

ON SUFFICIENT CONDITIONS FOR MEROMORPHIC STARLIKE FUNCTIONS

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ABSTRACT. The object of the present paper is to show certain sufficient conditions for starlikeness and close-to-convexity of meromorphic functions in the punctured unit disk.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the punctured unit disk $D = \{z : 0 < |z| < 1\}$. For f and g which are analytic in $U = \{z : |z| < 1\}$, we say that f is subordinate to g , written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

For $0 < \alpha \leq 1$, let $\mathcal{SMS}(\alpha)$ denote the class of functions $f \in \Sigma$ which are starlike of order α ; that is, which satisfy

$$-\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^\alpha \quad (z \in U). \quad (1.2)$$

We note that the equation (1.2) can be rewritten by the following form ;

$$\left| \arg \left(-\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).$$

Also, we note that if $\alpha = 1$, $\mathcal{SMS}(\alpha)$ coincides with Σ^* , the well known class of meromorphic starlike(univalent) functions with respect to origin.

In [1], Bajpai and Mehrok proved that the functions of the form (1.1) satisfying the condition

1991 Mathematics Subject Classification : 30C45.

Key words and phrases. subordinate, meromorphic starlike, meromorphic close-to-convex.

$$\operatorname{Re} \left\{ \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) - (\alpha + \beta) \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U)$$

are univalent and meromorphic starlike, where α and β are real numbers. For various other interesting developments involving analytic functions in the open unit disk U , the reader may be referred (for example) to the recent work of Nunokawa[3].

In this paper, we investigate some sufficient conditions for starlikeness and close-to-convexity of functions belonging to Σ .

2. Main results

In proving our theorems, we need the following lemma due to Nunokawa [2].

Lemma 2.1 *Let p be analytic in U , $p(0) = 1$ and $p(z) \neq 0$ in U . Suppose that there exists a point $z_0 \in U$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\delta \quad \text{for } |z| < |z_0| \quad (2.1)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\delta \quad (0 < \delta \leq 1). \quad (2.2)$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\delta k, \quad (2.3)$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2}\delta, \quad (2.4)$$

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2}\delta, \quad (2.5)$$

and

$$\{p(z_0)\}^{\frac{1}{\delta}} = \pm ia \quad (a > 0). \quad (2.6)$$

Applying Lemma 2.1, we have the following

Theorem 2.1. *Let p be analytic in U with $p(0) = 1$. If*

$$\left| \arg \left(\beta p(z) + \alpha \frac{zp'(z)}{p(z)} \right) \right| < \frac{\pi}{2} \gamma(\alpha, \beta, \delta) \quad (\alpha, \beta > 0, 0 < \delta < 1, z \in U), \quad (2.7)$$

where

$$\gamma(\alpha, \beta, \delta) = \frac{2}{\pi} \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta(1+\delta)^{\frac{1+\delta}{2}} (1-\delta)^{\frac{1-\delta}{2}} \cos \frac{\pi}{2} \delta} \right\}, \quad (2.8)$$

then

$$|\arg p(z)| < \frac{\pi}{2} \delta.$$

Proof. If there exists a point $z_0 \in U$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.1) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

From (2.7), we note that $p(z) \neq 0$ in U . In fact, if p has a zero of order m at $z = z_1 \in U$, then p can be written as

$$p(z) = (z - z_1)^m q(z) \quad (m \in N = \{1, 2, \dots\}),$$

where q is analytic in U and $q(z_1) \neq 0$. Hence we have

$$\beta p(z) + \alpha \frac{zp'(z)}{p(z)} = \frac{\alpha m z}{z - z_1} + \alpha \frac{zq'(z)}{q(z)} + \beta(z - z_1)^m q(z). \quad (2.9)$$

But choosing $z \rightarrow z_1$ suitably, the argument of the right hand side of (2.9) can take any value between 0 and 2π . This contradicts (2.7). Hence we have $p(z) \neq 0$ ($z \in U$). Then we obtain

$$\begin{aligned} \beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} &= \beta(\pm ia)^\delta + i\alpha\delta k \\ &= \beta a^\delta \cos \frac{\pi}{2} \delta + i \left\{ \beta a^\delta \sin \frac{\pi}{2} \delta + \alpha\delta k \right\}. \end{aligned}$$

Now we suppose that

$$\{p(z_0)\}^{\frac{1}{\delta}} = ia \quad (a > 0).$$

Then we have

$$\arg \left(\beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) = \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta k}{\beta \cos \frac{\pi}{2} \delta} \right\},$$

where

$$k a^{-\delta} \geq \frac{1}{2} (a^{1-\alpha} + a^{-1-\alpha}) \equiv g(a) \quad (a > 0).$$

Hence, by a simple calculation, we see that the function $g(a)$ takes the minimum value at $a = \sqrt{\frac{1+\alpha}{1-\alpha}}$. Hence we have

$$\begin{aligned} \arg \left(\beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) &\leq \tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta (1+\delta)^{\frac{1+\delta}{2}} (1-\alpha)^{\frac{1-\alpha}{2}} \cos \frac{\pi}{2} \delta} \right\} \\ &= \frac{\pi}{2} \gamma(\alpha, \beta, \delta), \end{aligned}$$

where $\gamma(\alpha, \beta, \delta)$ is given by (2.8). This evidently contradicts the assumption of Theorem 2.1.

Next, we suppose that

$$\{p(z_0)\}^{\frac{1}{\delta}} = -ia \quad (a > 0).$$

Applying the same method as the above, we have

$$\begin{aligned} \arg \left(\beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) &\geq -\tan^{-1} \left\{ \tan \frac{\pi}{2} \delta + \frac{\alpha \delta}{\beta (1+\delta)^{\frac{1+\delta}{2}} (1-\alpha)^{\frac{1-\alpha}{2}} \cos \frac{\pi}{2} \delta} \right\} \\ &= -\frac{\pi}{2} \gamma(\alpha, \beta, \delta), \end{aligned}$$

where $\gamma(\alpha, \beta, \delta)$ is given by (2.8), which is a contradiction to the assumption of Theorem 2.1. Therefore, we complete the proof of Theorem 2.1.

Taking $p(z) = -\frac{zf'(z)}{f(z)}$ in Theorem 2.1, we have

Corollary 2.1. *If $f \in \Sigma$ satisfies the condition*

$$\left| \arg \left\{ \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) - (\alpha + \beta) \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \gamma(\alpha, \beta, \delta) \quad (\alpha, \beta > 0, 0 < \delta < 1, z \in U),$$

where $\gamma(\alpha, \beta, \delta)$ is given by (2.8), then $f \in SMS(\delta)$.

Next, we prove

Theorem 2.2. Let $\alpha \geq 0$ or $\alpha \leq -2\beta$ ($\beta > 0$). If p satisfies the condition

$$(2.10) \quad \beta p(z) + \alpha \frac{zp'(z)}{p(z)} \neq ik \quad (z \in U),$$

where k is a real number with $|k| \geq \sqrt{(\alpha + 2\beta)\alpha}$. Then $\operatorname{Re} p(z) > 0$ ($z \in U$).

Proof. For the case $\alpha = 0$, it is obvious and so we suppose $\alpha \neq 0$. By using the same method of the proof in Theorem 2.1, we can see easily that $p(z) \neq 0$ in U . Suppose that there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for} \quad |z| < |z_0|,$$

$$\operatorname{Re} p(z_0) = 0 \quad \text{and} \quad p(z_0) = ia \quad (a \neq 0).$$

For the case $\alpha > 0$, from Lemma 2.1 with $\delta = 1$, we have

$$\beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} = i(\beta a + \alpha k),$$

and

$$\beta a + \alpha k \geq \frac{1}{2} \left((\alpha + 2\beta)a + \frac{\alpha}{a} \right) \geq \sqrt{(\alpha + 2\beta)\alpha} \quad \text{when } a > 0,$$

and

$$\beta a + \alpha k \leq -\frac{1}{2} \left((\alpha + 2\beta)|a| + \frac{\alpha}{|a|} \right) \leq -\sqrt{(\alpha + 2\beta)\alpha} \quad \text{when } a < 0,$$

which contradict (2.10). Therefore we have $\operatorname{Re} p(z) > 0$ in U . For the case $\alpha \leq -2\beta$, applying the same method as the above, we easily have the same conclusion. This completes the proof of our theorem.

Letting $p(z) = -\frac{zf'(z)}{f(z)}$ in Theorem 2.2, we easily have the following

Corollary 2.2. Let $\alpha \geq 0$ or $\alpha \leq -2\beta$ ($\beta > 0$). If $f \in \Sigma$ satisfies the condition

$$\alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) - (\alpha + \beta) \frac{zf'(z)}{f(z)} \neq ik \quad (z \in U),$$

where k is real number with $|k| \geq \sqrt{(\alpha + 2\beta)\alpha}$, then $f \in \Sigma^*$.

Making $\alpha = \beta = 1$ in Corollary 2.2, we obtain

Corollary 2.3. Let $f \in \Sigma$ and suppose that there exists a real number R for which

$$\left| \frac{zf''(z)}{f'(z)} - 2 \frac{zf'(z)}{f(z)} - R \right| < \sqrt{(R+1)^2 + 3} \quad (z \in U).$$

Then f is meromorphic starlike in U .

Putting $p(z) = -z^2 f'(z)$ in Theorem 2.2, we get

Corollary 2.4. Let $\alpha \geq 0$ or $\alpha \leq -2\beta$ ($\beta > 0$). If $f \in \Sigma$ satisfies the condition

$$\alpha \left(2 + \frac{zf''(z)}{f'(z)} \right) - \beta z^2 f'(z) \neq ik \quad (z \in U),$$

where k is given by Corollary 2.2. Then f is meromorphic univalent (or close-to-convex) in U .

Similarly, from Corollary 2.4, we have

Corollary 2.5. Let $f \in \Sigma$ and suppose that there exists a real number R for which

$$\left| \frac{zf''(z)}{f'(z)} - z^2 f'(z) - R \right| < \sqrt{(R+2)^2 + 3} \quad (z \in U).$$

Then f is meromorphic univalent (or close-to-convex) in U .

Acknowledgement

This work was partially supported by the Basic Science Research Program, Ministry of Education, Project No. BSRI-98-1440.

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