

Threefolds whose canonical bundles are not numerically effective

by

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In this note, we announce an application of the previous paper [3] with some examples. The proof will be published elsewhere.

§1. Announcement.

We assume that  $k$  is an algebraically closed field of characteristic 0 and  $X$  is a non-singular projective 3-fold over  $k$  whose canonical bundle  $K_X$  is not numerically effective. We use the terminology of [3]. By Corollary & [3],  $X$  has an extremal ray  $R$ , which we fix in this section.

Theorem 1. There exists a morphism  $\phi : X \longrightarrow Y$  to a projective variety  $Y$  such that (1)  $\phi_* \mathcal{O}_X = \mathcal{O}_Y$ , and (2) for any irreducible curve  $C$  in  $X$ ,  $[C] \in R$  if and only if  $\dim \phi(C) = 0$ . Furthermore, such a  $\phi$  is unique up to an isomorphism.

The structure of this  $\phi$  is given by the following theorems.

Theorem 2. The extremal ray  $R$  is not numerically effective if and only if  $\dim Y = 3$ . If these conditions are satisfied, then there exists an irreducible divisor  $D$  of  $X$  such that  $X$  is the blowing-up of  $Y$  by the ideal defining  $\phi(D)$  (given the reduced structure), and we have either

(1)  $\phi(D)$  is a non-singular curve and  $Y$  is non-singular;  
 $\phi|_D : D \longrightarrow \phi(D)$  is a  $\mathbb{P}^1$ -bundle and  $(D, \phi^{-1}(n)) = -1$  for any  $n \in \phi(D)$ ,

(2)  $Q = \phi(D)$  is a point and  $Y$  is non-singular;  $D \cong \mathbb{P}^2$

and  $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-1)$ ,

(3)  $Q = \phi(D)$  is an ordinary double point of  $Y$ ;  
 $D \cong \mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathcal{O}_D(D) \cong p_1^* \mathcal{O}_{\mathbb{P}}(-1) \otimes p_2^* \mathcal{O}_{\mathbb{P}}(-1)$ , where  $p_i$  is the  $i$ -th projection,

(4)  $Q = \phi(D)$  is a double point of  $Y$ ;  $D$  = an irreducible reduced singular quadric surface  $S$  in  $\mathbb{P}^3$ ,  $\mathcal{O}_D(D) \cong \mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}}(-1)$ , or

(5)  $Q = \phi(D)$  is a quadruple point of  $Y$ ;  $D \cong \mathbb{P}^2$ ,  
 $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-2)$ .

Let  $\mathcal{O}_{Y,Q}$  be the local ring of  $Y$  at  $Q$  for cases (3), (4), and (5) in Theorem 2. Then we have

Theorem 3. (1) The divisor class group of  $\mathcal{O}_{Y,Q}$  is 0 in cases (3) and (4), and  $\mathbb{Z}/2\mathbb{Z}$  in case (5), and

(2) the completion  $\mathcal{O}_{Y,Q}^\wedge$  of  $\mathcal{O}_{Y,Q}$  is given by

$$\mathcal{O}_{Y,Q}^\wedge \cong \begin{cases} k[[x,y,z,u]]/(x^2 + y^2 + z^2 + u^2) & \text{case (3),} \\ k[[x,y,z,u]]/(x^2 + y^2 + z^2 + u^3) & \text{case (4),} \\ k[[x,y,z]]^{(2)} & \text{case (5),} \end{cases}$$

where  $k[[x,y,z]]^{(2)}$  is the invariant subring of  $k[[x,y,z]]$  under the action of the involution  $(x,y,z) \mapsto (-x,-y,-z)$ .

The remaining cases are treated by

Theorem 4. If  $R$  is numerically effective, then  $Y$  is non-singular,  $\rho(X) = \rho(Y) + 1$ , and we have either

(1)  $\dim Y = 2$ , and for an arbitrary geometric point  $\eta$  of  $Y$ , the scheme-theoretic fiber  $X_\eta$  is isomorphic to a conic of  $\mathbb{P}_{k(\eta)}^2$ , where  $k(\eta)$  is the field of  $\eta$  (i.e.  $X_\eta$  is isomorphic to either a smooth conic, a reducible conic, or a double line,)

- (2)  $\dim Y = 1$ , and for an arbitrary geometric point  $\eta$  of  $Y$ ,  $X_\eta$  is an irreducible reduced surface such that  $\omega_{X_\eta}^{-1}$  is ample, or
- (3)  $\dim Y = 0$ , and  $X$  is a Fano 3-fold, (these 3-folds are classified by Iskovski [2].)

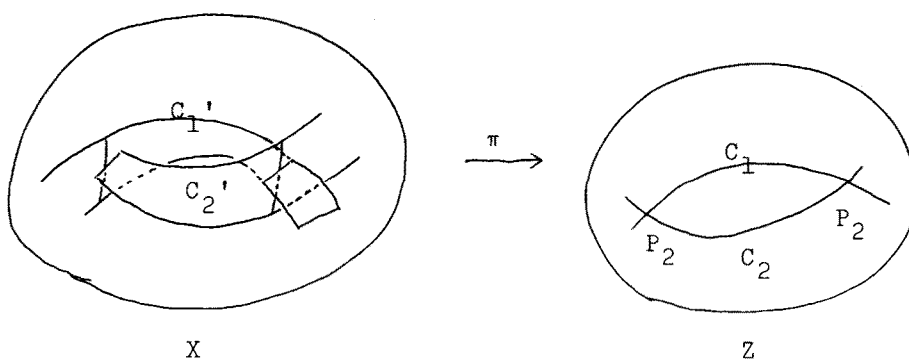
## §2. Exceptional divisors.

The most interesting part of section 1 is Theorem 2. Examples for Theorem 2 can be given by considering birational morphisms.

Theorem 5. Let  $\pi : X \longrightarrow Z$  be a birational morphism (which is not an isomorphism) of non-singular projective 3-folds. Then  $X$  contains an extremal rational curve  $\ell$  such that (1)  $\dim \pi(\ell) = 0$  and (2)  $\ell$  is not numerically effective. Hence the exceptional set of  $\pi$  contains a divisor described in Theorem 2.

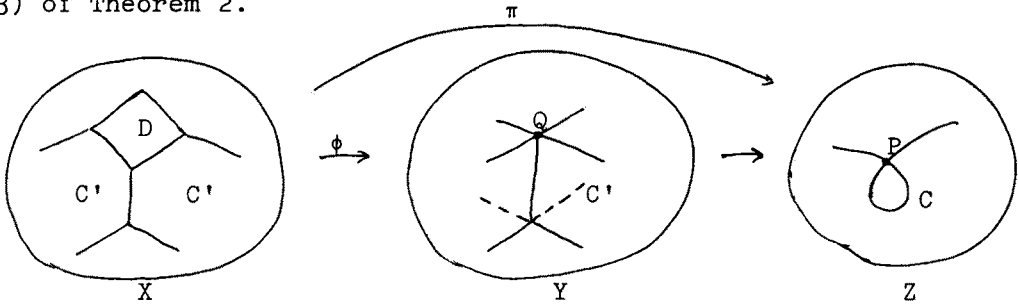
Examples 6. Let  $Z$  be a non-singular projective 3-fold.

(1): Let  $C_1$  and  $C_2$  be non-singular projective curves in  $Z$  intersecting transversally at 2 points  $P_1$  and  $P_2$ . If we operate Hironaka's twisted blowing-up to  $C_1$  and  $C_2$  (e.g. blowing up  $C_1$  first near  $P_1$  and  $C_2$  first near  $P_2$ ), then the "blowing up"  $\pi : X \longrightarrow Z$  does not have a divisor described in Theorem 2.



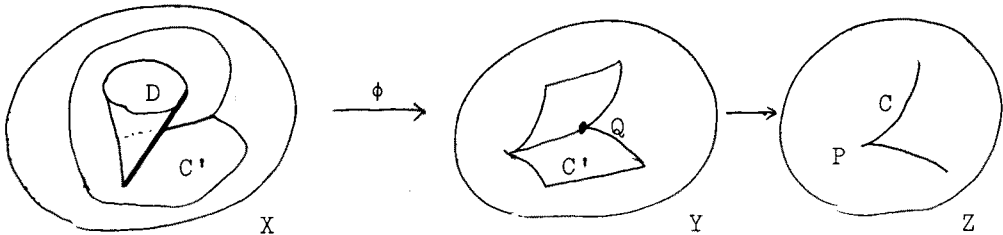
However, this does not contradict our theorems, because our  $X$  is not projective.

(2): Let  $C$  be an irreducible projective curve in  $Z$  with one ordinary double point  $P$  as singularities. If we blow up  $C$ , then the blown-up variety  $Y$  has one ordinary double point  $Q$  lying over  $P$  as singularities. If we resolve the singularity by blowing up  $Q$  and get a smooth 3-fold  $X$ ,  $\pi : X \rightarrow Z$  and  $\phi : X \rightarrow Y$ , then  $D = \phi^{-1}(Q)$  is the divisor described in case (3) of Theorem 2.

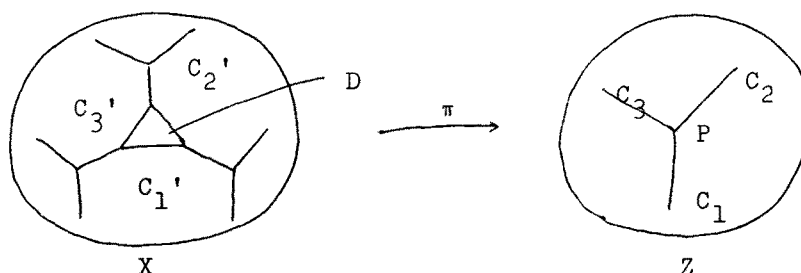


We remark that we can not start with an arbitrary ordinary double point because of Theorem 3, (1).

(3): Let  $C$  be an irreducible projective curve in  $Z$  with one ordinary cusp  $P$  as singularities. If we blow up  $C$ , the blown-up variety  $Y$  has one double point  $Q$  lying over  $P$  as singularities which falls in case (4) of Theorem 2. If we blow up  $Q$  to get a smooth 3-fold  $X$ ,  $\phi : X \rightarrow Y$  and  $\pi : X \rightarrow Z$ , then  $D = \phi^{-1}(Q)$  is the divisor described in case (4).



(4): Let  $C_1, C_2$ , and  $C_3$  be non-singular projective curves in  $Z$  intersecting transversally at one point  $P$ . If we operate Hironaka's modification in [1], we get a smooth <sup>projective</sup> 3-fold  $X$  and  $\pi : X \rightarrow Z$ , and  $D$ , the divisorial part of  $\pi^{-1}(P)$ , is the divisor given in case 5 of Theorem 2.



We will finish this section by proving Theorem 5. The proof consists of a few easy lemmas. We keep the notation of Theorem 5 till the end of this section.

Lemma 7.  $\pi_* : N(X) \rightarrow N(Z)$  has the property

$$\pi_* \overline{NE}(X) \subseteq \overline{NE}(Z).$$

Indeed, we have  $\pi_* NE(X) \subseteq NE(Z)$  by the definition of  $\pi_*$ , which implies Lemma 7 by continuity of  $\pi_*$ .

Lemma 8. There is an effective 1-cycle  $C$  on  $X$  such that  $\pi_* C = 0$  and  $(C \cdot c_1(X)) > 0$ .

Proof. Let  $E$  be the effective divisor on  $X$  such that  $\text{Supp } E$  is the exceptional set of  $\pi$  and  $K_X = \pi^* K_Z + E$ . We treat two cases.

Case 1:  $\dim \pi(\text{Supp } E) = 1$ .

By Bertini's theorem, there is a smooth hyperplane section  $L$

of  $Z$  such that  $\pi^{-1}(L)$  is irreducible and non-singular. Then we have

$$\begin{aligned} (K_X.E.\pi^*L) &= (\pi^*K_Z.E.\pi^*L) + (E^2.\pi^*L) \\ &= (E.\pi^*(K_Z.L)) + (\mathcal{O}_{\pi^{-1}(L)}(E)^2)_{\pi^{-1}(L)} \\ &= (\pi_*E.K_Z.L) + (\mathcal{O}_{\pi^{-1}(L)}(E)^2)_{\pi^{-1}(L)}. \end{aligned}$$

Since  $\pi_*E = 0$ , we have  $(\pi_*E.K_Z.L) = 0$ . We have

$$(\mathcal{O}_{\pi^{-1}(L)}(E)^2)_{\pi^{-1}(L)} < 0 \text{ because } C = E.\pi^*L (\neq 0, \text{ since}$$

$\dim \pi(\text{Supp } E) = 1$ ) is an exceptional divisor of  $\pi^{-1}(L) \rightarrow L$ .

Hence  $\pi_*C = 0$  and  $(K_X.C) < 0$ .

Case 2:  $\dim \pi(\text{Supp } E) = 0$ .

Let  $M$  be a smooth hyperplane section of  $X$ , hence  $M$  and  $E$  intersect properly and  $M.E \neq 0$ . Then we have

$$\begin{aligned} (K_X.E.M) &= (\pi^*K_Z.E.M) + (E^2.M) \\ &= (K_Z.\pi_*(E.M)) + (\mathcal{O}_M(E)^2)_M. \end{aligned}$$

Now  $\pi_*(E.M) = 0$  because  $\dim \pi(\text{Supp } E) = 0$ , and  $(\mathcal{O}_M(E)^2)_M < 0$  because  $E.M$  is an exceptional divisor of  $M \rightarrow \pi(M)$ . Hence  $C = E.M$  has the required property. q.e.d.

Lemma 9. There is an extremal rational curve  $\ell$  on  $X$  such that  $\pi_*\ell = 0$ .

Proof. Let  $H$  be an arbitrary ample divisor on  $X$  and  $\varepsilon$  a small enough positive number so that  $[C]$  given in Lemma 8 does not belong to  $\overline{NE}_\varepsilon(X, H)$ . By Theorem 3 in [3],  $[C]$  is written as

$$[C] = \sum_{i=1}^r a_i [\ell_i] + V,$$

where  $a_i \geq 0$ ,  $\ell_i$  are extremal rational curves for all  $i$ , and  $V \in \overline{NE}_\epsilon(X)$ . Hence  $\sum a_i \pi_*[\ell_i] + \pi_*V = 0$  and  $\pi_*[\ell_i], \pi_*V \in \overline{NE}(Z)$  by Lemma 7. Since  $Z$  is projective, we have  $a_i \pi_*[\ell_i] = 0$  for all  $i$  and  $\pi_*V = 0$  by Kleiman's criterion of projectivity:  $\overline{NE}(Z) \cap \{-\overline{NE}(Z)\} = \{0\}$ . Since  $[C] \notin \overline{NE}_\epsilon(X)$ , there is at least one  $j$  such that  $a_j \neq 0$ . Then  $\ell_j$  has the required property.

q.e.d.

Lemma 10. The curve  $\ell$  in Lemma 9 is not numerically effective.

If  $E$  is the effective divisor on  $X$  given in the proof of Lemma 8, then

$$\begin{aligned} (\ell.E) &= (\ell.K_X) - (\ell.\pi^*K_Z) \\ &= (\ell.K_X) - (\pi_*\ell.K_Z) < 0. \end{aligned}$$

Thus Theorem 5 is proved, and it is easy to check the assertions in Examples 6.

## References

- [1] H. Hironaka, An example of non-Kaehlerian complex-analytic deformation of Kaehlerian complex structures, Ann. Math. Vol. 75 (1962), 190-208.
- [2] V. A. Iskovskih, Fano 3-folds II, Math. USSR Izv. 11 (1977).
- [3] S. Mori, The cone of effective 1-cycles, in the same volume.