

MARTIN BOUNDARY FOR UNION OF CONVEX SETS

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1. Introduction

We study Martin boundary points of a proper subdomain in $\mathbb{R}^n$, where $n \geq 2$, that can be represented as the union of open convex sets. Especially, we give a certain sufficient condition for a boundary point to have exactly one (minimal) Martin boundary point.

In the 1970's, Ancona considered a bounded domain Ω that can be represented as the union of open balls with the same radius. He assumed that

- (A) if two balls tangent to each other at a boundary point ξ of Ω , then there is a truncated circular cone, with vertex at ξ and axis in the hyperplane tangent to such balls at ξ , included in Ω .

Under these assumption he showed that each boundary point has exactly one Martin boundary point and it is minimal ([4]).

However, this result is not applicable to domains with wedges. So we consider open convex sets rather than open balls with the same radius. Obviously, we need a different sufficient condition for a boundary point to have exactly one (minimal) Martin boundary point.

We write $\bar{E}$ and ∂E for the closure and the boundary of a set E , respectively. Let $x, y \in \mathbb{R}^n$ ($x \neq y$) and $r > 0$. We denote by $B(x, r)$ and $S(x, r)$ the open ball and the sphere of center x and radius r , respectively. For $\theta > 0$ let $\Gamma_\theta(x, y)$ stand for the open circular cone of vertex x , axis $\overline{xy}$ and aperture θ , i.e.,

$$\Gamma_\theta(x, y) := \{z \in \mathbb{R}^n : \angle zxy < \theta\}.$$

Let $\rho_0 > 0$ and $A_0 \geq 1$. We consider a proper subdomain D in $\mathbb{R}^n$ such that

- (I) D is the union of a family of open convex sets $\{C_\lambda\}_{\lambda \in \Lambda}$ such that $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0 \rho_0)$.
- (II) Let $\xi \in \partial D$. Then there are positive constants $\theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 \leq \rho_0 \cos \theta_1$ such that the union of truncated circular cones $\Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1)$ included in D is connected, i.e.,

$$\bigcup_{\substack{y \in D \\ \Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1) \subset D}} \Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1) \text{ is connected.}$$

Remark 1. We note that the union in the condition (II) is non-empty (Lemma 3.2). The condition (II) is the same as Ancona's when $A_0 = 1$ (Ancona's setting).

Throughout this note, we simply write a domain instead of a proper subdomain in $\mathbb{R}^n$. By a Greenian domain we mean a domain with the Green function.

The main result is as follows.

Theorem. *Let D be a Greenian domain satisfying (I). If $\xi \in \partial D$ satisfies (II), then there is exactly one Martin boundary point at ξ and it is minimal.*

Remark 2. We investigated in [3] that the number of minimal Martin boundary points at each boundary point of a John domain is estimated by the John constant. A bounded domain satisfying (I) is a John domain. As seen in Theorem, we obtain a better result under the condition (II).

Corollary. *Suppose that D is a bounded domain satisfying (I) and that each $\xi \in \partial D$ satisfies (II). Then the Martin boundary of D is homeomorphic to its Euclidean boundary. Moreover, each Martin boundary point is minimal.*

The following proposition implies the sharpness of bounds $\theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 \leq \rho_0 \cos \theta_1$ in the condition (II).

Proposition 1.1. *Let $A_0 > 1$. Suppose either*

- (i) $\theta_1 > \sin^{-1}(1/A_0)$, or
- (ii) $0 < \theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 > \rho_0 \cos \theta_1$.

Then there is a domain D satisfying (I) and $\xi \in \partial D$ satisfies (II), and yet ξ has multiple minimal Martin boundary points.

This note is organized as follows. In Section 2, we shall show a general fact for the support of the measure associated with a kernel function in the Martin representation. In Section 3, we shall show geometrical properties. In Section 4, we shall prove a Carleson type estimate after showing the upper bound of a non-negative subharmonic function on a bounded domain and showing the integrability of the negative power of the distance function. In Section 5, we shall show a (uniform) boundary Harnack principle. In Section 6, we shall prove Theorem and Corollary. In Section 7, we shall give examples for Proposition. In Section 8, we shall give a domain satisfying (I) and (II) at each boundary point but not a uniform domain.

By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If two positive functions f and g satisfy $A^{-1}f \leq g \leq Af$ for some constant $A \geq 1$, then we write $f \approx g$ and call A the constant of comparison.

2. General fact

In this section, we show a general fact for the support of the measure of a corresponding to a kernel function in the Martin representation. Let $\xi \in \partial D$ and $x_0 \in D$ be fixed. Let G denote the Green function for D . The Martin kernel (or the Martin boundary point) at ξ , written $K(\cdot, \xi)$, is given as a limit function of the Martin kernels $K(\cdot, y_j) := G(\cdot, y_j)/G(x_0, y_j)$ for some sequence $\{y_j\}$ in D converging to ξ . We say that a property holds quasi-everywhere if it holds except a polar set. A function h on D is called a kernel function at ξ if h is positive

and harmonic on D , satisfies $h(x_0) = 1$, vanishes quasi-everywhere on ∂D and is bounded on $D \setminus B(\xi, r)$ for each $r > 0$. We denote by Δ the Martin boundary of D , and by Δ_1 the subset of all minimal elements in Δ . We also write $\Delta(\xi)$ for the set of all Martin boundary points at ξ , and let $\Delta_1(\xi) := \Delta(\xi) \cap \Delta_1$. Let $E \subset D$ and $y \in \Delta_1$. We say that E is minimally thin at y if $\widehat{R}_{K(\cdot, y)}^E \neq K(\cdot, y)$. Here $\widehat{R}_u^E$ denotes the regularized reduced function of a non-negative superharmonic function u relative to E in D .

The following lemma will be used in the proof of Theorem (Section 6).

Lemma 2.1. *Let D be a Greenian domain and $\xi \in \partial D$. If h is a kernel function at ξ , then the support of the measure associated with it in the Martin representation is $\Delta_1(\xi)$. In particular, $\Delta_1(\xi)$ is non-empty.*

Proof. By the Martin representation, there is a unique measure μ on Δ_1 such that

$$h(x) = \int_{\Delta_1} K(x, y) d\mu(y) \quad \text{for } x \in D.$$

Let E be a compact subset of $\Delta \setminus \Delta(\xi)$ and let $\{E_j\}$ be a decreasing sequence of compact neighborhoods of E in the Martin topology such that $(E_1 \cap D) \cap B(\xi, r_1) = \emptyset$ for some $r_1 > 0$ and $\bigcap_j E_j = E$. Then we have ([5, Corollary 9.1.4])

$$\widehat{R}_h^{E_j \cap D}(x) = \int_{\Delta_1} \widehat{R}_{K(\cdot, y)}^{E_j \cap D}(x) d\mu(y) \quad \text{for } x \in D.$$

Noting that $\lim_{j \rightarrow \infty} \widehat{R}_h^{E_j \cap D}$ is bounded and harmonic on D and vanishes quasi-everywhere on ∂D since h is the kernel function at ξ , we have

$$(2.1) \quad 0 = \lim_{j \rightarrow \infty} \widehat{R}_h^{E_j \cap D}(x_0) = \int_{\Delta_1} \lim_{j \rightarrow \infty} \widehat{R}_{K(\cdot, y)}^{E_j \cap D}(x_0) d\mu(y)$$

by the monotone convergence. Let $y \in E \cap \Delta_1$. Then $E_j \cap D$ is not minimally thin at y for each j ([5, Lemma 9.1.5]), and so $\lim_{j \rightarrow \infty} \widehat{R}_{K(\cdot, y)}^{E_j \cap D}(x_0) = K(x_0, y) = 1$. Hence $\mu(E) = 0$ by (2.1). Thus the lemma follows. $\square$

3. Geometrical properties

Let Ω be a proper subdomain and $x, y \in \Omega$. We write $\delta_\Omega(x)$ for $\text{dist}(x, \partial\Omega)$, the distance from x to $\partial\Omega$, and define the quasi-hyperbolic metric between x and y by

$$k_\Omega(x, y) := \inf_{\gamma} \int_{\gamma} \frac{ds}{\delta_\Omega(z)},$$

where the infimum is taken over all rectifiable curves γ in Ω connecting x to y .

Throughout this section we suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). The main purpose of this section is to show the following lemma.

Lemma 3.1. *Let $\kappa = 6/\sin \theta_1$. There is a positive constant R_ξ with the following property. For each $0 < R < R_\xi$ there is $y_R \in D \cap S(\xi, R)$ such that $\delta_D(y_R) \geq A_\xi^{-1} R$ and*

$$k_{D \cap B(\xi, \kappa R)}(x, y_R) \leq A_\xi \log \frac{R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap B(\xi, R),$$

where $A_\xi \geq 1$ is independent of x and R .

Remark 3. In general, Lemma 3.1 does not hold for a John domain. We introduced in [3] a geometrical notion, a system of local reference points of order N . That is, for each $0 < R < R_\xi$ there are N points, say $y_R^1, \dots, y_R^N$, in $D \cap S(\xi, R)$ such that $\delta_D(y_R^i) \geq A_\xi^{-1} R$ for $i = 1, \dots, N$ and

$$\min_{i=1, \dots, N} \{k_{D \cap B(\xi, \kappa R)}(x, y_R^i)\} \leq A_\xi \log \frac{R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap B(\xi, R).$$

Lemma 3.1 is the case $N = 1$.

In order to prove Lemma 3.1, in view of translation and dilation, we may suppose that $\xi = 0$ and $\rho_1 = 1$ for simplicity. We briefly write $\Gamma(x, y)$ for $\Gamma_{\theta_1}(x, y)$. Let

$$\mathcal{Y} := \{y \in S(0, 1) : \Gamma(0, y) \cap B(0, 2) \subset D\}.$$

Then the union in the condition (II) is $\bigcup_{y \in \mathcal{Y}} \Gamma(0, y) \cap B(0, 2)$, written $\mathcal{C}(0)$. We prove Lemma 3.1 after showing some lemmas.

Lemma 3.2. *There is a positive constant $R_0 < \kappa^{-1}$ such that if $C_\lambda \cap B(0, R_0) \neq \emptyset$, then $C_\lambda \cap \mathcal{Y} \neq \emptyset$. In particular, $\mathcal{Y} \neq \emptyset$.*

Proof. We show this by leading a contradiction. Suppose that there is a sequence $\{C_{\lambda_j}\}$ such that $\text{dist}(0, C_{\lambda_j}) \rightarrow 0$ and $C_{\lambda_j} \cap \mathcal{Y} = \emptyset$. Let $B(z_j, \rho_0) \subset C_{\lambda_j} \subset B(z_j, A_0 \rho_0)$. Taking a subsequence if necessary, we may assume that z_j converges, say to z_0 . Let $x_j \in \partial C_{\lambda_j}$ be such that $x_j \rightarrow 0$. Then, by continuity of the angle $\angle \cdot x_j z_j$ and the distance $|\cdot - x_j|$,

$$\Gamma(0, z_0) \cap B(0, 2) \subset \bigcup_j (\Gamma(x_j, z_j) \cap B(x_j, 2)) \subset \bigcup_j C_{\lambda_j}.$$

Hence $\bigcup_j C_{\lambda_j} \cap \mathcal{Y} \neq \emptyset$, and this contradicts the assumption. Thus the lemma follows. $\square$

Let us take $y_1 \in \mathcal{Y}$ and fix. For $0 < R < 1$ we let $y_R := Ry_1$. Then $\delta_D(y_R) \geq R \sin \theta_1$.

Lemma 3.3. *There is a positive constant A such that if $0 < R < R_0$, then*

$$k_{D \cap B(0, \kappa R)}(Ry, y_R) \leq A \quad \text{for } y \in \mathcal{Y}.$$

Proof. Note that $\mathcal{C}(0) \cap S(0, 1)$ is connected since the cone $\mathcal{C}(0)$ is connected. We observe that there is a closed connected subset E of $\mathcal{C}(0) \cap S(0, 1)$ and $0 < r_0 \leq \sin \theta_1$ such that $\mathcal{Y} \subset E$ and $\text{dist}(E, \partial \mathcal{C}(0)) \geq r_0$. Then $y, y_1 \in E$. In view of the compactness of E , we can take a curve γ in $\mathcal{C}(0) \cap S(0, 1)$ joining y and y_1 such that $\delta_{\mathcal{C}(0)}(z) \geq r_0/2$ for all $z \in \gamma$ and $\ell(\gamma) \leq Ar_0$, where A depends only on a covering constant of E and $\ell(\gamma)$ denotes the length of a curve γ . Let γ_R be the image of γ in $S(0, R)$ under dilation. Then we have

$$k_{D \cap B(0, \kappa R)}(Ry, y_R) \leq \int_{\gamma_R} \frac{ds}{\delta_D(z)} \leq \frac{Ar_0 R}{r_0 R/2} = 2A.$$

Thus the lemma follows. $\square$

Let $[x, y]$ denote the (open) line segment between x and y . If C is a convex set, then the distance function δ_C is concave on $\overline{C}$, i.e.,

$$(3.1) \quad \delta_C(z) \geq \frac{|z - y|}{|x - y|} \delta_C(x) + \frac{|x - z|}{|x - y|} \delta_C(y) \quad \text{for } z \in [x, y],$$

whenever $x, y \in \overline{C}$ ($x \neq y$).

Lemma 3.4. *Let $0 < R < R_0$. If $C_\lambda \cap B(0, R) \neq \emptyset$ and $y \in C_\lambda \cap \mathcal{Y}$, then there exists $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3R/\sin \theta_1)$ such that*

$$\delta_{C_\lambda \cap \Gamma(0, y)}(w) \geq \frac{\sin \theta_1}{3} R.$$

Proof. We can take $w_1 \in C_\lambda \cap \overline{\Gamma(0, y)}$ with $|w_1| \leq R/\sin \theta_1$. In fact, if $x \in C_\lambda \cap B(0, R) \setminus \overline{\Gamma(0, y)}$, then we may take w_1 at which $[x, y]$ intersects $\partial \Gamma(0, y)$, so that

$$|w_1| = \frac{\text{dist}(w_1, [0, y])}{\sin \theta_1} \leq \frac{\text{dist}(x, [0, y])}{\sin \theta_1} \leq \frac{R}{\sin \theta_1}.$$

Note that $|w_1 - y| > 5R/\sin \theta_1$ since $R < \kappa^{-1} = 6^{-1} \sin \theta_1$. Let $w_2 \in [w_1, y] \subset C_\lambda \cap \Gamma(0, y)$ be such that $|w_1 - w_2| = R/\sin \theta_1$. Applying (3.1) to $C := \Gamma(0, y)$, we have

$$\delta_{\Gamma(0, y)}(w_2) \geq \frac{|w_1 - w_2|}{|w_1 - y|} \delta_{\Gamma(0, y)}(y) \geq \frac{R/\sin \theta_1}{1 + R/\sin \theta_1} \sin \theta_1 \geq \frac{2}{3} R.$$

Noting that $|w_2 - z_\lambda| \geq \rho_0 - 2R/\sin \theta_1 \geq 4R$ since $\rho_0 \geq 1 \geq 6R/\sin \theta_1$, we can take $w \in [w_2, z_\lambda] \subset C_\lambda$ with $|w_2 - w| = R/3$. Then (3.1) with $C := C_\lambda$ yields that

$$\delta_{C_\lambda}(w) \geq \frac{|w_2 - w|}{|w_2 - z_\lambda|} \delta_{C_\lambda}(z_\lambda) \geq \frac{R/3}{A_0 \rho_0} \rho_0 \geq \frac{\sin \theta_1}{3} R.$$

Hence we have

$$\delta_{\Gamma(0, y) \cap C_\lambda}(w) \geq \min \left\{ \frac{2}{3} R - \frac{R}{3}, \frac{\sin \theta_1}{3} R \right\} = \frac{\sin \theta_1}{3} R,$$

and

$$|w| \leq |w - w_2| + |w_2 - w_1| + |w_1| \leq \frac{R}{3} + \frac{R}{\sin \theta_1} + \frac{R}{\sin \theta_1} < \frac{3R}{\sin \theta_1}.$$

Thus the lemma follows. $\square$

Proof of Lemma 3.1. Let $x \in C_\lambda \cap B(0, R)$ and $y \in C_\lambda \cap \mathcal{Y}$. By Lemma 3.4, we can take $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3R/\sin \theta_1)$ with $\delta_{C_\lambda \cap \Gamma(0, y)}(w) \geq 3^{-1} R \sin \theta_1$. Then we have

$$\delta_D(z) \geq \delta_{C_\lambda}(z) \geq \frac{|x - z|}{|x - w|} \delta_{C_\lambda}(w) \geq \frac{\sin^2 \theta_1}{12} |x - z| \quad \text{for } z \in [x, w],$$

by (3.1) with $C := C_\lambda$. Since $[x, w] \subset B(0, \kappa R/2)$, it follows that

$$k_{D \cap B(0, \kappa R)}(x, w) \leq \int_{[x, w]} \frac{ds}{\delta_D(z)} \leq 1 + \int_{\frac{\delta_D(x)}{2}}^{|x-w|} \frac{12}{\sin^2 \theta_1} \frac{dt}{t} \leq A \log \frac{R}{\delta_D(x)} + A,$$

where A depends only on θ_1 . We also have $k_{D \cap B(0, \kappa R)}(w, Ry) \leq A$. In fact, since $\delta_{\Gamma(0, y)}(Ry) \geq R \sin \theta_1$, it follows from (3.1) with $C := \Gamma(0, y)$ that

$$\delta_D(z) \geq \delta_{\Gamma(0, y)}(z) \geq \frac{|w - z|}{|w - Ry|} \delta_{\Gamma(0, y)}(Ry) \geq \frac{\sin^2 \theta_1}{4} |w - z| \quad \text{for } z \in [w, Ry],$$

and so

$$k_{D \cap B(0, \kappa R)}(w, Ry) \leq \int_{[w, Ry]} \frac{ds}{\delta_D(z)} \leq 2 + \int_{\frac{\delta_D(w)}{2}}^{\delta_D(Ry)} \frac{4}{\sin^2 \theta_1} \frac{dt}{t} \leq A,$$

where A depends only on θ_1 . Hence we obtain from Lemma 3.3 that

$$\begin{aligned} k_{D \cap B(0, \kappa R)}(x, y_R) &\leq k_{D \cap B(0, \kappa R)}(x, w) + k_{D \cap B(0, \kappa R)}(w, Ry) + k_{D \cap B(0, \kappa R)}(Ry, y_R) \\ &\leq A \log \frac{R}{\delta_D(x)} + A. \end{aligned}$$

Thus Lemma 3.1 follows. $\square$

4. Carleson type estimate

In this section we show a Carleson type estimate. To this end, we prepare two lemmas. One is a refinement of Domar's theorem ([6, Theorem 2]). Another is the integrability of the negative power of the distance function. This is a local version of [1, Lemma 5].

We note first the following. Let Ω be a domain and $x, y \in \Omega$. We say that x and y are connected by a Harnack chain $\{B(x_j, \delta_\Omega(x_j))\}_{j=1}^N$ if $x \in B(x_1, \frac{1}{2}\delta_\Omega(x_1))$, $x_{j-1} \in B(x_j, \frac{1}{2}\delta_\Omega(x_j))$ for $j = 2, \dots, N$ and $x_N = y$. The number N is called the length of the Harnack chain. We observe that if $x \notin B(y, \frac{1}{2}\delta_\Omega(y))$, then the shortest length of the Harnack chain connecting x and y is comparable to $k_\Omega(x, y)$. Therefore, the Harnack inequality yields that there is a constant $A \geq 1$ depending only on the dimension such that if $x, y \in \Omega$ and h is a positive harmonic function on Ω , then

$$(4.1) \quad \exp(-Ak_\Omega(x, y) - 1) \leq \frac{h(x)}{h(y)} \leq \exp(Ak_\Omega(x, y) + 1).$$

Lemma 4.1. *Let Ω be a bounded domain. If u is a non-negative subharmonic function on Ω such that*

$$I := \int_{\Omega} (\log^+ u)^{n-1+\varepsilon} dx < \infty \quad \text{for some } \varepsilon > 0,$$

then there is a positive constant A depending only on ε and the dimension such that

$$(4.2) \quad u(x) \leq \exp\left(2 + A\left(\frac{I}{\delta_\Omega(x)^n}\right)^{1/\varepsilon}\right).$$

We show first the following lemma. We write $|E|$ for the volume of a set E .

Lemma 4.2. *Let u be a subharmonic function on Ω containing $\overline{B(x, R)}$. Suppose that $u(x) \geq t > 0$ and that*

$$(4.3) \quad R \geq L_n |\{y \in B(x, R) : t/e < u(y) \leq et\}|^{1/n},$$

where $L_n = (e^2/|B(0, 1)|)^{1/n}$. Then there exists $x' \in B(x, R)$ such that $u(x') > et$.

Proof. Suppose to the contrary that $u \leq et$ on $B(x, R)$. Noting that (4.3) is equivalent to

$$\frac{|\{y \in B(x, R) : t/e < u(y) \leq et\}|}{|B(x, R)|} \leq \frac{1}{e^2},$$

we have

$$\begin{aligned} t &\leq u(x) \leq \frac{1}{|B(x, R)|} \int_{B(x, R)} u(y) dy \\ &= \frac{1}{|B(x, R)|} \left(\int_{B(x, R) \cap \{u \leq t/e\}} u(y) dy + \int_{B(x, R) \cap \{t/e < u(y) \leq et\}} u(y) dy \right) \\ &\leq \frac{t}{e} + \frac{et}{e^2} < t. \end{aligned}$$

This is a contradiction, and the lemma follows. $\square$

Proof of Lemma 4.1. Since the right hand side of (4.2) is not less than e^2 , it is sufficient to show that

$$(4.4) \quad \delta_\Omega(x) \leq AI^{1/n}(\log u(x))^{-\varepsilon/n} \quad \text{whenever } u(x) > e^2.$$

Let $x_1 \in \Omega$ be such that $u(x_1) > e^2$, and let

$$R_j = L_n |\{y \in \Omega : e^{j-2}u(x_1) < u(y) \leq e^j u(x_1)\}|^{1/n}.$$

Let us show (4.4) for $x = x_1$. We can choose a finite or infinite sequence $\{x_j\}$ in Ω as follows. By Lemma 4.2, we can iteratively find $x_{j+1} \in B(x_j, R_j)$ with $u(x_{j+1}) > e^j u(x_1)$ whenever $\delta_\Omega(x_j) > R_j$. If $\delta_\Omega(x_j) \leq R_j$, then we stop this iteration, otherwise we continue.

We claim that

$$(4.5) \quad \delta_\Omega(x_1) \leq 2 \sum_{j=1}^{\infty} R_j.$$

Suppose first $\{x_j\}$ is finite. Noting that

$$(4.6) \quad \delta_\Omega(x_1) \leq \sum_{j=1}^{N-1} |x_j - x_{j+1}| + \delta_\Omega(x_N),$$

we obtain (4.5) by our choice of $\{x_j\}$. Suppose next $\{x_j\}$ is infinite. Since $u(x_j) \geq e^{j-1}u(x_1) \rightarrow \infty$, it follows from the local boundedness of a subharmonic function that x_j goes to the boundary. Hence $\delta_\Omega(x_N) \leq \delta_\Omega(x_1)/2$ for some N , and (4.5) follows from (4.6).

To obtain (4.4) for $x = x_1$, it is enough to show that

$$(4.7) \quad \sum_{j=1}^{\infty} R_j \leq AI^{1/n}(\log u(x_1))^{-\varepsilon/n}.$$

Let j_1 be the integer such that $e^{j_1} < u(x_1) \leq e^{j_1+1}$. Then $j_1 \geq 2$ and

$$R_j \leq L_n |\{y \in \Omega : e^{j_1+j-2} < u(y) \leq e^{j_1+j+1}\}|^{1/n}.$$

Since the family of intervals $\{(e^{j_1+j-2}, e^{j_1+j+1}]\}_j$ overlaps at most three times, it follows from Hölder's inequality that

$$\begin{aligned} \sum_{j=1}^{\infty} R_j &\leq 3L_n \sum_{j=j_1}^{\infty} |\{y \in \Omega : e^{j-1} < u(y) \leq e^j\}|^{1/n} \\ &\leq 3L_n \left(\sum_{j=j_1}^{\infty} \frac{1}{j^{(n-1+\varepsilon)/(n-1)}} \right)^{(n-1)/n} \left(\sum_{j=j_1}^{\infty} j^{n-1+\varepsilon} |\{y \in \Omega : e^{j-1} < u(y) \leq e^j\}| \right)^{1/n} \\ &\leq A j_1^{-\varepsilon/n} \left(\int_{\Omega} (\log^+ u(y))^{n-1+\varepsilon} dy \right)^{1/n} \\ &\leq A (\log u(x_1))^{-\varepsilon/n} I^{1/n}, \end{aligned}$$

where A depends only on ε and n . Thus (4.7) follows and Lemma 4.1 is proved. $\square$

Lemma 4.3. *Let D be a domain satisfying (I) and $\xi \in \partial D$. If $0 < R < \rho_0$, then there are positive constants τ and A depending only on A_0 and the dimension such that*

$$\int_{D \cap B(\xi, R)} \left(\frac{R}{\delta_D(x)} \right)^\tau dx \leq AR^n.$$

Proof. For each $j \in \mathbb{N} \cup \{0\}$ we put

$$V_j := \left\{ x \in D \cap B\left(\xi, R + \frac{A_0 + 1}{2^{j-1}} R\right) : \frac{R}{2^{j+1}} \leq \delta_D(x) < \frac{R}{2^j} \right\}.$$

Let $x \in \bigcup_{j=k+1}^\infty V_j$. Then there is C_λ so that $x \in C_\lambda$, and let $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0 \rho_0)$. Let $y, y' \in [x, z_\lambda]$ be such that $\delta_D(y) = R/2^k$ and $\delta_D(y') = (R/2^{k+1} + R/2^k)/2$. Then we see that $x \in \overline{B(y, A_0 R/2^k)}$ by (3.1), and that $B(y', R/2^{k+2}) \subset V_k \cap B(y, A_0 R/2^k)$. Hence we obtain

$$(4.8) \quad \left| B\left(y, \frac{5A_0 R}{2^k}\right) \right| \leq A_1 \left| V_k \cap B\left(y, \frac{A_0 R}{2^k}\right) \right|,$$

where A_1 depends only on A_0 and the dimension. We also have $\bigcup_{j=k+1}^\infty V_j \subset \bigcup_y \overline{B(y, A_0 R/2^k)}$, where y is the point associated with x as above. Hence the covering lemma yields that there is $\{y_j\}$ such that $\bigcup_{j=k+1}^\infty V_j \subset \bigcup_j \overline{B(y_j, 5A_0 R/2^k)}$ and $\{B(y_j, A_0 R/2^k)\}$ are mutually disjoint. Then we obtain from (4.8) that

$$\sum_{j=k+1}^\infty |V_j| = \left| \bigcup_{j=k+1}^\infty V_j \right| \leq \sum_j \left| B\left(y_j, \frac{5A_0 R}{2^k}\right) \right| \leq A_1 \sum_j \left| V_k \cap B\left(y_j, \frac{A_0 R}{2^k}\right) \right| \leq A_1 |V_k|.$$

Let $t = 1 + 1/2A_0$. Then

$$\begin{aligned} A_1 \sum_{k=0}^N t^{k+1} |V_k| &\geq \sum_{k=0}^N \sum_{j=k+1}^{N+1} t^{k+1} |V_j| = \sum_{j=1}^{N+1} \sum_{k=0}^{j-1} t^{k+1} |V_j| \geq \sum_{j=1}^N \sum_{k=0}^{j-1} t^{k+1} |V_j| \\ &= \sum_{j=1}^N \frac{t^{j+1} - t}{t-1} |V_j| = \frac{1}{t-1} \sum_{j=0}^N t^{j+1} |V_j| - \frac{t}{t-1} \sum_{j=0}^N |V_j|, \end{aligned}$$

and so

$$\sum_{j=0}^N t^{j+1} |V_j| \leq \frac{t}{1 - (t-1)A_1} \sum_{j=0}^N |V_j|.$$

Letting $N \rightarrow \infty$, we have

$$\sum_{j=0}^\infty t^{j+1} |V_j| \leq \frac{t}{1 - (t-1)A_1} \sum_{j=0}^\infty |V_j| \leq A |B(\xi, R + 2(A_0 + 1)R)| \leq AR^n.$$

Since $t^j < (R/\delta_D(x))^\tau \leq t^{j+1}$ for $x \in V_j$ with $\tau = \log t / \log 2 > 0$, we obtain

$$\int_{D \cap B(\xi, R)} \left(\frac{R}{\delta_D(x)} \right)^\tau dx \leq \sum_{j=0}^\infty t^{j+1} |V_j| \leq AR^n.$$

Thus the lemma follows. $\square$

Lemma 4.4 (Carleson type estimate). *Suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). Let $0 < R < R_\xi$. If h is a positive bounded harmonic function on $D \cap B(\xi, \kappa R)$ vanishing quasi-everywhere on $\partial D \cap B(\xi, \kappa R)$, then*

$$h(x) \leq Ah(y_R) \quad \text{for } x \in D \cap \overline{B(\xi, \kappa^{-1}R)},$$

where A is independent of x , R and h .

Proof. By (4.1) and Lemma 3.1 we have

$$(4.9) \quad \frac{h(x)}{h(y_R)} \leq A_2 \left(\frac{R}{\delta_D(x)} \right)^\alpha \quad \text{for } x \in D \cap B(\xi, R),$$

where A_2 and α are positive constants depending only on A_ξ and the dimension. We note that h has a non-negative subharmonic extension h^* to $B(\xi, R)$ with zero values on $B(\xi, R) \setminus \overline{D}$ ([5, Theorem 5.2.1]). Let $u = h^*/A_2h(y_R)$. Using the inequality

$$\left[\log \left(\frac{R}{\delta_D(x)} \right) \right]^n \leq \left(\frac{n}{\tau} \right)^\tau \left(\frac{R}{\delta_D(x)} \right)^\tau \quad \text{for } x \in D \cap B(\xi, R),$$

where $\tau > 0$ is as in Lemma 4.3, we obtain from (4.9) and Lemma 4.3 that

$$I = \int_{B(\xi, R)} (\log^+ u)^n dx \leq A \int_{D \cap B(\xi, R)} \left(\frac{R}{\delta_D(x)} \right)^\tau dx \leq AR^n.$$

Hence it follows from Lemma 4.1 that $u \leq A$ on $S(\xi, \kappa^{-1}R)$, and the maximum principle yields that

$$h(x) \leq Ah(y_R) \quad \text{for } x \in D \cap \overline{B(\xi, \kappa^{-1}R)}.$$

Thus the lemma follows. $\square$

5. Boundary Harnack principle

The purpose of this section is to show a (uniform) boundary Harnack principle, which is useful to obtain properties of Martin kernels. The proofs in this section are based on [2] for a uniform domain.

For $r > 0$ we let

$$U(r) := \{x \in D : \delta_D(x) < r\}.$$

We denote by $\omega(x, E, U)$ the harmonic measure of a set E for an open set U evaluated at x . We write $|E|$ for the volume of a set E . Let us start with an estimate of a harmonic measure.

Lemma 5.1. *Let D be a domain satisfying (I). Then there are constants $0 < \varepsilon_0 < 1$ and $A_3 \geq 1$ such that if $0 < r < \rho_0/2$, then*

$$\omega(x, U(r) \cap S(x, A_3r), U(r) \cap B(x, A_3r)) \leq \varepsilon_0 \quad \text{for } x \in U(r).$$

Proof. Let $x \in U(r)$. Then there is C_λ so that $x \in C_\lambda$, and let $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0)$. Take $w \in [x, z_\lambda]$ with $\delta_D(w) = 2r$. Then we have $|x - w| \leq 2A_0r$ by (3.1), and so $B(w, r) \subset B(x, 3A_0r) \setminus U(r)$. Hence there is $0 < \varepsilon_0 < 1$ depending only on A_0 and the dimension such that

$$\frac{|U(r) \cap B(x, 3A_0r)|}{|B(x, 3A_0r)|} \leq \varepsilon_0.$$

Let $A_3 := 3A_0 + 1$. We note that $\omega(\cdot, U(r) \cap S(x, A_3r), U(r) \cap B(x, A_3r))$ has a subharmonic extension ω to $B(x, A_3r)$ with zero values on $B(x, A_3r) \setminus \overline{U(r)}$ ([5, Theorem 5.2.1]). Hence

$$\omega(x) \leq \frac{1}{|B(x, 3A_0r)|} \int_{B(x, 3A_0r)} \omega(y) dy \leq \varepsilon_0.$$

Thus the lemma follows. $\square$

Lemma 5.2. *Let D be a domain satisfying (I) and A_3 be as in Lemma 5.1. Then there is a positive constant $A_4 \leq 1$ such that if $r > 0$ and $R > 0$, then*

$$(5.1) \quad \omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \leq \exp\left(A_3 - A_4 \frac{R}{r}\right) \quad \text{for } x \in U(r).$$

Proof. Note that if $R \leq A_3r$, then (5.1) clearly holds since the right hand side of (5.1) is not less than 1. Let $k \in \mathbb{N}$ be such that $kA_3r < R \leq (k+1)A_3r$. We claim that

$$(5.2) \quad \sup_{U(r) \cap B(x, R - jA_3r)} \omega(\cdot, U(r) \cap S(x, R), U(r) \cap B(x, R)) \leq \varepsilon_0^j$$

for $j = 0, \dots, k$, where ε_0 is as in Lemma 5.1. We show this by induction. If $j = 0$, then (5.2) clearly holds. We assume that (5.2) holds for $j - 1$, and show (5.2) for j . Let $y \in U(r) \cap S(x, R - jA_3r)$. Since $S(y, A_3r) \subset \overline{B(x, R - (j-1)A_3r)}$, it follows from the assumption, the maximum principle and Lemma 5.1 that

$$\begin{aligned} \omega(y, U(r) \cap S(x, R), U(r) \cap B(x, R)) &\leq \varepsilon_0^{j-1} \omega(y, U(r) \cap S(y, A_3r), U(r) \cap B(y, A_3r)) \\ &\leq \varepsilon_0^j. \end{aligned}$$

Since y is an arbitrary point in $U(r) \cap S(x, R - jA_3r)$, the maximum principle yields (5.2) for j . Finally, noting that $R/A_3r \leq 2k$, we obtain from (5.2) with $j := k$ that

$$\omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \leq \exp((\varepsilon_0 - 1)k) \leq \exp\left(\frac{\varepsilon_0 - 1}{2A_3} \frac{R}{r}\right).$$

Thus the lemma follows. $\square$

Lemma 5.3. *Suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). Let $0 < R < R_\xi$. If h is a positive bounded harmonic function on $D \cap B(\xi, \kappa R)$ vanishing quasi-everywhere on $\partial D \cap B(\xi, \kappa R)$, then*

$$\omega(x, D \cap S(\xi, \kappa^{-1}R), D \cap B(\xi, \kappa^{-1}R)) \leq A \frac{h(x)}{h(y_R)} \quad \text{for } x \in D \cap B(\xi, \kappa^{-2}R),$$

where A is independent of x , R and h .

Proof. By Lemma 4.4, we have $h \leq Ah(y_R)$ on $D \cap B(\xi, \kappa^{-1}R)$. Let A_5 be such that $A_5h/h(y_R) \leq e^{-1}$ on $D \cap B(\xi, \kappa^{-1}R)$, and put $u := A_5h/h(y_R)$. Then it follows from (4.1) and Lemma 3.1 that

$$(5.3) \quad u(x) \geq A \left(\frac{\delta_D(x)}{R} \right)^\alpha \quad \text{for } x \in D \cap B(\xi, \kappa^{-1}R).$$

Let $D_j := \{x \in D : \exp(-2^{j+1}) \leq u(x) < \exp(-2^j)\}$ and $U_j := \{x \in D : u(x) < \exp(-2^j)\}$. Then, by (5.3), we have

$$U_j \cap B(\xi, \kappa^{-1}R) \subset V_j := \left\{ x \in D : \delta_D(x) \leq A_6 R \exp\left(-\frac{2^j}{\alpha}\right) \right\}.$$

Let $\{R_j\}$ be a sequence defined by $R_0 := \kappa^{-1}R$ and

$$R_j := \left(\kappa^{-1} - \frac{6(\kappa^{-1} - \kappa^{-2})}{\pi^2} \sum_{k=1}^j \frac{1}{k^2} \right) R.$$

Then $R_j \downarrow \kappa^{-2}R$. We briefly write $\omega_0 := \omega(\cdot, D \cap S(\xi, \kappa^{-1}R), D \cap B(\xi, \kappa^{-1}R))$, and put

$$d_j := \begin{cases} \sup_{D_j \cap B(\xi, R_j)} \frac{\omega_0}{u} & \text{if } D_j \cap B(\xi, R_j) \neq \emptyset, \\ 0 & \text{if } D_j \cap B(\xi, R_j) = \emptyset. \end{cases}$$

It suffices to show that $\sup_{j \geq 0} d_j$ is bounded by a constant independent of R and u . Let $j > 0$ and $x \in U_j \cap B(\xi, R_j)$. Then the maximum principle yields that

$$(5.4) \quad \omega_0(x) \leq \omega(x, U_j \cap S(\xi, R_{j-1}), U_j \cap B(\xi, R_{j-1})) + d_{j-1}u(x).$$

Since $B(x, R_{j-1} - R_j) \subset B(\xi, R_{j-1})$, the first term of the right hand side of (5.4) is not greater than

$$\omega(x, V_j \cap S(x, R_{j-1} - R_j), V_j \cap B(x, R_{j-1} - R_j)) \leq \exp \left(A_3 - A_4 \frac{R_{j-1} - R_j}{A_6 R \exp(-2^j/\alpha)} \right)$$

by Lemma 5.2. Let us divide the both sides of (5.4) by $u(x)$ and take the supremum over $D_j \cap B(\xi, R_j)$. Then we have

$$d_j \leq \exp \left(2^{j+1} + A_3 - A_4 \frac{6(\kappa^{-1} - \kappa^{-2}) \exp(2^j/\alpha)}{\pi^2 A_6 j^2} \right) + d_{j-1}.$$

Since $d_0 \leq e^2$, we obtain

$$d_j \leq \sum_{i=1}^{\infty} \exp \left(2^{i+1} + A_3 - A_4 \frac{6(\kappa^{-1} - \kappa^{-2}) \exp(2^i/\alpha)}{\pi^2 A_6 i^2} \right) + d_0 < \infty.$$

Thus the lemma follows. $\square$

Lemma 5.4 (Boundary Harnack principle). *Suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). Let $0 < R < R_\xi$. If u and v are positive bounded harmonic functions on $D \cap B(\xi, \kappa R)$ vanishing quasi-everywhere on $\partial D \cap B(\xi, \kappa R)$, then*

$$\frac{u(y)}{v(y)} \approx \frac{u(y')}{v(y')} \quad \text{for } y, y' \in D \cap B(\xi, \kappa^{-2}R),$$

where the constant of comparison is independent of y, y', R, u and v .

Proof. By Lemma 4.4, the maximum principle and Lemma 5.3, we have

$$u(y) \leq Au(y_R) \omega(y, D \cap S(\xi, \kappa^{-1}R), D \cap B(\xi, \kappa^{-1}R)) \leq Au(y_R) \frac{v(y)}{v(y_R)}$$

for $y \in D \cap B(\xi, \kappa^{-2}R)$. Changing the roles of u and v , we have

$$v(y') \leq Av(y_R) \frac{u(y')}{u(y_R)} \quad \text{for } y' \in D \cap B(\xi, \kappa^{-2}R).$$

Hence two inequalities above yield the lemma. $\square$

Remark 4. We note that the constant of comparison and R_ξ in Lemma 5.4 depends on ξ . For a bounded uniform domain, these constants could be taken uniformly for ξ ([2, Theorem 1]). Using this fact, the first author showed the uniqueness of a kernel function at ξ ([2, Lemma 4 and Proof of Theorem 3]). However, in view of Lemma 2.1, we need not take those constants uniformly in order to prove Theorem. We also note that there is a bounded domain satisfying (I) and each boundary point satisfies (II) but not a uniform domain (see example in Section 8).

6. Proof of Theorem and Corollary

Suppose that D is a domain satisfying (I) and that $\xi \in \partial D$ satisfies (II). We note first that every Martin kernel at ξ is a kernel function at ξ . In fact, let $R > 0$ be small enough and $x \in D \setminus B(\xi, \kappa R)$. Applying Lemma 5.4 to $u := G(x, \cdot)$ and $v := G(x_0, \cdot)$, we see that each Martin kernel at ξ is bounded on $D \setminus B(\xi, \kappa R)$, and is kernel function at ξ .

Proof of Theorem. Let $u, v \in \Delta_1(\xi)$ and $R > 0$ be small enough. Then, by definition of the Martin kernel at ξ , there are sequences $\{y_j\}$ and $\{y'_j\}$ in D converging to ξ such that $K(\cdot, y_j) \rightarrow u$ and $K(\cdot, y'_j) \rightarrow v$, respectively. Since $K(x, y_j) \approx K(x, y'_j)$ for $x \in D \setminus B(\xi, \kappa R)$ by Lemma 5.4 if j is sufficiently large, we have $u(x) \approx v(x)$ for $x \in D \setminus B(\xi, \kappa R)$. Since the constant of comparison is independent of R , it follows from the minimality of u and v and $u(x_0) = 1 = v(x_0)$ that $u \equiv v$. Hence $\Delta_1(\xi)$ is a singleton. Furthermore, it follows from Lemma 2.1 that $\Delta(\xi) = \Delta_1(\xi)$. Theorem is proved. $\square$

Proof of Corollary. Let $x \in D$. By Theorem, we see that $K(x, \cdot)$ extends continuously to $\bar{D} \setminus \{x_0\}$. Moreover, it follows from the first paragraph of this section that $K(\cdot, \xi_1) \neq K(\cdot, \xi_2)$ if $\xi_1, \xi_2 \in \partial D$ are distinct. Thus Corollary follows. $\square$

7. Remark for bounds in condition (II)

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and let

$$\mathbb{H}_+ := \{x \in \mathbb{R}^n : x_n > 0\} \quad \text{and} \quad \mathbb{H}_- := \{x \in \mathbb{R}^n : x_n < 0\}.$$

In view of dilation, we give examples for $\rho_0 = 1$.

Example of (i) ($\theta_1 > \sin^{-1}(1/A_0)$). Let $w_0 = (0, \dots, 0, A_0)$ and let V_1 be the convex hull of $B(w_0, 1) \cup \{0\}$. We consider the domain

$$D := \left(B(0, A_0 + 1) \setminus (\overline{B(0, A_0 - 1)} \cap \mathbb{H}_+) \right) \cup V_1.$$

Then D satisfies (I) and the union $\mathcal{C}(0)$ in the condition (II) at 0 is $B(0, 2\rho_1) \cap \mathbb{H}_-$, that is, the origin satisfies (II). But there are two minimal Martin boundary points at the origin.

Example of (ii) for $1 < A_0 \leq 2$ ($0 < \theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 > \rho_0 \cos \theta_1$). Let

$$\begin{aligned} w_1 &= (0, 0, \dots, 0, 1), \quad w_2 = (\sqrt{1 - (2 - A_0)^2}, 0, \dots, 0, -1) \quad \text{and} \\ w_3 &= (\sqrt{1 - (2 - A_0)^2}, 0, \dots, 0, A_0 - 1). \end{aligned}$$

Let V_2 be the convex hull of $B(w_2, 1) \cup \{w_3\}$. We consider the domain

$$D := \left(B(0, 5) \setminus \overline{(B(0, 3) \cap \mathbb{H}_+)} \right) \cup B(w_1, 1) \cup V_2.$$

Then D satisfies (I) and $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbb{H}_-$. But there are two minimal Martin boundary points at the origin.

Example of (ii) for $A_0 > 2$ ($0 < \theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 > \rho_0 \cos \theta_1$). Let $w'_1 = (0, \dots, 0, 1)$, $w'_2 = (1, 0, \dots, 0, 1 - A_0)$ and $w'_3 = (1, 0, \dots, 0, 1)$. Let V_3 be the convex hull of $B(w'_2, 1) \cup \{w'_3\}$. We consider the domain

$$D := \left(B(0, 5) \setminus \overline{(B(0, 3) \cap \mathbb{H}_+)} \right) \cup B(w_1, 1) \cup V_3.$$

Then D satisfies (I) and $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbb{H}_-$. But there are two minimal Martin boundary points at the origin.

It is easy to check, in each case, that D is represented as the union of balls $B(z_\lambda, 1)$ and V_i , and that V_i includes a ball of radius 1 and is included in a ball of radius A_0 with the same center. We also observe that any truncated circular cone $\Gamma_{\theta_1}(0, y) \cap B(0, 2\rho_1)$ is not included in $D \cap \mathbb{H}_+$, so that $\mathcal{C}(0) = B(0, 2\rho_1) \cap \mathbb{H}_-$. Moreover, we observe that one limit function obtained by approaching from $D \cap \mathbb{H}_+$ is bounded on $D \cap \mathbb{H}_-$ and another limit function obtained by approaching from $D \cap \mathbb{H}_-$ is bounded on $D \cap \mathbb{H}_+$, so that the origin has two minimal Martin boundary points.

8. Example of a domain satisfying (I) and (II) but not a uniform domain

A domain Ω is called a uniform if there exists a positive constant A with the following property. For each pair of points $x_1, x_2 \in \Omega$ there is a rectifiable curve γ in Ω joining x_1 and x_2 such that

- (i) $\ell(\gamma) \leq A|x_1 - x_2|$,
- (ii) $\min\{\ell(\gamma(x_1, z)), \ell(\gamma(z, x_2))\} \leq A\delta_\Omega(z)$ for all $z \in \gamma$,

where $\ell(\gamma)$ and $\gamma(z, w)$ are the length of γ and the subarc of γ between z and w , respectively.

For simplicity, we give an example when $n = 2$.

Example. Let $a = (0, 2)$, $b = (0, -2)$ and $c = (-2, 0)$. Suppose

$$\Omega := B(a, 2) \cup B(b, 2) \cup B(c, 2).$$

Then Ω satisfies (I) and each boundary point satisfies (II) but not a uniform domain.

In fact, let $p = (0, 1)$ and $w = (x, y)$ be a point in $S(p, 1)$ such that $x > 0$ and $0 < y < 1$, and let $\bar{w} = (x, -y)$. Then $y = 1 - (1 - x^2)^{1/2}$. Let γ_w be an arbitrary rectifiable curve in Ω joining w and $\bar{w}$. Then γ_w must hit y -axis $\{x = 0\}$, and we have

$$\ell(\gamma_w) \geq \text{dist}(w, \{x = 0\}) = x = \frac{x}{1 - (1 - x^2)^{1/2}} y = \frac{1}{2} \frac{x}{1 - (1 - x^2)^{1/2}} |w - \bar{w}|.$$

This inequality shows that a constant A satisfying (i) does not exist since

$$\lim_{x \rightarrow 0^+} \frac{x}{1 - (1 - x^2)^{1/2}} = +\infty.$$

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