

A Variational Problem Governed by a Differential Inclusion in a Banach Space

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1 Introduction

Let $\mathcal{X}$ be a real separable reflexive Banach space. A correspondence (= multi-valued mapping) $\Gamma : [0, T] \times \mathcal{X} \rightarrow \mathcal{X}$ and a function $u : [0, T] \times \mathcal{X} \times \mathcal{X} \rightarrow \overline{\mathbb{R}}$ are assumed to be given. A double arrow $\rightarrow$ indicates the domain and the range of a correspondence. The compact interval $[0, T]$ is endowed with the Lebesgue measure dt . $\mathcal{L}$ denotes the σ -field of the Lebesgue-measurable sets of $[0, T]$.

Let $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ be the Sobolev space consisting of functions of $[0, T]$ into $\mathcal{X}$ (cf. Appendix) And let $\Delta(a)$ be the set of all the solutions in the Sobolev space $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ of a differential inclusion :

$$(*) \quad \dot{x}(t) \in \Gamma(t, x(t)), x(0) = a,$$

where $\dot{x}$ denotes the derivative of x and a is a fixed vector in $\mathcal{X}$. And consider a variational problem :

$$(\#) \quad \text{Minimize}_{x \in \Delta(a)} \int_0^T u(t, x(t), \dot{x}(t)) dt.$$

The object of this paper is to discuss a couple of existence problems as follows :

- (i) the existence of a solution for the differential inclusion $(*)$, and
- (ii) the existence of an optimal solution for the variational problem $(\#)$.

In Maruyama [14] [15], I presented a solution of these problems in the special case $\mathcal{X} = \mathbb{R}^\ell$ by making use of the convenient properties of the weak convergence in the Sobolev space $\mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$; i.e. if a sequence $\{x_n\}$ in $\mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$, weakly converges to some $x^* \in \mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$, then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that

$$\begin{aligned} z_n &\rightarrow x^* \quad \text{uniformly on } [0, T], \text{ and} \\ \dot{z}_n &\rightarrow \dot{x}^* \quad \text{weakly in } \mathcal{L}^2([0, T], \mathbb{R}^\ell). \end{aligned} \tag{W}$$

However it deserves a special notice that this property does not hold in the space $\mathcal{W}^{1,2}([0, T], \mathcal{X})$ if $\dim \mathcal{X} = \infty$. Taking account of this fact, I provided a new convergence result to overcome this difficulty in the case $\mathcal{X}$ is a real separable Hilbert space in Maruyama [17]. And I also gave a existence theory for the problems (i) and (ii) being based upon this new tool in the framework of a separable Hilbert space in Maruyama [17],[18].

The purpose of the present paper is to generalize my previous results to the case $\mathcal{X}$ is a real separable reflexive Banach space. Papageorgiou [19] also gave an elegant extension of my results in Maruyama [14],[15] to the infinite dimensional case. The present paper might be regarded as an alternative approach to Papageorgiou's theory.

Let me mention about another improvement added on this occasion. In Maruyama [17], I imposed a very restrictive requirement on the continuity of the correspondence Γ ; i.e.

the correspondence $x \mapsto \Gamma(t, x)$ is upper hemi-continuous for each fixed $t \in [0, T]$ with respect to the weak topology for the domain and the strong topology for the range.

I have to admit frankly that this is a very unpleasant assumption. In the present paper, I propose the upper hemi-continuity of $x \mapsto \Gamma(t, x)$ with respect to the “weak-weak” combination of topologies instead of the “weak-strong” combination.

2 A Convergence Theorem in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$

As I have already said, any weakly convergent sequence $\{x_n\}$ in the Sobolev space $\mathcal{W}^{1,2}([0, T], \mathbb{R}^\ell)$ has a subsequence which satisfies the property (W) in section 1.

On the other hand, let $\mathcal{X}$ be a real Banach space with the Radon-Nikodým property (RNP). Then any absolutely continuous function $f : [0, T] \rightarrow \mathcal{X}$ is Fréchet-differentiable a.e. (If the Banach space $\mathcal{X}$ does not have RNP, this property does not hold. For a counter-example, see Komura [13].) Let $\{x_n\}$ be a sequence in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ which weakly converges to some $x^* \in \mathcal{W}^{1,p}([0, T], \mathcal{X})$.

We should keep in mind that it is not necessarily true that the sequence $\{x_n\}$ has a subsequence $\{z_n\}$ which satisfies the property (W) if $\dim \mathcal{X} = \infty$ even in the case $p = 2$.

Counter-Example (Cecconi[9], pp.28-29) Let $\mathcal{H}$ be a real separable Hilbert space and $\{\varphi_n; n = 1, 2, \dots\}$ a complete orthonormal system of $\mathcal{H}$. (cf. Yosida [28] P.89.) Define a sequence $\{x_n : [0, T] \rightarrow \mathcal{H}\}$ by

$$x_n(t) = t\varphi_n \quad (n = 1, 2, \dots).$$

We also define the function $x^* : [0, 1] \rightarrow \mathcal{H}$ by $x^*(t) \equiv 0$. Then x_n 's as well as x^* are elements of $\mathcal{W}^{1,2}([0, T], \mathcal{H})$. It follows from the Riemann-Lebesgue lemma that the sequence $\{x_n\}$ weakly converges to x^* in $\mathcal{W}^{1,2}([0, 1], \mathcal{H})$. However there is no subsequence of $\{x_n\}$ which converges strongly (hence uniformly) to x^* in $\mathcal{L}^2([0, 1], \mathcal{H})$.

The following theorem cultivated to overcome this difficulty is a generalization of Theorem 1 of Maruyama [18].

Henceforth we denote by $\mathcal{X}_s$ (resp. $\mathcal{X}_w$) a Banach space $\mathcal{X}$ endowed with the strong (resp. weak) topology.

THEOREM 1. Let $\mathcal{X}$ be a real separable reflexive Banach space. And consider a sequence $\{x_n\}$ in the Sobolev space $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ ($p \geq 1$). Assume that

- (i) the set $\{x_n(t)\}_{n=1}^\infty$ is bounded (and hence relatively compact) in $\mathcal{X}_w$ for each $t \in [0, T]$, and
- (ii) there exists some function $\psi \in \mathcal{L}^p([0, T], (0, +\infty))$ such that

$$\| \dot{x}_n(t) \| \leq \psi(t) \quad \text{a.e.}$$

Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^* \in \mathcal{W}^{1,p}([0, T], \mathcal{X})$ such that

- (a) $z_n \rightarrow x^*$ uniformly in $\mathcal{X}_w$ on $[0, T]$, and
- (b) $\dot{z}_n \rightarrow \dot{x}^*$ weakly in $\mathcal{L}^p(0, T], \mathcal{X})$.

Remark Since $\mathcal{X}$ is separable and reflexive, the following results holds true. Assume that $p \geq 1$.

[I] $\mathcal{L}^p([0, T], \mathcal{X})$ is separable.

[II] $\mathcal{L}^p([0, T], \mathcal{X})'$ is isomorphic to $\mathcal{L}^q([0, T], \mathcal{X}')$, where $1/p + 1/q = 1$ and “ , ” denotes the dual space.

[III] Any absolutely continuous function $f : [0, T] \rightarrow \mathcal{X}$ is Fréchet-differentiable a.e. and the “fundamental theorem of calculus” , i.e.

$$f(t) = f(0) + \int_0^t \dot{f}(\tau) d\tau; t \in [0, T]$$

is valid.

Proof of Theorem 1. (a) To start with, we shall show the equicontinuity of $\{x_n\}$. Since ψ is integrable, there exists some $\delta > 0$ for each $\varepsilon > 0$ such that

$$\|x_n(t) - x_n(s)\| \leq \int_s^t \|\dot{x}_n(\tau)\| d\tau \leq \int_s^t \psi(\tau) d\tau \leq \varepsilon \quad \text{for all } n$$

provided that $|t - s| \leq \delta$. This proves the equicontinuity of $\{x_n\}$ in the strong topology for $\mathcal{X}$. Hence $\{x_n\}$ is also equicontinuous in the weak topology for $\mathcal{X}$.

Taking account of this fact as well as the assumption (i), we can claim, thanks to the Ascoli-Arzelà theorem (cf. Schwartz[21] p.78), that $\{x_n\}$ is relatively compact in $\mathcal{C}([0, T], \mathcal{X}_w)$ (the set of continuous functions of $[0, T]$ into $\mathcal{X}_w$) with respect to the topology of uniform convergence.

By the assumption (i), $\{x_n(0)\}$ is bounded in $\mathcal{X}$, say

$$\sup_n \|x_n(0)\| \leq C < +\infty.$$

And the assumption (ii) implies that

$$\left\| \int_0^t \dot{x}_n(\tau) d\tau \right\| \leq \|\psi\|_1 \quad \text{for all } t \in [0, T].$$

Hence

$$\begin{aligned} \sup_n \|x_n(t)\| &= \sup_n \left\| x_n(0) + \int_0^t \dot{x}_n(\tau) d\tau \right\| \leq C + \|\psi\|_1 \\ &\quad \text{for all } t \in [0, T]. \end{aligned}$$

Thus each x_n can be regarded as a mapping of $[0, T]$ into the set

$$M = \{w \in \mathcal{X} \mid \|w\| \leq C + \|\psi\|_1\}.$$

The weak topology on M is metrizable because M is bounded and $\mathcal{X}$ is a

separable reflexive Banach space. Hence if we denote by M_w the space M endowed with the weak topology, then the uniform convergence topology on $\mathcal{C}([0, T], M_w)$ is metrizable.

Since we can regard $\{x_n\}$ as a relatively compact subset of $\mathcal{C}([0, T], M_w)$, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ which uniformly converges to some $x^* \in \mathcal{C}([0, T], \mathcal{X}_w)$.

(b) Since

$$\|\dot{y}_n(t)\| \leq \psi(t) \quad \text{a.e.,}$$

the sequence $\{w_n : [0, T] \rightarrow \mathcal{X}\}$ defined by

$$w_n(t) = \frac{\dot{y}_n(t)}{\psi(t)}; \quad n = 1, 2, \dots$$

is contained in the unit ball of $\mathcal{L}^\infty([0, T], \mathcal{X})$ which is weak*-compact (as the dual space of $\mathcal{L}^1([0, T], \mathcal{X}')$) by Alaoglu's theorem. Note that the weak* topology on the unit ball of $\mathcal{L}^\infty([0, T], \mathcal{X})$ is metrizable since $\mathcal{L}^1([0, T], \mathcal{X}')$ is separable. Hence $\{w_n\}$ has a subsequence $\{w_{n'}\}$ which converges to some $w^* \in \mathcal{L}^\infty([0, T], \mathcal{X})$ in the weak* topology. We shall write $\dot{z}_n = \dot{y}_{n'} = \psi \cdot w_{n'}$.

If we define an operator $A : \mathcal{L}^\infty([0, T], \mathcal{X}) \rightarrow \mathcal{L}^p([0, T], \mathcal{X})$ by

$$A : g \mapsto \psi \cdot g,$$

then A is continuous in the weak* topology for $\mathcal{L}^\infty$ and the weak topology for $\mathcal{L}^p$. In order to see this, let $\{g_\lambda\}$ be a net in $\mathcal{L}^\infty([0, T], \mathcal{X})$ such that $w^* - \lim_\lambda g_\lambda = g^* \in \mathcal{L}^\infty([0, T], \mathcal{X})$; i.e.

$$\int_0^T \langle \alpha(t), g_\lambda(t) \rangle dt \rightarrow \int_0^T \langle \alpha(t), g^*(t) \rangle dt \quad \text{for all } \alpha \in \mathcal{L}^1([0, T], \mathcal{X}').$$

Then it is quite easy to verify that

$$\begin{aligned} \int_0^T \langle \beta(t), \psi(t)g_\lambda(t) \rangle dt &= \int_0^T \langle \psi(t)\beta(t), g_\lambda(t) \rangle dt \\ &\rightarrow \int_0^T \langle \psi(t)\beta(t), g^*(t) \rangle dt \\ \text{for all } \beta &\in \mathcal{L}^q([0, T], \mathcal{X}'), \quad 1/p + 1/q = 1 \end{aligned}$$

since $\psi \cdot \beta \in \mathcal{L}^1([0, T], \mathcal{X}')$. This proves the continuity of A .

Hence

$$\dot{z}_n = \psi \cdot w_{n'} \rightarrow \psi \cdot w^* \quad \text{weakly in } \mathcal{L}^p([0, T], \mathcal{X}), \quad (1)$$

which implies

$$\langle \theta, \int_s^t \dot{z}_n(\tau) d\tau \rangle = \int_s^t \langle \theta, \dot{z}_n(\tau) \rangle d\tau \rightarrow \int_s^t \langle \theta, \psi(\tau) \cdot w^*(\tau) \rangle d\tau \quad \text{for all } \theta \in \mathcal{X}'. \quad (2)$$

On the other hand, since

$$z_n(t) - z_n(s) = \int_s^t \dot{z}_n(\tau) d\tau \quad \text{for all } n,$$

and $z_n(t) - z_n(s) \rightarrow x^*(t) - x^*(s)$ in $\mathcal{X}_w$, we get

$$\langle \theta, \int_s^t \dot{z}_n(\tau) d\tau \rangle = \langle \theta, z_n(t) - z_n(s) \rangle \rightarrow \langle \theta, x^*(t) - x^*(s) \rangle \quad \text{for all } \theta \in \mathcal{X}'. \quad (3)$$

(2) and (3) imply that

$$\langle \theta, x^*(t) - x^*(s) \rangle = \langle \theta, \int_s^t \psi(\tau) \cdot w^*(\tau) d\tau \rangle \quad \text{for all } \theta \in \mathcal{X}',$$

from which we can deduce the equality

$$x^*(t) - x^*(s) = \int_s^t \psi(\tau) \cdot w^*(\tau) d\tau. \quad (4)$$

By (1) and (4), we get the desired result :

$$\dot{z}_n \rightarrow \dot{x}^* = \psi \cdot w^* \quad \text{weakly in } \mathcal{L}^p([0, T], \mathcal{X}).$$

□

In the proof of our Theorem 1, we are making use of some ideas of Aubin and Cellina [1] (pp.13-14). However their reasoning does not seem to be perfectly sound.

3 Differential Inclusions (1)

In this section, we prepare several lemmas which are to play crucial roles in the existence theory for differential inclusions.

Throughout this section, $\mathcal{X}$ is assumed to be a real separable reflexive Banach space.

Let us begin by specifying some assumptions imposed on the correspondence $\Gamma : [0, T] \times \mathcal{X}_w \rightarrow \mathcal{X}_w$. Special attentions should be paid to the fact that both of the domain and the range of Γ are endowed with the weak topologies.

Assumption 1. Γ is compact-convex-valued ; i.e. $\Gamma(t, x)$ is a non-empty, compact and convex subset of $\mathcal{X}_w$ for all $t \in [0, T]$ and all $x \in \mathcal{X}$.

Assumption 2. The correspondence $x \mapsto \Gamma(t, x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t \in [0, T]$; i.e. for any fixed $(t, x) \in [0, T] \times \mathcal{X}_w$ and for any neighborhood V of $\Gamma(t, x) \subset \mathcal{X}_w$, there exists some neighborhood U of x such that $\Gamma(t, z) \subset V$ for all $z \in U$.

Assumption 3. The graph of the correspondence $t \mapsto \Gamma(t, x)$ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable for each fixed $x \in \mathcal{X}$ where $\mathcal{B}(\mathcal{X}_w)$ denotes the Borel σ -field on $\mathcal{X}_w$. (For the concept of "measurability" of a correspondence, the best reference is Castaing-Valadier [8] Chap.III.)

Assumption 4. Γ is $\mathcal{L}^p$ -integrably bounded ; i.e. there exists $\psi \in \mathcal{L}^p([0, T], (0, +\infty))$ ($p > 1$) such that $\Gamma(t, x) \subset S_{\psi(t)}$ for every $(t, x) \in [0, T] \times \mathcal{X}$, where $S_{\psi(t)}$ is the closed ball in $\mathcal{X}$ with the center 0 and the radius $\psi(t)$.

The following lemma is essentially due to Castaing [5].

LEMMA 1 (Castaing [5]) Suppose that a correspondence $\Gamma : \mathcal{X} \rightarrow \mathcal{X}$ satisfies the Assumptions 1-3, and that a function $x : [0, T] \rightarrow \mathcal{X}$ is Bochner-integrable. Then there exists a closed-valued correspondence $\Sigma : [0, T] \rightarrow \mathcal{X}_w$ such that

$$\Sigma(t) \subset \Gamma(t, x(t)) \quad \text{for all } t \in [0, T],$$

and the graph $G(\Sigma)$ of Σ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable.

Proof. Let $\{x_n : [0, T] \rightarrow \mathcal{X}\}$ be a sequence of simple functions which satisfies that

$$\|x_n(t) - x(t)\| \rightarrow 0 \quad \text{for each } t \in [0, T] \quad \text{as } n \rightarrow \infty.$$

(For the existence of such a sequence, see Yosida [28] p.133.)

Define a correspondence $\Gamma_n : [0, T] \rightarrow \mathcal{X}_w$ by

$$\Gamma_n : t \mapsto \Gamma(t, x_n(t)); \quad n = 1, 2, \dots$$

Then it can be shown that the graph $G(\Gamma_n)$ of each Γ_n is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable. In order to confirm it, we denote by $\{y_1, y_2, \dots, y_k\}$ the image of $[0, T]$ by the simple function x_n ; i.e.

$$x_n([0, T]) = \{y_1, y_2, \dots, y_k\}.$$

Furthermore if we define a correspondence $\Phi_j : [0, T] \rightarrow \mathcal{X}_w$ ($j = 1, 2, \dots, k$) by

$$\Phi_j : t \mapsto \Gamma(t, y_j),$$

then the graph $G(\Phi_j)$ of Φ_j is obviously $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable. The graph $G(\Gamma_n)$ of Γ_n can be expressed as

$$G(\Gamma_n) = \cup_{j=1}^k G[\Phi_j|_{x_n^{-1}(\{y_j\})}],$$

where $\Phi_j|_{x_n^{-1}(\{y_j\})}$ is the restriction of the correspondence Φ_j to the set $x_n^{-1}(\{y_j\}) = \{t \in [0, T] \mid x_n(t) = y_j\}$. Since $G[\Phi_j|_{x_n^{-1}(\{y_j\})}]$ ($j = 1, 2, \dots, k$) is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable, so is $G(\Gamma_n)$.

Since $\|x_n(t) - x(t)\| \rightarrow 0$ for each $t \in [0, T]$ as $n \rightarrow \infty$, the set $\{x_1(t), x_2(t), \dots, x_n(t)\}$ is weakly compact for each $t \in [0, T]$. Furthermore, by the Assumptions 1-2, the correspondence Γ is compact-valued and u.h.c. in the second variable. Consequently the set

$$\cup_{n=1}^{\infty} \Gamma(t, x_n(t))$$

is relatively compact in $\mathcal{X}_w$ (for each $t \in [0, T]$). Taking account of the fact that the weak topology of a weakly compact subset of a separable Banach space is metrizable, we can conclude, by Baire's category theorem, that the set

$$\Sigma(t) \equiv \cap_{n=1}^{\infty} \overline{\cup_{m=n}^{\infty} \Gamma(t, x_m(t))}^w$$

is non-empty (for each $t \in [0, T]$), where $\overline{}^w$ denotes the closure operation with respect to the weak topology.

The correspondence $\Sigma : [0, T] \rightarrow \mathcal{X}_w$ is closed-valued and its graph is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable. Finally the inclusion

$$\Sigma(t) \subset \Gamma(t, x(t)) \quad \text{for each } t \in [0, T]$$

is clear because Γ is compact-valued and u.h.c. □

We can show the Next lemma in a similar way as in Maruyama[17], taking account of [III] of the Remark on page 4.

LEMMA 2 Let A be a non-empty compact and convex set in $\mathcal{X}_w$, and X a subset of $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ ($p > 1$) defined by

$$X = \{x \in \mathcal{W}^{1,p} \mid \|\dot{x}(t)\| \leq \psi(t) \text{ a.e., } x(0) \in A\},$$

where $\psi \in \mathcal{L}^p([0, T], (0, +\infty))$. Then X is non-empty convex and compact in $\mathcal{X}_w$.

Proof. Since it is obvious that X is non-empty and convex, we have only to show the weak compactness of X .

It is not hard to show the boundedness of X . Let x be any element of X . Then x can be represented in the form

$$x(t) = a + \int_0^t \dot{x}(\tau) d\tau; t \in [0, T]$$

(a is a point of A) by [III] of the Remark on page 3. It follows that

$$\begin{aligned} \|x(t)\| &= \|a + \int_0^t \dot{x}(\tau) d\tau\| \leq \|a\| + \int_0^t \|\dot{x}(\tau)\| d\tau \\ &\leq \|a\| + \int_0^t \psi(\tau) d\tau \leq B + \int_0^T \psi(\tau) d\tau, \end{aligned}$$

where $B = \sup_{a \in A} \|a\| < +\infty$. Consequently we have the evaluation :

$$\sup_{x \in X} \|x\|_p^p \leq [B + \int_0^T \psi(\tau) d\tau]^p \cdot T < +\infty,$$

where $\|\cdot\|_p$ denotes the $\mathcal{L}^p$ -norm. Since the right-hand side is independent of x , X is bounded in $\mathcal{L}^p$. On the other hand, the set $\{\dot{x} \mid x \in X\}$ is also bounded by $\|\psi\|_p$. Therefore we can claim that X is bounded in $\mathcal{W}^{1,p}$.

$\mathcal{W}^{1,p}$ is reflexive because $\mathcal{X}$ is reflexive and $p > 1$. Hence the bounded set X is weakly relatively compact in $\mathcal{W}^{1,p}$.

To show the weak compactness of X , we need only to show the weak closedness of X . However X is weakly closed if and only if X is strongly closed since X is convex. Let $\{x_n\}$ be a sequence in X which strongly converges to x^* in $\mathcal{W}^{1,p}$. Then $\{\dot{x}_n\}$ has a subsequence, say $\{\dot{x}_{n'}\}$, which converges to $\dot{x}^*$ a.e. Since $\|\dot{x}_{n'}(t)\| \leq \psi(t)$ a.e., it follows that

$$\|\dot{x}^*(t)\| \leq \psi(t) \text{ a.e.}$$

Finally it is clear that $x^*(0) \in A$. Then we obtain $x^* \in X$. This proves that X is strongly closed in $\mathcal{W}^{1,p}$. $\square$

We denote by $B(0; \mathcal{X}_w)$ a neighborhood base of the zero element of $\mathcal{X}_w$ which consists of convex sets. The following lemma plays a crucial role in the

subsequent arguments although its proof is easy.

LEMMA 3 Suppose that the Assumptions 1-2 are satisfied. Let (t^*, x^*) be any point of $[0, T] \times \mathcal{X}$. Define, for any $V \in \mathcal{B}(0; \mathcal{X}_w)$, a subset $K(t^*; x^*, V)$, of $[0, T] \times \mathcal{X}$ by

$$K(t^*; x^*, V) = \{(t, x) \in [0, T] \times \mathcal{X} \mid x \in x^* + V, t = t^*\}.$$

Then we have

$$\Gamma(t^*, x^*) = \cap_{V \in \mathcal{B}(0; \mathcal{X}_w)} \overline{\text{co}} \Gamma(K(t^*; x^*, V)).$$

(Here we do not have to distinguish the convex closure with respect to the strong topology and that with respect to weak topology. So I simply denote it by $\overline{\text{co}}$.)

LEMMA 4 Suppose that the Assumptions 1,2 and 4 (with $p > 1$) are satisfied. Let A be a non-empty convex compact subset of $\mathcal{X}_w$. Then the set

$$H \equiv \{(a, x, y) \in A \times X \times X \mid \dot{y}(t) \in \Gamma(t, x(t)) \text{ a.e. and } x(0) = y(0) = a\}$$

is weakly compact in $A \times X \times X$. (The set X is defined in Lemma 2.)

Proof. Since we have already known that $A \times X \times X$ is weakly compact in $\mathcal{X} \times \mathcal{W}^{1,p} \times \mathcal{W}^{1,p}$, it is enough to show that H is a weakly closed subset of $A \times X \times X$.

Since $\mathcal{W}^{1,p}$ is a reflexive Banach space, the dual of which is separable, the weak topology on the bounded set X is metrizable. So we are permitted to use a sequence argument.

Let $\{q_n \equiv (a_n, x_n, y_n)\}$ be a sequence in H which weakly converges to some $q^* = (a^*, x^*, y^*)$ in $A \times X \times X$. We have to show that $q^* \in H$. And it is enough to check that

$$\dot{y}^*(t) \in \Gamma(t, x^*(t)) \text{ a.e.}$$

The set $\{x_n(t)\}$ is relatively compact in $\mathcal{X}_w$ (for each $t \in [0, T]$) since we have the evaluation:

$$\|x_n(t)\| \leq \|a\| + \int_0^t \|\dot{x}_n(\tau)\| d\tau \leq \|a\| + \int_0^T \psi(\tau) d\tau$$

by the Assumption 4. Hence, thanks to Theorem 1, $\{q_n\}$ has a subsequence (no change in notation) such that

$$x_n(t) \rightarrow x^*(t) \text{ uniformly in } \mathcal{X}_w, \text{ and} \quad (1)$$

$$\dot{y}_n(t) \rightarrow \dot{y}^*(t) \text{ weakly in } \mathcal{L}^p. \quad (2)$$

Then applying Mazur's theorem, we can choose, for each $j \in \mathbb{N}$, some finite elements

$$\dot{y}_{n_j+1}, \dot{y}_{n_j+2}, \dots, \dot{y}_{n_j+m(j)}$$

of $\{\dot{y}_n\}$ and numbers

$$\alpha_{ij} \geq 0, 1 \leq i \leq m(j), \sum_{i=1}^{m(j)} \alpha_{ij} = 1$$

such that

$$\left\| \dot{y}^* - \sum_{i=1}^{m(j)} \alpha_{ij} \dot{y}_{n_j+i} \right\|_p \leq \frac{1}{j}, n_{j+1} > n_j + m(j).$$

Denoting

$$\eta_j(t) = \sum_{i=1}^{m(j)} \alpha_{ij} \dot{y}_{n_j+i}(t),$$

we obtain

$$\eta_j(t) \in \text{co}(\cup_{i=1}^{m(j)} \Gamma(t, x_{n_j+i}(t))).$$

Since $\{\eta_j\}$ has a subsequence which converges to $\dot{y}^*$ a.e., we may assume, without loss of generality, that

$$\|\eta_j(t) - \dot{y}^*(t)\| \rightarrow 0 \quad \text{a.e.} \quad (3)$$

On the other hand, for each $V \in \mathcal{B}(0; \mathcal{X}_w)$, there exists some $n_0(V) \in \mathbb{N}$ such that

$$x_n(t) \in x^*(t) + V \quad \text{for all } n \geq n_0(V) \quad \text{and for all } t \in [0, T].$$

That is ,

$$(t, x_n(t)) \in K(t; x^*(t), V) \quad \text{for all } n \geq n_0(V) \quad \text{and for all } t \in [0, T].$$

Hence we have

$$\eta_j(t) \in \text{co}\Gamma(K(t; x^*(t), V)) \quad \text{a.e.}$$

for sufficiently large j . Passing to the limit, we obtain

$$\dot{y}^*(t) \in \overline{\text{co}}\Gamma(K(t; x^*(t), V)) \quad \text{a.e.} \quad (4)$$

by (3). Since (4) holds true for all $V \in B(0; \mathcal{X}_w)$, it follows that

$$y^*(t) \in \bigcap_{V \in B(0; \mathcal{X}_w)} \overline{\text{co}} \Gamma(K(t; x^*(t), V) = \Gamma(t, x^*(t)) \quad \text{a.e.}$$

The last equality in (5) comes from Lemma 3. Thus we have proved that $(a^*, x^*, y^*) \in H$. $\square$

4 Differential Inclusions (2)

$\mathcal{X}$ is still assumed to be a real separable reflexive Banach space in this section.

We are now going to find out a solution of $(*)$ in the Sobolev space $\mathcal{W}^{1,p}([0, T], \mathcal{X})$, $p > 1$. Define a set $\Delta(a)$ in $\mathcal{W}^{1,p}$ by

$$\Delta(a) = \{x \in \mathcal{W}^{1,p} \mid x \text{ satisfies } (*) \text{ a.e.}\}$$

for a fixed $a \in \mathcal{X}$. The following theorem tells us that $\Delta(a) \neq \emptyset$ and that Δ depends continuously, in some sense, upon the initial value a .

THEOREM 2. Suppose that the correspondence Γ satisfies the Assumptions 1-4. Let A be a non-empty, convex and compact subset of $\mathcal{X}_w$. Then

- (i) $\Delta(a^*) \neq \emptyset$ for any $a^* \in A$, and
- (ii) the correspondence $\Delta : A \rightarrow \mathcal{W}^{1,p}$ is compact-valued and u.h.c. on A_w , in the weak topology for $\mathcal{W}^{1,p}$.

The proof is essentially the same as in Maruyama [17].

Proof. (i) Fix any $a^* \in A$. If we define a set $X(a^*) \subset X$ by $X(a^*) = \{x \in X \mid x(0) = a^*\}$, then $X(a^*)$ is convex and weakly compact in $\mathcal{W}^{1,p}$. Furthermore we define a correspondence $\Phi : X(a^*)_w \rightarrow X(a^*)_w$ by

$$\Phi(x) = \{y \in X(a^*) \mid \dot{y}(t) \in \Gamma(t, x(t)) \quad \text{a.e.}\}.$$

Then the problem is simply reduced to finding out a fixed point of Φ .

1° $\Phi(x) \neq \emptyset$ for every $x \in X(a^*)$ — This fact can be proved through the Measurable Selection Theorem.

Let x be any element of $X(a^*)$. Then by Lemma 1, there exists a closed-valued correspondence $\Sigma : [0, T] \rightarrow \mathcal{X}_w$ such that $\Sigma(t) \subset \Gamma(t, x(t))$ for all $t \in [0, T]$, and its graph is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable. We also note that $\mathcal{X}_w$ is a Souslin space. Thanks to Saint-Beuve's measurable selection theorem (Saint-Beuve [20]), Σ admits a $(\mathcal{L}, \mathcal{B}(\mathcal{X}_w))$ -measurable selection $\sigma : [0, T] \rightarrow \mathcal{X}$. Since

$\mathcal{X}$ is separable, σ is $(\mathcal{L}, \mathcal{B}(\mathcal{X}_s))$ -mesurable. (cf. Yosida [28] p.131.) By the Assumption 4, σ is clearly integrable. If we define a function $y : [0, T] \rightarrow \mathcal{X}$ by

$$y(t) = a^* + \int_0^t \sigma(\tau) d\tau,$$

then $y \in \Phi(x)$.

2° Φ is convex-compact-valued. — This is not hard.

3° Φ is u.h.c. — If we define the a^* -selection H_{a^*} of H by $H_{a^*} = \{(a, x, y) \in H \mid a = a^*\}$, then H_{a^*} is obviously weakly compact in $A \times X \times X$. And the graph $G(\Phi)$ of Φ is expressed as $G(\Phi) = \text{proj}_{X \times X} H_{a^*}$, the projection of H_{a^*} into $X \times X$, which is also closed.

Summing up — Φ is convex-compact-valued and u.h.c. Applying now the Fan-Glicksberg Fixed-Point Theorem to the correspondence Φ , we obtain an $x^* \in X(a^*)$ such that $x^* \in \Phi(x^*)$; i.e.

$$\dot{x}^*(t) \in \Gamma(t, x^*(t)) \quad \text{a.e.} \quad \text{and} \quad x^*(0) = a^*.$$

This proves (i).

(ii) Since the compactness of $\Delta(a)$ ($a \in A$) can be verified by applying Mazur's theorem and making use of the Assumptions 1-2, we may omit the details. Hence we have only to show the u.h.c. of Δ . However it is also obvious because the graph $G(\Delta)$ of Δ can be expressed as

$$G(\Delta) = \text{proj}_{A \times X} \{(a, x, y) \in H \mid x = y\},$$

which is closed in $A \times X$.

□

I am much indebted to Castaing-Valadier [7] for various important ideas embodied in the proof of Theorem 2.

Remark. Among other things, the assumption that the set $\Gamma(t, x)$ is always convex is seriously restrictive, especially from the viewpoint of applications. However there seems to be no easy way to wipe out the convexity assumption. (See De Blasi [10] and Tateishi [23].)

Here it may be suggestive for us to glimpse the special case in which Γ is a (single-valued) mapping. A related result was obtained by Szepe [23]. (I am indebted to Professor Tosio Kato for this reference.)

COROLLARY 1. Let $f : [0, T] \times \mathcal{X}_w \rightarrow \mathcal{X}_w$ be a (single-valued) mapping which satisfies the following three conditions.

- (i) The function $x \mapsto f(t, x)$ is continuous for each fixed $t \in [0, T]$.
- (ii) The function $t \mapsto f(t, x)$ is measurable for each fixed $x \in \mathcal{X}$.
- (iii) There exists $\psi \in \mathcal{L}^p([0, T], (0, +\infty))$, $p > 1$ such that $f(t, x) \in S_{\psi(t)}$ for every $(t, x) \in [0, T] \times \mathcal{X}$; i.e. $\sup_{x \in \mathcal{X}} \|f(t, x)\| \leq \psi(t)$ for all $t \in [0, T]$.

Then the differential equation

$$(*) \quad \dot{x} = f(t, x), \quad x(0) = a \text{ (fixed vector in } \mathcal{X} \text{)}$$

has at least a solution in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$. (A solution of $(*)$ is a function $x \in \mathcal{W}^{1,p}$ which satisfies $(*)$ a.e.)

5 Variational problem governed by an Differential Inclusion

Let $\mathcal{X}$ be a real separable reflexive Banach space throughout this section, too. Assume that $u : [0, T] \times \mathcal{X}_w \times \mathcal{X}_s \rightarrow (-\infty, +\infty]$ is a given proper function. Consider a variational problem :

$$(\#) \quad \text{Minimize}_{x \in \Delta(a)} J(x) = \int_0^T u(t, x(t), \dot{x}(t)) dt,$$

where $\Delta(a)$ is the set of all the solutions of the differential inclusion $(*)$ discussed in the preceding sections.

In order to examine the existence of a solution of the problem $(\#)$, we have to check a couple of points as usual ; i.e.

- (I) the compactness of $\Delta(a)$ for some suitable topology, and
- (II) the lower semi-continuity of the functional J for the same topology.

Since we have already proved that $\Delta(a)$ is weakly compact in $\mathcal{W}^{1,p}([0, T], \mathcal{X})$ under certain conditions, we are concentrating on the second point (II) in this section. In this context, the theorem due to Castaing-Clauzure [6] provides the most crucial key. Related results are also obtained by Balder [2], Maruyama [16] and Valadier [25].

DEFINITION Let (Ω, ξ, μ) be a measure space, S a topological space, and $\mathcal{V}$ a real Banach space. A function $f : \Omega \times S \times \mathcal{V} \rightarrow \overline{\mathbb{R}}$ is assumed to be given. We denote by $\mathcal{M}(\Omega, S)$ the set of all the $(\xi \otimes \mathcal{B}(S))$ -measurable functions. ($\mathcal{B}(S)$ denotes the Borel σ -field on S .) f is said to have the lower compactness property if $\{f^-(\omega, \varphi_n(\omega), \theta_n(\omega))\}$ is weakly relatively compact in $\mathcal{L}^1(\Omega, \overline{\mathbb{R}})$ for any sequence $\{(\varphi_n, \theta_n)\}$ in $\mathcal{M}(\Omega, S) \times \mathcal{L}^p(\Omega, \mathcal{V})$ ($p \geq 1$) which satisfies the following three conditions:

- (a) $\{\varphi_n\}$ converges in measure to some $\varphi^* \in \mathcal{M}(\Omega, S)$,
- (b) $\{\theta_n\}$ converges weakly to some $\theta^* \in \mathcal{L}^p(\Omega, \mathcal{V})$, and
- (c) there exists some $C < +\infty$ such that

$$\sup_n \int_{\Omega} f(\omega, \varphi_n(\omega), \theta_n(\omega)) d\mu \leq C.$$

The following theorem is a variation of a result due to Castaing-Clauzure [6] in the spirit of Ioffe [12]. See also Valadier [27].

THEOREM 3 Let (Ω, ξ, μ) be a finite complete measure space, S a metrizable Souslin space, and $\mathcal{V}$ a separable reflexive Banach space. Suppose that a proper function $f : \Omega \times S \times \mathcal{V} \rightarrow \overline{\mathbb{R}}$ satisfies the following conditions:

- (i) f is a normal integrand ; i.e.
 - (a) f is $(\xi \otimes \mathcal{B}(S) \otimes \mathcal{B}(\mathcal{V}), \mathcal{B}(\overline{\mathbb{R}}))$ -measurable, and
 - (b) the function $(\xi, v) \mapsto f(\omega, \xi, v)$ is lower semi-continuous for any fixed $\omega \in \Omega$,
- (ii) the function $v \mapsto f(\omega, \xi, v)$ is convex for any fixed $(\omega, \xi) \in \Omega \times S$, and
- (iii) f has the lower compactness property.

Let $\{\varphi_n\}$ be a sequence in $\mathcal{M}(\Omega, S)$ which converges in measure to some $\varphi^* \in \mathcal{M}(\Omega, S)$. Let $\{\theta_n\}$ be a sequence in $\mathcal{L}^p(\Omega, \mathcal{V})$ ($1 \leq p < +\infty$) which converges weakly to some $\theta^* \in \mathcal{L}^p(\Omega, \mathcal{V})$. Then we have

$$\int_{\Omega} f(\omega, \varphi^*(\omega), \theta^*(\omega)) d\mu \leq \liminf_n \int_{\Omega} f(\omega, \varphi_n(\omega), \theta_n(\omega)) d\mu.$$

Remark 1° A normal integrand $f : \Omega \times S \times \mathcal{V} \rightarrow \overline{\mathbb{R}}$ which also satisfies the condition (ii) is called a *convex normal integrand*.

2° Ioffe [8] established a fundamental theorem on the lower semi-continuity of a nonlinear integral functional as above in the case both of S and $\mathcal{V}$ are finite dimensional Euclidean spaces. Theorem 3 is an extension of Ioffe's result to the case of nonlinear integral functional defined on the space of Bochner integrable functions.

LEMMA 5 Suppose that the Assumptions 1-4 are satisfied. Let $\{x_n\}$ be a sequence in $\Delta(a) \subset \mathcal{W}^{1,p}([0, T], \mathcal{X})$ ($p > 1$). Let $u : [0, T] \times \mathcal{X}_w \times \mathcal{X}_s \rightarrow \overline{\mathbb{R}}$ be a proper convex normal integrand with the lower compactness property. Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and $x^* \in \Delta(a)$ such that

$$J(x^*) \leq \liminf_n J(z_n), \quad (1)$$

where

$$J(x) = \int_0^T u(t, x(t), \dot{x}(t)) dt.$$

Proof. By the Assumption 4, all the images of x_n 's are contained in some closed ball $\overline{B}$ with the center 0 ; i.e.

$$x_n(t) \in \overline{B} \quad \text{for all } t \in [0, T] \quad \text{and } n.$$

Hence we may restrict the domain of u to $[0, T] \times \overline{B}_w \times \mathcal{X}$, provided that the sequence $\{x_n\}$ is concerned. Denoting $\overline{u} = u|_{[0, T] \times \overline{B} \times \mathcal{X}}$, (restriction of u to $[0, T] \times \overline{B} \times \mathcal{X}$) we have to show that there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^* \in \Delta(a)$ such that

$$\int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t)) dt \leq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t)) dt,$$

which is equivalent to (1).

The set $\overline{B}$ endowed with the weak topology is metrizable and compact. Hence it is a Polish space. According to Theorem 1, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and $x^* \in \mathcal{W}^{1,p}([0, T], \mathcal{X})$ such that

- (a) $z_n \rightarrow x^*$ uniformly in $\overline{B}_w$, and
- (b) $\dot{z}_n \rightarrow \dot{x}^*$ weakly in $\mathcal{L}^p([0, T], \mathcal{X})$.

(a) implies, of course, that $z_n \rightarrow x^*$ in measure. Thus applying Theorem 3, we obtain the relation

$$\int_0^T \overline{u}(t, x^*(t), \dot{x}^*(t)) dt \leq \liminf_n \int_0^T \overline{u}(t, z_n(t), \dot{z}_n(t)) dt.$$

Finally we have to prove that $x^* \in \Delta(a)$. By (a), it follows that

$$\lim_{n \rightarrow \infty} \langle z_n(t), \eta(t) \rangle = \langle x^*(t), \eta(t) \rangle$$

for any $t \in [0, T]$ and $\eta \in \mathcal{L}^q([0, T], \mathcal{X}')$, where $1/p + 1/q = 1$. Since $z_n(t) \in \overline{B}$, there exists some positive constant $C < \infty$ such that

$$| \langle z_n(t), \eta(t) \rangle | \leq C \| \eta(t) \|.$$

Hence we have, by the Bounded Convergence Theorem, that

$$\lim_{n \rightarrow \infty} \int_0^T \langle z_n(t), \eta(t) \rangle dt = \int_0^T \langle x^*(t), \eta(t) \rangle dt$$

for any $\eta \in \mathcal{L}^q([0, T], \mathcal{X}')$.

This proves that $z_n \rightarrow x^*$ weakly in $\mathcal{L}^p$.

Combining this result with (b), we can conclude that $\{z_n\}$ weakly converges to x^* in $\mathcal{W}^{1,p}$. Since $\Delta(a)$ is weakly closed, $x^* \in \Delta(a)$. $\square$

Let $\{x_n\}$ be a minimizing sequence of the problem (#). Then, by Lemma 5, $\{x_n\}$ has a subsequence (without change of notation) such that

$$J(x^*) \leq \liminf_n J(x_n)$$

for some $x^* \in \Delta(a)$. It is also obvious that

$$\inf_{x \in \Delta(a)} J(x) = \liminf_n j(x_n) \leq J(x^*).$$

Thus we have proved that x^* is a solution of the problem (#). Summing up

THEOREM 4 Suppose that Assumptions 1-4 with $p > 1$ are satisfied for a correspondence $\Gamma : [0, T] \times \mathcal{X} \rightarrow \mathcal{X}$. Furthermore let $u : [0, T] \times \mathcal{X}_w \times \mathcal{X}_s \rightarrow \bar{\mathbb{R}}$ be a normal convex integrand with the lower compactness property. Then the problem (#) has a solution.

Appendix

Banach Space-valued Sobolev Spaces

This appendix aims at a brief summary of the concepts and basic facts in the theory of Banach space-valued Sobolev spaces. (cf. Schwartz [22], Barbu [3].)

1. Let $p = (p_1, p_2, \dots, p_\ell)$ be an ℓ -tuple of non-negative integers. The number $|p| = p_1 + p_2 + \dots + p_\ell$ is called the order of p . We denote by D^p the differential operator

$$D^p = \frac{\partial^{p_1+p_2+\dots+p_\ell}}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_\ell^{p_\ell}}$$

Let Ω be an open set of $\mathbb{R}^\ell$ and K a compact subset of Ω . We denote by $\mathcal{D}_K(\Omega)$ the set of all the infinitely differentiable real-valued functions $\varphi : \Omega \rightarrow \mathbb{R}$ whose supports are contained in K ; i.e.

$$\mathcal{D}_K(\Omega) = \{\varphi \in \mathcal{C}^\infty(\Omega, \mathbb{R}) \mid \text{supp } \varphi \subset K\}.$$

Under the topology generated by the family of seminorms :

$$p_{K,m}(\varphi) = \sup_{\substack{x \in K \\ |p| \leq m}} |D^p \varphi(x)|, \quad m = 1, 2, \dots,$$

$\mathcal{D}_K(\Omega)$ becomes a locally convex Hausdorff topological vector space (LCHTVS).

The space $\mathcal{D}(\Omega) = \cup\{\mathcal{D}_K(\Omega) \mid K \text{ is a compact subset of } \Omega\}$ is also a vector space. And the space $\mathcal{D}(\Omega)$ endowed with the strict inductive limit topology defined by $\{\mathcal{D}_K(\Omega) \mid K \text{ is a compact subset of } \Omega\}$ is a LCHTVS, called the **Schwartz space**. It is well-known that a net $\{\varphi_\alpha\}$ in $\mathcal{D}(\Omega)$ converges to some $\varphi^* \in \mathcal{D}(\Omega)$ if and only if there exists some compact subset K of Ω with

$$\text{supp } \varphi_\alpha \subset K \quad \text{for all } \alpha,$$

and

$$D^p \varphi_\alpha \rightarrow D^p \varphi^* \quad \text{uniformly on } \Omega$$

for every index $p = (p_1, p_2, \dots, p_\ell)$

2. Let $\mathcal{X}$ be a real Banach space. Any continuous linear operator $S : \mathcal{D}(\Omega) \rightarrow \mathcal{X}$ is called a $\mathcal{X}$ -valued distribution and the set of all the $\mathcal{X}$ -valued distributions is denoted by $\mathcal{D}'(\Omega \mid \mathcal{X})$.

If $f : \Omega \rightarrow \mathcal{X}$ is a locally Bochner-integrable function, the operator $S_f : \mathcal{D}(\Omega) \rightarrow \mathcal{X}$ defined by

$$S_f : \varphi \mapsto \int_{\Omega} f(\omega) \varphi(\omega) d\omega, \quad \varphi \in \mathcal{D}(\Omega)$$

is an $\mathcal{X}$ -valued distribution. ($d\omega$ is the Lebesgue measure on Ω .) Identifying f and S_f , we can safely say that any locally Bochner-integrable function is an $\mathcal{X}$ -valued distribution.

The value of $S \in \mathcal{D}'(\Omega \mid \mathcal{X})$ at $\varphi \in \mathcal{D}(\Omega)$ is sometimes denoted by $\langle S, \varphi \rangle$ instead of $S(\varphi)$.

Let S be an $\mathcal{X}$ -valued distribution and D^p an differential operator. Then the operator $D^p S : \mathcal{D}(\Omega) \rightarrow \mathcal{X}$ defined by

$$\varphi \mapsto (-1)^{|p|} \langle S, D^p \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega)$$

is also an $\mathcal{X}$ -valued distribution, called the **distributional derivative** (or the **derivative in sense of distribution**) of S ; i.e.

$$\langle D^p S, \varphi \rangle = (-1)^{|p|} \langle S, D^p \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega).$$

An $\mathcal{X}$ -valued distribution is infinitely differentiable in the sense of distribution.

3. The $\mathcal{X}$ -valued Sobolev space $\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ ($p \geq 1$) is the set of all the functions $f : \Omega \rightarrow \mathcal{X}$ such that its distributional derivative $D^s f$ exists and belongs to $\mathcal{L}^p(\Omega, \mathcal{X})$ for all $s = (s_1, s_2, \dots, s_\ell)$ with $|s| \leq k$.

$\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ is clearly a vector space. In fact, it becomes a Banach space under the norm :

$$\|f\|_{k,p} = \left(\sum_{|s| \leq k} \int_{\Omega} \|D^s f(\omega)\|^p d\omega \right)^{1/p}$$

If $\mathcal{X}$ is a Hilbert space and $p = 2$, $\mathcal{W}^{k,2}(\Omega, \mathcal{X})$ is also a Hilbert space under the inner product :

$$\langle f, g \rangle_{k,p} = \sum_{|s| \leq k} \int_{\Omega} \langle D^s f(\omega), D^s g(\omega) \rangle d\omega.$$

Finally, we state three results which are to play some roles in this paper.

FACT 1 If $\mathcal{X}$ is a separable Banach space, then $\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ ($p \geq 1$) is also separable.

FACT 2 If $\mathcal{X}$ is a separable reflexive Banach space and $p > 1$, then $\mathcal{W}^{k,p}(\Omega, \mathcal{X})$ is reflexive.

Let $\Omega = (0, T)$. We denote by $\mathcal{W}^{k,p}([0, T], \mathcal{X})$ the set of all the functions $f : [0, T] \rightarrow \mathcal{X}$ such that

- a The derivatives $D^j f$ (defined a.e.) are absolutely continuous for $j = 1, 2, \dots, k-1$, and
- b $D^j f \in \mathcal{L}^p([0, T], \mathcal{X})$ for $j = 0, 1, 2, \dots, k$.

FACT 3 Let $\mathcal{X}$ be a Banach space with the Radon-Nikodým property. Then the following two statements are equivalent for a function $f \in \mathcal{L}^p([0, T], \mathcal{X})$ ($p \geq 1$).

- (i) $f \in \mathcal{W}^{k,p}([0, T], \mathcal{X})$.
- (ii) There exists some $f_1 \in \mathcal{W}^{k,p}([0, T], \mathcal{X})$ such that $f(t) = f_1(t)$ a.e. $\omega \in (0, T)$.

Thus we may assume, without loss of generality, that each element of $\mathcal{W}^{k,p}((0, T), \mathcal{X})$ is defined on the closed interval $[0, T]$ rather than $(0, T)$. When we wish to emphasize this aspect, we use the notation $\mathcal{W}^{k,p}([0, T], \mathcal{X})$ rather than $\mathcal{W}^{k,p}((0, T), \mathcal{X})$.

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