

An extension of the univalence criteria of Nehari and Ozaki

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Abstract

In this paper, we obtain a sufficient condition for the univalence of analytic functions in the open unit disk $\mathbb{U}$. This condition involves two arbitrary functions $g(z)$ and $h(z)$ analytic in $\mathbb{U}$. Replacing $g(z)$ and $h(z)$ by some particular functions, we find the well-known conditions for univalence established by Z.Nehari (Bull. Amer. Math. Soc. 55(1949)) and S.Ozaki (Proc. Amer. Math. Soc. 33(1972)). Likewise we find other new sufficient conditions.

1 Introduction

We denote by $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$ the disk of z -plane, where $r \in (0, 1]$, $\mathbb{U}_1 = \mathbb{U}$ and $I = [0, \infty)$. Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in $\mathbb{U}$ with the normalizations $f(0) = 0$ and $f'(0) = 1$. In the present paper, we consider the following conditions for univalence of functions $f(z)$ belonging to the class $\mathcal{A}$.

Theorem 1.1. ([1]) *Let $f(z) \in \mathcal{A}$. If, for all $z \in \mathbb{U}$, $f(z)$ satisfies*

$$|\{f; z\}| \leq \frac{2}{(1 - |z|^2)^2}, \quad (1.1)$$

where

$$\{f; z\} = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2, \quad (1.2)$$

then the function $f(z)$ is univalent in $\mathbb{U}$.

Theorem 1.2. ([2]) *Let $f(z) \in \mathcal{A}$. If, for all $z \in \mathbb{U}$, $f(z)$ satisfies*

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < 1, \quad (1.3)$$

then the function $f(z)$ is univalent in $\mathbb{U}$.

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Example 1.1. If we take Koebe function $f(z) = \frac{z}{(1-z)^2}$ which is the extremal function for the class of starlike functions in $\mathbb{U}$, then

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| = |-z^2| < 1 \quad (z \in \mathbb{U}).$$

2 Preliminaries

Our considerations are based on the theory of Löwner chains. We first recall here the following basic result of this theory by Pommerenke.

Theorem 2.1. ([4]) Let $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots$, $a_1(t) \neq 0$ be analytic in $\mathbb{U}_r$ for all $t \in I$, locally absolutely continuous in I , and locally uniform with respect to $\mathbb{U}_r$. For almost all $t \in I$ suppose that

$$z \frac{\partial L(z, t)}{\partial z} = p(z, t) \frac{\partial L(z, t)}{\partial t} \quad (\forall z \in \mathbb{U}_r),$$

where $p(z, t)$ is analytic in $\mathbb{U}$ and satisfies the condition $\operatorname{Re} p(z, t) > 0$ for all $z \in \mathbb{U}$, $t \in I$. If $|a_1(t)| \rightarrow \infty$ for $t \rightarrow \infty$ and $\{L(z, t)/a_1(t)\}$ forms a normal family in $\mathbb{U}_r$, then, for each $t \in I$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk $\mathbb{U}$.

3 Main results

Main theorem of our paper is contained in

Theorem 3.1. Let $f(z) \in \mathcal{A}$. If, for some analytic functions $g(z) = 1 + b_1 z + \dots$ and $h(z) = c_0 + c_1 z + \dots$ in $\mathbb{U}$, the following inequalities

$$\left| \frac{f'(z)}{g(z)} - 1 \right| < 1, \quad (3.1)$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{g(z)} - 1 \right) |z|^4 + z(1 - |z|^2)|z|^2 \left(2 \frac{f'(z)h(z)}{g(z)} + \frac{g'(z)}{g(z)} \right) \right. \\ & \quad \left. + z^2(1 - |z|^2)^2 \left(\frac{f'(z)h(z)^2}{g(z)} + \frac{g'(z)h(z)}{g(z)} - h'(z) \right) \right| \leq |z|^2 \end{aligned} \quad (3.2)$$

hold true for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

Proof. Let us consider the function $h_1(z, t)$ given by

$$h_1(z, t) = 1 + (e^t - e^{-t})zh(e^{-t}z).$$

For all $t \in I$ and $z \in \mathbb{U}$ we have $e^{-t}z \in \mathbb{U}$ and from the analyticity of $h(z)$ in $\mathbb{U}$ it follows that $h_1(z, t)$ is also analytic in $\mathbb{U}$. Since $h_1(0, t) = 1$, there exists a disk $\mathbb{U}_r$, $0 < r < 1$ in which $h_1(z, t) \neq 0$ for all $t \in I$. Then the function $L(z, t)$ defined by

$$L(z, t) = f(e^{-t}z) + \frac{(e^t - e^{-t})zg(e^{-t}z)}{1 + (e^t - e^{-t})zh(e^{-t}z)}$$

is analytic in $\mathbb{U}_r$ for all $t \in I$ and has the following form

$$L(z, t) = a_1(t)z + a_2(t)z^2 + \dots,$$

where $a_1(t) = e^t$, $a_1(t) \neq 0$ for all $t \in I$ and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$.

From the analyticity of $L(z, t)$ in $\mathbb{U}_r$, it follows that there exists a number r_1 , $0 < r_1 < r$, and a constant $K = K(r_1)$ such that

$$|L(z, t)/a_1(t)| < K \quad (\forall z \in \mathbb{U}_{r_1}, t \in I).$$

In consequence, the family $\{L(z, t)/a_1(t)\}$ is normal in $\mathbb{U}_{r_1}$. From the analyticity of $\frac{\partial L(z, t)}{\partial t}$, for all fixed numbers $T > 0$ and r_2 , $0 < r_2 < r_1$, there exists a constant $K_1 > 0$ (that depends on T and r_2) such that

$$\left| \frac{\partial L(z, t)}{\partial t} \right| < K_1 \quad (\forall z \in \mathbb{U}_{r_2}, t \in [0, T]).$$

It follows that the function $L(z, t)$ is locally absolutely continuous in I , locally uniform with respect to $\mathbb{U}_{r_2}$. Let us define the functions $p(z, t)$ and $w(z, t)$ by

$$p(z, t) = z \frac{\partial L(z, t)}{\partial z} \bigg/ \frac{\partial L(z, t)}{\partial t}$$

and

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}.$$

Then the function $p(z, t)$ is analytic in $\mathbb{U}_{r_3}$, $0 < r_3 < r_2$, and the function $p(z, t)$ has an analytic extension with positive real part in $\mathbb{U}$, for all $t \in I$, if the function $w(z, t)$ can be continued analytically in $\mathbb{U}$ and $|w(z, t)| < 1$ for all $z \in \mathbb{U}$ and $t \in I$.

After simple computation, we obtain that

$$\begin{aligned} w(z, t) = & \left(\frac{f'(e^{-t}z)}{g(e^{-t}z)} - 1 \right) e^{-2t} + (1 - e^{-2t})e^{-t}z \left(\frac{2f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)}{g(e^{-t}z)} \right) \\ & + (1 - e^{-2t})^2 z^2 \left(\frac{f'(e^{-t}z)h(e^{-t}z)^2}{g(e^{-t}z)} + \frac{g'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} - h'(e^{-t}z) \right). \end{aligned} \quad (3.3)$$

From (3.1) and (3.2), we deduce that $g(z) \neq 0$ for all $z \in \mathbb{U}$ and then the function $w(z, t)$ is analytic in $\mathbb{U}$. In view of (3.1) and (3.3), we have

$$w(0, t) = 0 \quad \text{and} \quad |w(z, 0)| = \left| \frac{f'(z)}{g(z)} - 1 \right| < 1. \quad (3.4)$$

If $t > 0$ is a fixed number and $z \in \mathbb{U}$, $z \neq 0$, then the function $w(z, t)$ is analytic in $\bar{\mathbb{U}}$ because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{\mathbb{U}}$, and it is known that

$$|w(z, t)| = \max_{|\zeta|=1} |w(\zeta, t)| = |w(e^{i\theta}, t)|, \quad \theta = \theta(t) \in \mathcal{R}. \quad (3.5)$$

Let us denote by $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and, from (3.3), we get

$$|w(e^{i\theta}, t)| = \left| \left(\frac{f'(u)}{g(u)} - 1 \right) |u|^2 + (1 - |u|^2) u \left(\frac{2f'(u)h(u)}{g(u)} + \frac{g'(u)}{g(u)} \right) \right. \\ \left. + (1 - |u|^2)^2 \frac{u^2}{|u|^2} \left(\frac{f'(u)h(u)^2}{g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u) \right) \right|.$$

Since $u \in \mathbb{U}$, the relation (3.2) implies $|w(e^{i\theta}, t)| \leq 1$ and, from (3.4) and (3.5), we conclude that $|w(z, t)| < 1$ for all $z \in \mathbb{U}$ and $t \in I$. This gives us that $L(z, t)$ is the Löwner chain and hence the function $L(z, 0) = f(z)$ is univalent in $\mathbb{U}$. $\square$

We can get some corollaries for special cases of functions $g(z)$ and $h(z)$. So in the particular case $g(z) = f'(z)$ as a direct consequence of Theorem 3.1, we get

Theorem 3.2. *Let $f \in \mathcal{A}$. If, for an analytic function $h(z) = c_0 + c_1 z + \dots$ in $\mathbb{U}$, $f(z)$ satisfies*

$$\left| (1 - |z|^2) |z|^2 \left(2h(z) + \frac{f''(z)}{f'(z)} \right) \right. \\ \left. + z(1 - |z|^2)^2 \left(h(z)^2 + \frac{f''(z)h(z)}{f'(z)} - h'(z) \right) \right| \leq |z| \quad (3.6)$$

for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

If we take

$$h(z) = -\frac{1}{2} \frac{f''(z)}{f'(z)} \quad (3.7)$$

in Theorem 3.2, then we have

Corollary 3.1. ([1]) *If $f(z) \in \mathcal{A}$ satisfies the inequality (1.1) for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.*

Proof. For the function $h(z)$ defined by (3.7), the Schwartzian derivative (1.2) shows that

$$h(z)^2 + \frac{f''(z)h(z)}{f'(z)} - h'(z) = \frac{1}{2} \left[\frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right] = \frac{1}{2} \{ f; z \}.$$

and then the inequality (3.6) becomes (1.1). $\square$

In the particular case $g(z) = \left(\frac{f(z)}{z} \right)^2$ in Theorem 3.1, we have

Theorem 3.3. Let $f(z) \in \mathcal{A}$. If, for an analytic function $h(z) = c_0 + c_1z + \dots$ in $\mathbb{U}$, $f(z)$ satisfies

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < 1 \quad (3.8)$$

and

$$\begin{aligned} & \left| \left(\frac{z^2 f'(z)}{f(z)^2} - 1 \right) |z|^4 + 2z(1 - |z|^2)|z|^2 \left(\frac{z^2 f'(z)h(z)}{f(z)^2} + \frac{f'(z)}{f(z)} - \frac{1}{z} \right) \right. \\ & \left. + z^2(1 - |z|^2)^2 \left[\frac{z^2 f'(z)h(z)^2}{f(z)^2} + 2h(z) \left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) - h'(z) \right] \right| \leq |z|^2 \end{aligned} \quad (3.9)$$

for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

We remark that the inequality (3.8) is just the inequality (1.3) and we will get Ozaki's univalent criterion for a particular choice of the function $h(z)$. So, if we take in Theorem 3.3

$$h(z) = \frac{1}{z} - \frac{f(z)}{z^2}, \quad (3.10)$$

then we obtain

Corollary 3.2. ([2]) If $f(z) \in \mathcal{A}$ satisfies the inequality (1.3) for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

Proof. For the function $h(z)$ defined by (3.10), we see that

$$\frac{z^2 f'(z)h(z)}{f(z)^2} + \frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{z f'(z)}{f(z)^2} - \frac{1}{z}$$

and

$$\frac{z^2 f'(z)h(z)^2}{f(z)^2} + 2h(z) \left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) - h'(z) = \frac{f'(z)}{f(z)^2} - \frac{1}{z^2}.$$

The inequality (3.9) becomes

$$\left| \left(\frac{z^2 f'(z)}{f(z)^2} - 1 \right) (|z|^4 + 2|z|^2(1 - |z|^2) + (1 - |z|^2)^2) \right| \leq |z|^2,$$

and then

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| \leq |z|^2. \quad (3.11)$$

It is easy to prove that if the inequality (1.3) is true, then the inequality (3.11) is also true. Indeed, if we put

$$w(z) = \frac{z^2 f'(z)}{f(z)^2} - 1,$$

then the function $w(z)$ is analytic in $\mathbb{U}$ and, since $f(z) \in \mathcal{A}$, we observe that

$$w(z) = d_2 z^2 + d_3 z^3 + \dots,$$

which shows that $w(0) = w'(0) = 0$. By inequality (1.3), we have $|w(z)| < 1$. Thus the Schwartz's lemma gives us that $|w(z)| < |z|^2$. $\square$

Finally, we give an example for Corollary 3.2.

Example 3.1. *Let us consider the function $f(z)$ given by*

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} \frac{1}{n(n^2-1)} z^n}.$$

Then we have that

$$\frac{z^2 f'(z)}{f(z)^2} - 1 = - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} z^n,$$

which gives that

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1.$$

Therefore, the function $f(z)$ is univalent in $\mathbb{U}$.

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