

Aging Transitions in Multi-layer Networks of Coupled Oscillators

Kai Morino^{1,a}, Gouhei Tanaka², Kazuyuki Aihara^{1,2}

¹Graduate School of Information Science and Technology, The University of Tokyo, Tokyo, 113-8656, Japan,
^amorino@sat.t.u-tokyo.ac.jp

²Institute of Industrial Science, The University of Tokyo, Tokyo, 153-8505, Japan

Introduction

Coupled oscillators have been used to understand many real-world phenomena, e.g., behaviors in networks of biological oscillatory elements [1]. Since living things eventually die and lose their activity, behaviors of mixed populations composed of active and inactive elements are interesting phenomena. Recently, mixed populations of coupled active and inactive oscillators have received much attention.

When the proportion of inactive elements in a mixed population increases beyond a critical value, global oscillations of the network vanish away: this phenomenon is called an *aging transition* [2]. Aging transitions in mixed populations with active and inactive oscillators have been studied first by Daido and Nakanishi in globally coupled models [2, 3]. Several related studies with regular networks have followed this seminal work [4, 5, 6, 7, 8]. Another possible networks structure found in many real systems such as the cortical column is a multi-layer one. Therefore, aging transitions with active and inactive oscillators in multi-layer networks would be worth consideration.

Our study aims to generalize the framework of the aging transition, because multi-layer models include conventional single-layer models as a special case. Two main important results are reported in our presentation. One is that the interlayer coupling affects the robustness of multi-layer networks. The other is that an increase of mismatch of oscillator types among locally connected oscillators decreases the robustness.

Multi-layer Networks of Coupled Oscillators

Let us introduce a multi-layer network model with L layers. Each layer is composed of N globally coupled elements. The state of the n th element on the l th layer is denoted by a complex variable $z_n^{(l)}$ ($1 \leq l \leq L, 1 \leq n \leq N$). For simplicity, we assume that the interlayer connections exist among the elements with the same index n . An example of a network with $L = 2$ and $N = 4$ is shown in Fig. 1(a). Based on these assumptions, the time evolution equation of $z_n^{(l)}$ is described as follows:

$$\dot{z}_n^{(l)} = F(z_n^{(l)}) + \frac{K}{N} \sum_j (z_j^{(l)} - z_n^{(l)}) + D \cdot G(\mathbf{z}_n), \quad (1)$$

where the structure of interlayer connections is characterized by the function $G(\cdot)$, $\mathbf{z}_n$ is defined as $\{z_n^{(j)}\}_{j=1}^L$, and K and D are the coupling strengths of intralayer and interlayer connections, respectively. We analyze the two-layer network ($L = 2$) in detail. Three important types of coupling structures for $z_n^{(l)}$ ($l = 1, 2$) are described as follows:

$$G(z_n^{(l)}, z_n^{(3-l)}) = \begin{cases} \frac{1}{2}(z_n^{(l)} + z_n^{(3-l)}), & \text{(Case I)} \\ \frac{1}{2}z_n^{(3-l)}, & \text{(Case II)} \\ \frac{1}{2}(z_n^{(3-l)} - z_n^{(l)}). & \text{(Case III)} \end{cases} \quad (2)$$

Cases I, II, and III correspond to mean field, chain, and diffusion interlayer couplings, respectively. Equation (1) with $D = 0$ and $L = 1$ coincides with Daido and Nakanishi's model (the single-layer network) [2]. Each element in our model is assumed to be Stuart-Landau (SL) oscillator: $F(z) = (\alpha + i\Omega - |z|^2)z$, where α is a control parameter and Ω is a natural frequency. The single SL oscillator is active for $\alpha = a > 0$ and inactive for $\alpha = -b < 0$. We set the parameter values at $a = 2$, $b = 1$, and $\Omega = 3$. Hereafter, we denote the proportion of inactive oscillators to the total population by p , which is assumed to be the same value in every layer. By using the order parameter $|Z| := |\sum_{l=1}^L \sum_{n=1}^N z_n^{(l)}|/NL$ for the whole network activity, an aging transition point p_c is defined as a value of p at which $|Z|$ changes from positive to zero by increasing p [2]. The aging transition point p_c can be understood as an index of robustness of a network: a higher value of p_c implies a more robust network. We set $N = 3000$ and $K = 6$ unless otherwise noted in this abstract.

Results

(i) Interlayer coupling scheme affects the robustness

Representing an active oscillator as A and an inactive one as I , the possible combinations of the pairs of oscillators connected between the two layers are classified into (A, A) , (A, I) , (I, A) , and (I, I) . By assuming

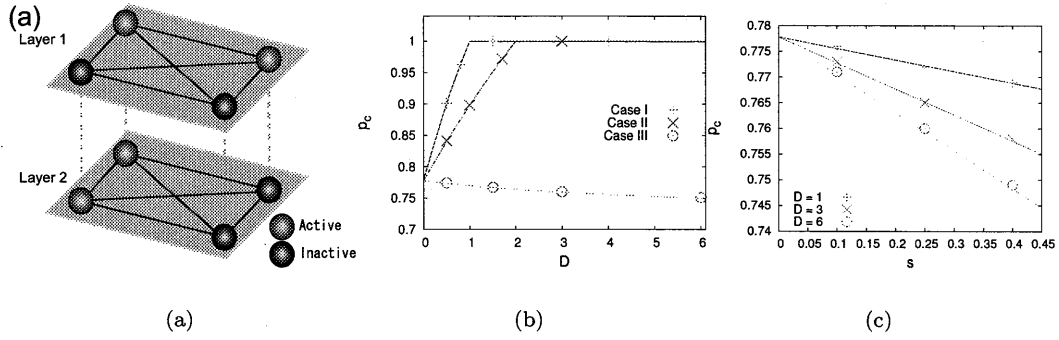


Figure 1: (a) An example of a layered network with $L = 2$ and $N = 4$. The red and blue circles indicate active and inactive oscillators, respectively. In this case, $p = 0.5$. The black solid and green dashed lines indicate global intralayer couplings and local interlayer couplings, respectively. (b) The values of aging transition point p_c are plotted against the coupling strength D in Case I (+), II (x), and III (o). The results obtained from the reduced and original systems are indicated by lines and symbols, respectively. (c) The values of p_c are plotted against s for the values of D of 1.0 (+), 3.0 (x), and 6.0 (o) for Case III.

that all the states of oscillators in each subpopulation are identical, we can reduce Eq. (1) into a four-dimensional system.

Figure 1(b) shows the values of p_c , which are obtained from the reduced (lines) and original (symbols) systems against D for the three Cases. For $D \geq 0$, the values of p_c obtained by the reduced and original systems agree with each other. At $D = 0$, the values of p_c in all the three cases coincide with that in the single-layer case [2] where $p_c = \frac{a(K+b)}{(a+b)K} = 7/9$. In Cases I and II, the value of p_c is greater than $7/9$ for $D \geq 0$, that is, the two-layer network is more robust than the single-layer network. On the contrary, in Case III, the value of p_c is smaller than $7/9$, and so the two-layer network is less robust than the single-layer network. The interlayer coupling scheme affects the robustness of multi-layer networks as mentioned above.

(ii) The mismatch of oscillator types weakens the robustness

Let us define a new parameter s , which accounts for a mismatch of pairs connected by interlayer couplings in two-layer networks. The definition of s is the proportion of mismatched pairs (A, I) and (I, A) in the total number N . Thus, the range of s is restricted to $0 \leq s \leq 1 - |2p - 1|$. We can also reduce the system into a four-dimensional system in this case.

The value of p_c linearly decreases with increase of s , as shown in Fig. 1(c) for Case III. This suggests that the mismatch of pairs decreases the robustness of the network in Case III. Moreover, in Cases I and II, the values of p_c linearly decrease with increase of s as well. This relationship also holds for multi-layer networks with $L \geq 3$. The larger values of L , the values of p_c are smaller and the values of s are larger. This shows that the mismatch of oscillator types weakens the robustness of multi-layer networks.

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