

SATO'S PRINCIPLE FOR MICROLOCALIZATION AT THE BOUNDARY OF A CONVEX SET

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INTRODUCTION

Roughly speaking, Sato's fundamental microlocalization principle asserts that $Pu=f$ implies for any partial differential operator P and any solution u , that one has the inclusion $S.S.(u) \subset S.S.(f) \cup \{(x,\eta) : \overset{\circ}{P}(x,\eta) = 0\}$ in the cotangential spherical bundle of the base space. Here, we are going to see that such a formula remains true if one considers a certain type of microlocalization at the boundary that allows to characterize the possible decompositions in sums of holomorphic functions in special imaginary conic domains admitted by a real analytic function or an hyperfunction defined over a convex set Ω of $\mathbb{R}^n$ near a point $x \in \partial\Omega$. To establish this fact, we are basically going to investigate the conditions under which the morphism induced by a linear differential operator with constant coefficients constitutes an isomorphism of the stalks of the microlocalization sheaf $\mathcal{C}^b$.

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GEOMETRICAL BACKGROUND

From now, let Ω denote a convex open set of $\mathbb{R}^n$, F be its closure in the radial compactification $\mathbb{D}^n \simeq \mathbb{R}^n \cup S_{n-1}^\infty$ of $\mathbb{R}^n$ and $\partial\Omega$ its boundary in $\mathbb{D}^n$. We denote by $z=x+iy$ the points of $F+i\mathbb{R}^n$ and by $\zeta=\xi+i\eta$ the directions of $\mathbb{C}^n$, i.e. the elements of $\mathbb{C}^n \setminus \{0\}$. We identify $\mathbb{C}^n$ provided with the hermitian product $\langle z, \zeta \rangle = \sum z_j \overline{\zeta_j}$ with the euclidean space $\mathbb{R}^{2n}$ provided with the scalar product $\text{Re} \langle z, \zeta \rangle$. We shall denote indifferently by P or $P(D)$ the linear partial differential operator $\sum_{|\alpha| \leq m} c_\alpha D_x^\alpha$ over Ω with constant coefficients c_α or its natural extension to the complex domain $\sum c_\alpha D_z^\alpha$ (where $D_{z_j} = (D_{x_j} - iD_{y_j})/2$). The characteristic variety of P will be denoted by

$$\text{Char}(P) = \{ \zeta \in \mathbb{C}^n \setminus \{0\} : \overset{\circ}{P}(\zeta) = 0 \},$$

where $\overset{\circ}{P}$ is the principal part of the operator. Throughout this paper, we will also mean by ω any open subset of F whose intersection with Ω is convex, by S_{n-1} the unit sphere of $\mathbb{R}^n$ and by S_{n-1}^* the unit cosphere of $\mathbb{R}^n$.

Let us now give a brief description of the sheaves of microlocalization we are going to deal with. If $\Gamma \supset \Gamma'$ are open convex cones of $\mathbb{R}^n$ with vertex 0, we denote by $\Lambda(\omega, \Gamma, \Gamma')$ the profile $\left[\bigcup_{x \in \omega \cap \Omega} \{x\} + i\Gamma \right] \cup \left[\bigcup_{x \in \omega \setminus \Omega} \{x\} + i\Gamma' \right]$ and call a tuboid of profile $\Lambda(\omega, \Gamma, \Gamma')$ any intersection of an open convex neighborhood of $\omega \cap \Omega$ in $\Omega + i\mathbb{R}^n$ with an open subset V of $\Lambda(\omega, \Gamma, \Gamma')$ such that, given any compact set $K \subset \Lambda(\omega, \Gamma, \Gamma')$, one can find $\rho_0 > 0$ such that $x + i\rho y$ belongs to V for every $x + iy \in K$ and every ρ in $]0, \rho_0]$. If A and O denote respectively the sheaf of real analytic functions over Ω and the sheaf of holomorphic functions

over $\Omega + i\mathbb{R}^n$, we denote by $\mathcal{O}_0$ [resp. $\mathcal{O}_1, \mathcal{O}_2$] the sheaf over $SF := F \times S_{n-1}$ that associates to any $SF \cap \Lambda(\omega, \Gamma, \Gamma)$ the space $\varinjlim \mathcal{O}(V)$, where V runs through the family of tuboids of profile $\Lambda(\omega, \mathbb{R}^n, \mathbb{R}^n)$ [resp. $\Lambda(\omega, \mathbb{R}^n, \Gamma), \Lambda(\omega, \Gamma, \Gamma)$]. The quotient sheaves $\mathcal{O}_{1,0} = \mathcal{O}_1 / \mathcal{O}_0$ and $\mathcal{O}_{2,0} = \mathcal{O}_2 / \mathcal{O}_0$ allow to define the sheaves $\mathcal{C}^{b,k}$ ($k=1,2$) which vanish over $\Omega \times S_{n-1}^*$ but, whose stalk at any point $(x_0, \eta_0) \in \partial\Omega \times S_{n-1}^*$ is defined by the formula

$$\mathcal{C}_{(x_0, \eta_0)}^{b,k} = \lim_{\vec{m}} \frac{\mathcal{O}_{k,0}[SF \cap \Lambda(\omega_m, \Gamma_m, \Gamma_m)]}{\delta_{j=1}^n \mathcal{O}_{k,0}[SF \cap \Lambda(\omega_m, \Gamma_{j,m}, \Gamma_{j,m})]} \quad (1)$$

with the following notations. The family ω_m ($m \in \mathbb{N}$) is a decreasing sequence of open neighborhoods of x_0 in F such that $\omega_m \cap \Omega$ is convex for any m , δ is the Čech coboundary operator (i.e. alternate sum of restrictions); if for any $\eta \in S_{n-1}^*$, we denote by E_η the open half-space $\{y \in \mathbb{R}^n : \langle y, \eta \rangle > 0\}$, then Γ_m and $\Gamma_{j,m}$ are defined by $\Gamma_m := \bigcap_{j=1}^n E_{\eta_{j,m}}$ and $\Gamma_{j,m} := \bigcap_{k \neq j} E_{\eta_{k,m}}$, where, for each m in $\mathbb{N}$, $\{\eta_{1,m}, \dots, \eta_{n,m}\}$ is a set of linearly independent points of S_{n-1}^* verifying

$$\begin{cases} \lim_{m \rightarrow \infty} \eta_{j,m} = \eta_0, \quad \forall j \in \{1, \dots, n\} \\ \{\eta_0, \eta_{1,m+1}, \dots, \eta_{n,m+1}\} \in \gamma_m := \left\{ \sum_{j=1}^n r_j \eta_{j,m} : r_j > 0 \right\}, \quad \forall m. \end{cases}$$

It is then possible to define the morphisms ρ^k, σ^k and $\tilde{\sigma}^k$ ($k=1,2$) making exact the rows of the commutative diagrams

$$\begin{array}{ccccccc} 0 & \rightarrow & \tau_* \mathcal{O}_0 & \xrightarrow{\rho^1} & i_* A & \xrightarrow{\sigma^1} & \pi_* \mathcal{C}^{b,1} \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & \tau_* \mathcal{O}_0 & \xrightarrow{\rho^2} & i_* B & \xrightarrow{\sigma^2} & \pi_* \mathcal{C}^{b,2} \rightarrow 0 \end{array} \quad (2)$$

and

$$\begin{array}{ccccccc}
0 \rightarrow \ker \tilde{\sigma}^1 & \longrightarrow & \pi^{-1} \iota_* A & \xrightarrow{\tilde{\sigma}^1} & C^{b,1} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow \ker \tilde{\sigma}^2 & \longrightarrow & \pi^{-1} \iota_* B & \xrightarrow{\tilde{\sigma}^2} & C^{b,2} & \rightarrow & 0
\end{array} \quad (3)$$

where ι is the imbedding $\Omega + i\mathbb{R}^n \rightarrow F + i\mathbb{R}^n$, τ the projection $SF \rightarrow F$, π the projection $S^*F := F \times S_{n-1}^* \rightarrow F$, the ρ^k 's are restriction morphisms, the σ^k 's decomposition morphisms and the $\tilde{\sigma}^k$'s the compositions of the respective $\pi^{-1}\sigma^k$'s with the natural morphisms $\pi^{-1}\pi_* C^{b,k} \rightarrow C^{b,k}$. We may then define the wave-front sets up to the boundary by $W.F.^b(f) = \text{supp}_{C^{b,k}}(\sigma^k f)$ for $f \in A$ if $k=1$ and $f \in B$ if $k=2$.

In what follows, we are going to develop the complete proofs only in the case of A and $C^{b,1}$ (that we are going to denote now by C^b for short). The case of hyperfunctions is entirely similar and the proofs even contain some simplifications related to the use of $\mathcal{O}_2$ instead of that of $\mathcal{O}_1$.

SOME PRELIMINARY PROPOSITIONS

PROPOSITION 1. *Any operator P induces an endomorphism $\tilde{P}$ of C^b verifying $\tilde{\sigma} \circ P = \tilde{P} \circ \tilde{\sigma}$ (P denotes also the trivial extension of the original P as an endomorphism of $\pi^{-1}\iota_* A$). Moreover, $\tilde{P}$ induces a surjective endomorphism of $C^b_{(x_0, \eta_0)}$ for any point (x_0, η_0) in S^*F such that x_0 admits a basis of neighborhoods $\{\omega\}$ verifying $PA(\omega \cap \Omega) = A(\omega \cap \Omega)$.*

Proof. By the definition of $\mathcal{O}_{1,0}$, it is clear that P induces an endomorphism of this sheaf deduced from the usual action of a differential operator over holomorphic functions. The cohomological definition of C^b in terms of derived categories allows to deduce from this morphism the endomorphism $\tilde{P}: C^b \rightarrow C^b$ by derivation of functors.

The equality $\tilde{P} \circ \tilde{\sigma} = \tilde{\sigma} \circ P$ may then be verified only in the stalks. As the derivation of functors commutes with restrictions, it is clear that P coincides in the stalks with the application of P to elements of the numerator of (1). By commutativity of the diagram

$$\begin{array}{ccc} \pi^{-1} \pi_* C^b & \xrightarrow{\pi^{-1} \pi_* \tilde{P}} & \pi^{-1} \pi_* C^b \\ \downarrow & & \downarrow \\ C^b & \xrightarrow{\tilde{P}} & C^b \end{array}$$

and by the stability of the stalks under π^{-1} , we are lead to prove that $\sigma_{x_0} \circ P = (\pi_* \tilde{P})_{x_0} \circ \sigma_{x_0}$ holds for any $x_0 \in F$. By use of Čech cohomology, it is possible to prove that the image under σ of any $f \in (i_* A)(\omega) = A(\omega \cap \Omega)$ [where ω is any neighborhood of x_0 such that $\omega \cap \Omega$ is convex] may be represented by a vector $(([f_1], \dots, [f_{n+1}]))$ with the f_j 's sections of $\mathcal{O}_1$ over domains of SF whose projections over S_{n-1} are such that their polars constitute a covering of S_{n-1}^* for every x in ω . In the above representation, $[]$ means the equivalence class for $\mathcal{O}_0$, $(())$ the equivalence class for Čech coboundaries and one has

$$f = \sum_{j=1}^{n+1} f_j \quad \text{over } \omega \cap \Omega. \quad (*)$$

We get therefore the equality $(\pi_* \tilde{P}) \circ \sigma(f) = (([Pf_1], \dots, [Pf_{n+1}]))$. On the other side, the equality $Pf = \sum_{j=1}^{n+1} Pf_j$ and the uniqueness of the decomposition $(*)$ modulo Čech coboundaries gives also $\sigma \circ P(f) = (([Pf_1], \dots, [Pf_{n+1}])),$ which is enough.

The surjectivity of $\tilde{P}_{(x_0, \eta_0)}$ when $PA(\omega \cap \Omega) = A(\omega \cap \Omega)$ holds for a basis of neighborhoods of x_0 is then a trivial consequence of the exact sequence of microlocalization (3). $\square$

LEMMA 2. Let ω_0 be an open subset of F , $\eta_1, \dots, \eta_n \in S_{n-1}^*$ be linearly independent and Γ be the open cone $\bigcap_{j=1}^n E_{\eta_j}$. For any open convex set ω relatively compact in ω_0 and any tuboid V of profile $\Lambda(\omega_0, \mathbb{R}^n, \Gamma)$, there exists a convex tuboid V' of profile $\Lambda(\omega, \Gamma, \Gamma)$ contained in V and which is constituted by the intersection of a convex neighborhood of $\omega \cap \Omega$ in $\Omega + i\mathbb{R}^n$ with a domain of product type with respect to real and imaginary variables.

Proof. As convex neighborhood of $\omega \cap \Omega$ that will allow to define V' , let us consider the intersection of the convex complex neighborhood of $\omega_0 \cap \Omega$ that defines V with $\omega + i\mathbb{R}^n$. From now, let us also denote for short by V and V' the other two open sets whose intersections with the neighborhoods of $\omega_0 \cap \Omega$ and $\omega \cap \Omega$ we just mentioned constitute the tuboids of the statement.

Let us denote by η and $\eta_{j,m}$ the traces on the cosphere of $\{\lambda \Sigma \eta_j : \lambda < 0\}$ and $\{\lambda(\eta_j + m^{-1}\eta) : \lambda > 0\}$; it is direct to verify that for any $m \in \mathbb{N}$ larger than 1, the $\{\eta_{1,m}, \dots, \eta_{n,m}\}$ still constitute bases of the dual of $\mathbb{R}^n$ and that the cones $\gamma_m := \bigcap_{j=1}^n E_{\eta_{j,m}}$ verify $\cup \gamma_m = \Gamma$ as well as $\gamma_m \subset \gamma_{m+1} \subset \dots \subset \Gamma$ for every $m \geq 2$.

For any $r_m > 0$, we have $\bigcap_{j=1}^n \{y \in \mathbb{R}^n : |y - r_m \eta_{j,m}| < r_m\} \subset \gamma_m$ and therefore, it is straightforward to prove the existence of a sequence of positive numbers r_m that decreases to 0 and such that one has:

$$(\alpha) \quad B_m := \bar{\omega} + i \bigcap_{j=1}^n \{y \in \mathbb{R}^n : |y - r_m \eta_{j,m}| < r_m\} \subset V, \quad \forall m \geq 2.$$

Let us now also prove the following two relations:

$$\left. \begin{aligned} (\beta) \quad & \forall K \subset \bar{\omega} + i\gamma_m, \exists \rho > 0 \text{ s.t. } \{x + ip'y : x + iy \in K, 0 < \rho' \leq \rho\} \subset B_m \\ (\gamma) \quad & \exists \delta_m > 0 \text{ s.t. } \{x + iy \in B_m : \langle y, \eta \rangle > -\delta_m\} \subset B_{m+1}. \end{aligned} \right\}$$

If (β) does not hold, we can find $x_q + iy_q \in K$ such that $x_q + iq^{-1}y_q$

does not belong to B_m , hence such that

$$|y_q|^2 \geq 2 r_m q \inf \{ \langle y, \eta_{j_0, m} \rangle : x+iy \in K \}$$

holds for at least one $j_0 \leq n$. As this lower bound is strictly positive, we get a contradiction by making $q \rightarrow \infty$. If (γ) does not hold, we can find $x_q + iy_q \in B_m \setminus B_{m+1}$ verifying $\langle y_q, \eta \rangle \geq -q^{-1}$. As we have $y_q \in \gamma_m \subset \Gamma$ for every q , as up to a positive coefficient independent of q we also have $\sum_{j=1}^n \langle y_q, \eta_j \rangle \leq q^{-1}$ and as up to the choice of a subsequence we may also suppose $x_q + iy_q \rightarrow x_0 + iy_0 \in \bar{\omega} + i\bar{\gamma}_m$, we get $0 \leq \lim_{q \rightarrow \infty} \langle y_q, \eta_j \rangle = \langle y_0, \eta_j \rangle \leq \sum_{j=1}^n \langle y_0, \eta_j \rangle = 0$ for every j which implies $y_0 = 0$. Considering now the compact set $K := \{x_q + iy_q | y_q|^{-1} : q \in \mathbb{N}\} \subset \bar{\omega} + i\bar{\gamma}_m \subset \bar{\omega} + i\bar{\gamma}_{m+1}$, we get by (β) a number $\rho > 0$ such that $|y_q| \leq \rho$ implies $x_q + iy_q \in B_{m+1}$. Hence a contradiction because $y_q \rightarrow 0$.

We are now going to modify the B_m 's in order to get an increasing sequence of convex sets open in $\bar{\omega} + i\mathbb{R}^n$ verifying

$$(\delta) \quad B'_m \subset \bigcup_{m'=2}^m B_{m'}, \quad \forall m \geq 2,$$

and for which there exists a decreasing sequence of positive numbers $\varepsilon_m \rightarrow 0$ such that the following equality holds:

$$(\varepsilon) \quad \{x+iy \in B'_m : \langle y, \eta \rangle > -\varepsilon_m\} = \{x+iy \in B_m : \langle y, \eta \rangle > -\varepsilon_m\}.$$

As it is clear that $B'_2 = B_2$ and $\varepsilon_2 = 1$ are suitable, let us proceed by induction and suppose $B'_2, \dots, B'_{m-1}$ and $\varepsilon_2, \dots, \varepsilon_{m-1}$ already determined. Combining (γ) and (ε) , we may find $\delta \in]0, \varepsilon_{m-1}[$ such that $\{x+iy \in B'_{m-1} : \langle y, \eta \rangle > -\delta\} \subset B_m$ holds. Let us now prove the existence of $\varepsilon_m \in]0, \delta[$ such that the convex hull of $B'_{m-1} \cup \{x+iy \in B_m : \langle y, \eta \rangle > -\varepsilon_m\}$ is contained in $B_m \cup B'_{m-1}$. If this does not occur, we can find sequences $\theta_q \in [0, 1]$, $z_q = x_q + iy_q \in B'_{m-1}$ and $z'_q = x'_q + iy'_q \in B_m$ such that $\langle y'_q, \eta \rangle > -q^{-1}$ and $\theta_q z_q + (1 - \theta_q) z'_q \notin B_m \cup B'_{m-1}$. Let us first remark that we may suppose $\theta_q \rightarrow \theta_0 \in [0, 1]$, $z_q \rightarrow x_0 + iy_0$ belonging to $\bar{\omega} + i\bar{\gamma}_{m-1}$ and $z'_q \rightarrow x'_0 \in \bar{\omega}$ because one gets directly $\lim y'_q = 0$ by the same argumentation as above. If the y_q 's verify

$\langle y_q, \eta \rangle > -\delta$ for q large enough, we are lead to a first contradiction by convexity of B_m which implies $\theta_q z_q + (1-\theta_q) z'_q \in B_m$. We may therefore suppose $\langle y_q, \eta \rangle \leq -\delta$ for any q . As we have $\langle y'_q, \eta \rangle > -\delta/2$ for q large enough, there exist $\mu_q \in [0,1]$ such that $-\delta/2 = \langle \mu_q y_q + (1-\mu_q) y'_q, \eta \rangle$. If we suppose $\mu_q \rightarrow \mu_0 \in [0,1]$, we have necessarily $\mu_0 \in]0,1[$. As B'_{m-1} is open and convex in $\bar{\omega} + i\mathbb{R}^n$ and admits both $\tilde{x}_0 = \theta_0 x_0 + (1-\theta_0) x'_0$ and $\tilde{x}_0 + i y_0$ as points of its closure, it contains $\tilde{x}_0 + i \mu_0 y_0$ and consequently $\tilde{x}_0 + i(\mu_q y_q + (1-\mu_q) y'_q)$ for q large enough. Let us first consider the possibility $\theta_q \in [0, \mu_q]$; as we may write

$$\theta_q y_q + (1-\theta_q) y'_q = \left(1 - \frac{\theta_q}{\mu_q}\right) y'_q + \frac{\theta_q}{\mu_q} (\mu_q y_q + (1-\mu_q) y'_q)$$

with $\tilde{x}_0 + i y'_q \in B_m$ and $\tilde{x}_0 + i[\mu_q y_q + (1-\mu_q) y'_q] \in \{x + i y \in B'_{m-1} : \langle y, \eta \rangle > -\delta\} \subset \{z \in B_m : \langle y, \eta \rangle > -\delta\}$, we get another contradiction by convexity of B_m . The second possibility $\theta_q \in]\mu_q, 1]$ provides also a contradiction if one writes

$$\theta_q y_q + (1-\theta_q) y'_q = \left(\frac{\theta_q - \mu_q}{1 - \mu_q}\right) y_q + \left(\frac{1 - \theta_q}{1 - \mu_q}\right) [\mu_q y_q + (1-\mu_q) y'_q]$$

because y_q and the factor between brackets belong to B'_{m-1} which is convex.

The conclusion follows then directly by taking as B'_m the convex hull we just considered and as ε_m , the number of which we proved the existence. As a matter of fact, the union of the B'_m 's will provide a convex set which is of required type as one can verify easily by use of (β) ; it is then straightforward to verify that such a set is of product type with respect to the real and imaginary variables. $\square$

PROPOSITION 3. If (x_0, η_0) is a point of $\partial\Omega \times S_{n-1}^*$ such that $\tilde{P}(\eta_0) \neq 0$, then $\tilde{P}$ induces a bijective endomorphism of $C_{(x_0, \eta_0)}^b$.

Proof. Let γ_0 be an open convex salient cone of the dual of $\mathbb{R}^n$ containing η_0 and consider the representation of $C_{(x_0, \eta_0)}^b$ given by formula (1) of the introduction. Using the notations of that paragraph, we may of course suppose that the closures in the complement of the origin of the cones $\gamma_m = \{\sum r_j \eta_{j,m} : r_j > 0\}$ are contained in γ_0 and that $\tilde{P}$ does not vanish over γ_0 .

Any element of $C_{(x_0, \eta_0)}^b$ appears then like the equivalence class of a function f holomorphic over a tuboid V of profile $\Lambda(\omega_m, \mathbb{R}^n, \Gamma_m)$. As it is trivial to prove by using the theory of inductive limits that the restriction map corresponding to the inclusion $\omega_{m+1} + i\Gamma_m \subset \omega_m + i\Gamma_m$ induces in $C_{(x_0, \eta_0)}^b$ the identity operator, lemma 2 allows to suppose that V contains a tuboid V' of profile $\Lambda(\omega_m, \Gamma_m, \Gamma_m)$ which is a convex intersection of a convex neighborhood of $\omega_m \cap \Omega$ with a domain $\tilde{V}'$ of product type. By Malgrange-Ehrenpreis principle, we can solve the equation $Pu = f$ over V' . Let us then remark that $\omega_m \cap \Omega$ is contained in the boundary of V' and that in a neighborhood of any point of $\omega_m \cap \Omega$, V' coincides with $\tilde{V}'$. It is then easy to verify that V' fulfills the condition $C(x, I)$ stated in 4.1 of [1] for $I = -i(\overline{\gamma_m} \setminus \{0\}) \subset -i\gamma_0$. By homogeneity, $\tilde{P}$ does not vanish on I and therefore, theorem 4.1 of [1] asserts that u extends holomorphically on a neighborhood of $\omega_m \cap \Omega$.

The surjectivity of $\tilde{P}_{(x_0, \eta_0)}$ will then follow directly from proposition 1 because u constitutes clearly a section of $\mathcal{O}_1$ over $\omega_m \times (S_{n-1} \cap \Gamma_m)$ such that $\tilde{P}(\tilde{\sigma}u) = \tilde{\sigma}(Pu) = \tilde{\sigma}f$ holds in $C^b[\omega_m \times (S_{n-1} \cap \Gamma_m)]$ and because the decomposition of f is unique modulo Čech coboundaries.

To prove the injectivity of $\tilde{P}_{(x_0, \eta_0)}$, let us again denote by f a representing function of an element of $C_{(x_0, \eta_0)}^b$, i.e. a holomorphic funct-

ion defined over a tuboid of profile $\Lambda(\omega_m, \mathbb{R}^n, \Gamma_m)$. By homogeneity of $\tilde{P}$, we may find $a \in]0, 1[$ such that $\tilde{P}$ does not vanish over $T(\eta_0) := \{\zeta \in \mathbb{C}^n : |\xi|^2 + |\eta|^2 = 1, |\xi| < a, \eta \in \pm \overline{\gamma_{m-1}} \setminus \{0\}\}$.

If $\tilde{P}(\tilde{\sigma}f) = \tilde{\sigma}(Pf)$ vanishes in $C^b_{(x_0, \eta_0)}$, we may suppose that $Pf \in \bigoplus_{j=1}^n \mathcal{O}_{1,0}[SF \cap \Lambda(\omega_m, \mathbb{R}^n, \Gamma_{j,m})]$; this means the existence of n tuboids V_j of profile $\Lambda(\omega_m, \mathbb{R}^n, \Gamma_{j,m})$ and of functions $g_j \in \mathcal{O}(V_j)$ and $g_0 \in \mathcal{O}_0[SF \cap \Lambda(\omega_m, \Gamma_m, \Gamma_m)]$ such that $Pf = \sum_{j=0}^n g_j$ over $\omega_m \cap \Omega$.

By a restriction affecting only the real variables and by a procedure similar to the one we used to prove the surjectivity of $\tilde{P}$ (use of lemma 2), we may suppose that g_0 is defined over a convex complex neighborhood $\tilde{V}_0$ of $\omega_m \cap \Omega$ and that the V_j 's contain respectively a tuboid $\tilde{V}_j$ of profile $\Lambda(\omega_m, \Gamma_{j,m}, \Gamma_{j,m})$ composed by the intersection of a convex complex neighborhood of $\omega_m \cap \Omega$ with a domain of product type. By Malgrange-Ehrenpreis principle, we may again solve the equations $Pf_j = g_j$ over those $\tilde{V}_j$'s; we hence obtain $P(f - \sum_{j=0}^n f_j) = 0$ on $V' := V \cap \bigcap_{j=0}^n \tilde{V}_j$. According to theorem 2.1 of [1], we may extend $f - \sum_{j=0}^n f_j$ to any open convex set V'' containing V' such that each hyperplane of $\mathbb{R}^{2n}$ whose normal is characteristic and that intersects V'' , intersects also V' .

We are going to take

$$V'' := \bigcup_{\zeta \in \text{Char}(P)} \bigcap_{z' \in V'} \{z \in \mathbb{C}^n : \text{Re}\langle z - z', \zeta \rangle = 0\}$$

as such a V'' . As a matter of fact, V'' is open because otherwise, we could find a point $z_0 \in V''$ and a sequence $z_m \notin V''$ converging to z_0 . It should therefore exist some $\zeta_m \in \text{Char}(P)$ such that any $z' \in V'$ verifies $\text{Re}\langle z_m - z', \zeta_m \rangle \neq 0$. By convexity of V' and up to the extraction of a subsequence, we may suppose $\zeta_m \rightarrow \zeta_0 \in \text{Char}(P)$ and $V' \subset \{z : \text{Re}\langle z_m - z, \zeta_m \rangle > 0\}$ for any m .

By taking the limit and using the fact that V' is open, we obtain $V' \subset \{z : \operatorname{Re}\langle z_0 - z, \zeta_0 \rangle > 0\}$, which contradicts $z_0 \in V''$.

The convexity of V'' will follow from the convexity of V' as one may verify directly. More important is the fact that V'' contains $\omega_m \cap \Omega$. As a matter of fact, we shall prove this by distinguishing the two cases $|\xi| < a$ and $|\xi| \geq a$. In the first one, we get necessarily $\eta \neq \overline{\gamma}_m$ and hence, we may find y_0 in Γ_m such that $\langle y_0, \eta \rangle = 0$. Any $x \in \omega_m \cap \Omega$ is then the center of a compact ball b contained in $\omega_m \cap \Omega$; we may then find $\rho_0 > 0$ such that $x' + i\rho y_0$ belongs to V' for any x' in b and any $\rho \in]0, \rho_0]$. As there exists at least one x' in b such that $\langle x - x', \xi \rangle = 0$, the point $z := x' + i\rho y_0 \in V'$ verifies $\operatorname{Re}\langle x - z, \zeta \rangle = 0$. In the second opportunity ($|\xi| \geq a$), there exists certainly $r > 0$ such that the function $\langle x, \cdot, \xi \rangle$ takes all the values between $-r$ and r in b . Let us then consider a point y of Γ_m ; there exists again $\rho_0 > 0$ such that $x' + i\rho y$ belongs to V' for any x' in b and ρ in $]0, \rho_0]$. Hence we may choose ρ in order to get $|\langle \rho y, \eta \rangle| < r$ and then x' in b to get $\langle x - x', \xi \rangle = \rho \langle y, \eta \rangle$, which is also sufficient.

The injectivity of $\tilde{P}_{(x_0, \eta_0)}$ follows then directly because the function $f - \sum_{j=0}^n f_j$ will then constitute in fact a section of $\mathcal{O}_0$. $\square$

SATO'S PRINCIPLE FOR C^b .

THEOREM. If u and f are simultaneously real analytic functions (or hyperfunctions) over the intersection $\omega \cap \Omega$ and verify $Pu = f$, the following inclusions hold:

$$W.F.^b(f) \subset W.F.^b(u) \subset W.F.^b(f) \cup \{(x, \eta) \in \partial\Omega \times S_{n-1}^* : \tilde{P}(\eta) = 0\}$$

Proof. The first inclusion is a trivial consequence of

the linearity of P . Let us now consider a point (x_0, η_0) which does not belong neither to $W.F.^b(f)$ nor to $\partial\Omega \times \text{Char}(P)$. If we identify u , f , Pu and Pf with their respective images in $(\pi^{-1}_*A)_{(x_0, \eta_0)}$, we get by proposition 1 the relation $\tilde{P}(\tilde{\sigma}u) = \tilde{\sigma}(Pu) = \tilde{\sigma}(f) = 0$ and by proposition 3, this implies that $u=0$, i.e. $(x_0, \eta_0) \notin W.F.^b(u)$. $\square$

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