graded by grading $\deg X^\alpha = i$,
where $\alpha = (\alpha_1, \dots, \alpha_n, i) \in \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}^{n+1}$.
- $\mathcal{P}$ is normal (resp. very ample) $\iff$
 $K[\mathcal{P}]$ (resp. m^k) is generated by degree one elements
 (resp. for $k \gg 0$, where $m = K[\mathcal{P}]_+$).

From Algebraic Geometry

X : toric variety of dim n / K , L : ample line bundle on X

$\mathcal{P} \subset \mathbb{R}^n$: the corresponding integral polytope to (X, L)

Then

$$H^0(X, L) \cong \bigoplus_{v \in \mathcal{P} \cap \mathbb{Z}^n} K e_v$$

$$R(X, L) := \bigoplus_{i \geq 0} H^0(X, L^{\otimes i}) \cong \overline{K[\mathcal{P}]}$$

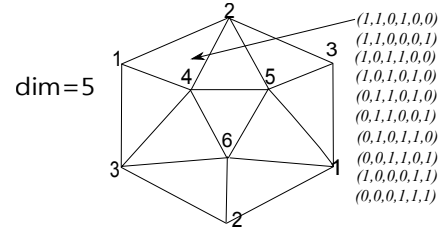
- $\mathcal{P}$ is normal $\iff L$ is *normally generated*
(in the sense of Mumford)
- $\mathcal{P}$ is very ample $\iff L$ is very ample

3. EXAMPLES

Our main interest is the existence of

**non-normal very ample
integral polytopes !!**

EXAMPLE (Bruns–Gubeladze '02)



- non-normal : $(1, 1, 1, 1, 1, 1) =$

$$1/2((1, 1, 0, 1, 0, 0) + (0, 1, 1, 0, 1, 0) + (0, 0, 1, 1, 0, 1) + (1, 0, 0, 0, 1, 1))$$

- very ample ; $(1, 1, 1, 1, 1, 1) + (1, 1, 0, 1, 0, 0) =$

$$(1, 0, 1, 1, 0, 0) + (0, 1, 0, 1, 1, 0) + (1, 1, 0, 0, 0, 1) \text{ and so on...}$$

- the number of holes = 1

EXAMPLE (Bruns–Gubeladze '09)

$$\text{conv} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 3 & 4 & 1 & 2 \end{pmatrix}, \quad \dim = 3,$$

$$\circ (1, 1, 4) = \frac{1}{2}((0, 0, 1) + (1, 0, 2) + (0, 1, 3) + (1, 1, 2))$$

$$\circ (1, 1, 4) + (0, 0, 0) = (0, 0, 1) + (0, 0, 1) + (1, 1, 2) \dots\dots$$

- the number of holes = 1

REMARK (Ogata '11)

In [3], infinitely many non-normal very ample integral polytopes of dim 3 are constructed.

4. MAIN RESULT

How about higher dimensions ?

Theorem

Let n and h be integers with $n \geq 3$ and $h \geq 1$. Then there exists a non-normal very ample integral polytope of dimension n having exactly h holes.

REMARK (Katthän '12)

$\mathcal{P}$: non-normal very ample integral polytope

$\implies K[\mathcal{P}]$ is **NEVER** Cohen–Macaulay !!

REFERENCE

- [1] W. Bruns and J. Gubeladze, “Polytopes, rings and K-theory”, Springer–Verlag, Heidelberg, 2009.
- [2] D. Cox, J. Little and H. Schenck, “Toric varieties”, American Mathematical Society, 2011.
- [3] S. Ogata, Very ample but not normal lattice polytopes, to appear in *Beiträge Algebra Geom.*, (2011)