

Asymptotic efficiency of $\{C_n\}$ -consistent estimators

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$x_1, x_2, \dots, x_n$ は, 密度関数 $f(x, \theta)$ をもつ母集団からの任意標本とする. ($\theta \in \Theta$).

$f(x, \theta)$ は次の (i) ~ (v) を満たす.

(i) $\{x | f(x, \theta)\}$ は θ と独立である.

(ii) $f(x, \theta)$ は θ に関して 4 回連続偏微分可能で, その偏導関数は連続である.

(iii)

$$E_{\theta}(|\log f(x, \theta)|) < \infty$$

$$0 < I(\theta) = -E_{\theta}\left(\frac{\partial^2}{\partial \theta^2} \log f(x, \theta)\right)$$

(iv) 任意の $\theta \in \Theta$ に対して, θ の近傍 Θ_0 が存在して,

$$\left|\frac{\partial^i}{\partial \theta^i} \log f(x, \theta)\right| < G(x) \quad (i=1, 2, 3),$$

$$\left|\frac{\partial^i}{\partial \theta^i} \log f(x, \theta)\right| < H(x) \quad (i=4) \quad (\forall \theta \in \Theta_0)$$

かつ

$$E_{\theta}[\{G(x)\}^4] < \infty, \quad E_{\theta}\{H(x)\} < \infty$$

(V) 最尤推定量 $\hat{\theta}_{ML}$ は $o(n^{-\frac{1}{2}})$ まで Θ_0 の中で一様に Edgeworth 展開可能である.

order $\{c_n\}$ の一致推定量 $\hat{\theta}_n$ が, 任意の θ の近傍 Θ_0 で一様に

$$\begin{aligned} & \lim_{n \rightarrow \infty} c_n^{k-1} |P_\theta(\hat{\theta}_n \leq \theta) - g(c_n^{-1}, \theta)| \\ &= \lim_{n \rightarrow \infty} c_n^{k-1} |P_\theta(\hat{\theta}_n \geq \theta) - 1 + g(c_n^{-1}, \theta)| = 0 \end{aligned}$$

を満たすとき, $\hat{\theta}_n$ は k -th order asymptotically $g(c_n^{-1}, \theta) - \frac{1}{2}$ biased estimator であるといい, そのような推定量の全体を $C(g(c_n^{-1}, \theta), k)$ で表わすことにする.

$\hat{\theta}_n \in C(g(c_n^{-1}, \theta), k)$ が, 任意の $a, b (> 0)$ および任意の θ に対して,

$$\begin{aligned} & \lim_{n \rightarrow \infty} c_n^{k-1} \{P_\theta(-a \leq c_n(\hat{\theta}_n - \theta) \leq b) - \max_{\hat{\theta}_n \in C(g(c_n^{-1}, \theta), k)} P_\theta(-a \leq c_n(\hat{\theta}_n - \theta) \leq b)\} \\ & \geq 0 \end{aligned}$$

を満たすとき, $\hat{\theta}_n$ は k -th order asymptotically efficient in $C(g(c_n^{-1}, \theta), k)$ であるという. $g(c, \theta)$ は $C(|c| \leq c_0)$, θ に関して偏微分可能で, その偏導関数は連続とする.

$\hat{\theta}_n \in C(g(n^{-\frac{1}{2}}, \theta), 2)$ ならば, 最強力検定の方法により任意の $t > 0$ に対して,

$$P_\theta(\sqrt{n}(\hat{\theta}_n - \theta) \leq t) \leq P_\theta(T_n \geq a_n)$$

に代し

$$T_n = \sum \log \frac{f(x_i, \theta)}{f(x_i, \theta + \frac{t}{\sqrt{n}})},$$

a_n は、十分大きい n に対して、

$$P_{\theta + \frac{t}{\sqrt{n}}} (T_n \geq a_n) = g(n^{-\frac{1}{2}}, \theta + \frac{t}{\sqrt{n}}) + o(n^{-\frac{1}{2}})$$

を満たす。

一方 (iv) により、 T_n は $o(n^{-\frac{1}{2}})$ まで、 θ に関して局所一様に Edgeworth 展開可能であるから、

$$E_{\theta + \frac{t}{\sqrt{n}}} (T_n) = -\frac{t^2}{2} I(\theta) - \frac{t^3}{6\sqrt{n}} (3J(\theta) + 2K(\theta)) + o(n^{-\frac{1}{2}})$$

$$V_{\theta + \frac{t}{\sqrt{n}}} (T_n) = I(\theta)t^2 + \frac{t^3}{\sqrt{n}} (J(\theta) + K(\theta)) + o(n^{-\frac{1}{2}})$$

$$E_{\theta + \frac{t}{\sqrt{n}}} \{ (T_n - E_{\theta + \frac{t}{\sqrt{n}}} (T_n))^3 \} = -\frac{t^3}{\sqrt{n}} K(\theta) + o(n^{-\frac{1}{2}})$$

に代し

$$J(\theta) = E_{\theta} \left(\frac{\partial^2}{\partial \theta^2} \log f(x, \theta) \cdot \frac{\partial}{\partial \theta} \log f(x, \theta) \right)$$

$$K(\theta) = E_{\theta} \left\{ \left(\frac{\partial}{\partial \theta} \log f(x, \theta) \right)^3 \right\}$$

を用いて

$$b_n = \frac{a_n + \frac{1}{2} I(\theta) t^2}{\sqrt{I(\theta)} t}$$

とおくと

$$-b_n = U(g_{00}(\theta)) + \frac{1}{\sqrt{n}} \frac{g_{10}(\theta) + g_{01}(\theta)t}{\phi(U(g_{00}(\theta)))} + \frac{t^2}{\sqrt{n}} \frac{3J(\theta) + K(\theta)}{I(\theta)}$$

$$+ \frac{t}{2\sqrt{n}} \frac{J(\theta) + K(\theta)}{I(\theta)} U(g_{00}(\theta)) + \frac{1}{\sqrt{n}} \frac{K(\theta)}{6\sqrt{I(\theta)} I(\theta)} [\{U(g_{00}(\theta))\}^2 - 1]$$

$$+ o(n^{-\frac{1}{2}})$$

$$t \in \mathbb{R}, \quad g_{00}(\theta) = g(0, \theta), \quad g_{10}(\theta) = \frac{\partial}{\partial \theta} g(0, \theta), \quad g_{01}(\theta) = \frac{\partial}{\partial \theta} g(0, \theta)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt, \quad \phi(x) = \Phi'(x), \quad u(g) = \Phi^{-1}(g)$$

従って

$$E_{\theta}(T_n) = \frac{t^2}{2} I(\theta) + \frac{t^3}{6\sqrt{n}} (3J(\theta) + K(\theta)) + o(n^{-\frac{1}{2}})$$

$$V_{\theta}(T_n) = t^2 I(\theta) + \frac{t^3}{\sqrt{n}} J(\theta) + o(n^{-\frac{1}{2}})$$

$$E_{\theta}[\{T_n - E_{\theta}(T_n)\}^3] = -\frac{t^3}{\sqrt{n}} K(\theta) + o(n^{-\frac{1}{2}})$$

よって

$$P_{\theta}(T_n \geq a_n) = 1 - P_{\theta}\left(\frac{T_n - \frac{1}{2}I(\theta)t^2}{\sqrt{I(\theta)}t} \leq b_n - \sqrt{I(\theta)}t\right)$$

$$= \Phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}t)$$

$$+ \frac{1}{\sqrt{n}} \phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}t) \left\{ \frac{g_{10}(\theta) + t g_{01}(\theta)}{\phi(u(g_{00}(\theta)))} \right.$$

$$\left. + \frac{tK(\theta)}{6I(\theta)} u(g_{00}(\theta)) + \frac{t^2(3J(\theta) + 2K(\theta))}{6\sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})$$

よって $t > 0$ のとき

$$P_{\theta}(\sqrt{n}(\hat{\theta}_n - \theta) \leq t) \leq \Phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}t)$$

$$+ \frac{1}{\sqrt{n}} \phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}t) \left\{ \frac{g_{10}(\theta) + t g_{01}(\theta)}{\phi(u(g_{00}(\theta)))} + \frac{tK(\theta)}{6I(\theta)} u(g_{00}(\theta)) \right.$$

$$\left. + \frac{t^2(3J(\theta) + K(\theta))}{6\sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})$$

同様にして, $t > 0$ のとき

$$\begin{aligned}
 P_0(\sqrt{n}(\hat{\theta}_n - \theta) \leq -t) &\geq \Phi(u(g_{00}(\theta)) - \sqrt{I(\theta)}t) \\
 &+ \frac{1}{\sqrt{n}} \phi(u(g_{00}(\theta)) - \sqrt{I(\theta)}t) \left\{ \frac{g_{10}(\theta) - g_{01}(\theta)t}{\phi(u(g_{00}(\theta)))} - \frac{tK(\theta)}{6I(\theta)} u(g_{00}(\theta)) \right. \\
 &\quad \left. + \frac{t^2(3J(\theta) + 2K(\theta))}{6\sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})
 \end{aligned}$$

定理 1. θ の推定量 $\hat{\theta}_n$ が, 任意の t に対して,

$$\begin{aligned}
 P_0(\sqrt{n}(\hat{\theta}_n - \theta) \leq t) &= \Phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}t) \\
 &+ \frac{1}{\sqrt{n}} \phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}t) \left\{ \frac{g_{10}(\theta) + g_{01}(\theta)t}{\phi(u(g_{00}(\theta)))} + \frac{tK(\theta)}{6I(\theta)} u(g_{00}(\theta)) \right. \\
 &\quad \left. + \frac{t^2(3J(\theta) + 2K(\theta))}{6\sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})
 \end{aligned}$$

を満たすならば, $\hat{\theta}_n$ は, *second order asymptotically efficient* in $C(g(n^{-\frac{1}{2}}, \theta), 2)$

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ならば,

$$Z_n = \sqrt{nI(\theta)} \left(\hat{\theta}_n + \frac{u(g_{00}(\theta))}{\sqrt{nI(\theta)}} - \theta \right)$$

とおくとき,

$$E_{\theta}(Z_n) = \frac{1}{\sqrt{n}} \left\{ \frac{g_{01}(\theta) u(g_{00}(\theta))}{\phi(u(g_{00}(\theta))) \sqrt{I(\theta)}} - \frac{g_{10}(\theta)}{\phi(u(g_{00}(\theta)))} \right. \\ \left. - \frac{3J(\theta) + K(\theta)}{6I(\theta) \sqrt{I(\theta)}} \{u(g_{00}(\theta))\}^2 - \frac{3J(\theta) + 2K(\theta)}{6I(\theta) \sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})$$

$$V_{\theta}(Z_n) = 1 + \frac{2}{\sqrt{n}} \left\{ - \frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta))) \sqrt{I(\theta)}} + \frac{2J(\theta) + K(\theta)}{2I(\theta) \sqrt{I(\theta)}} u(g_{00}(\theta)) \right\} + o(n^{-\frac{1}{2}})$$

$$E_{\theta}[\{Z_n - E_{\theta}(Z_n)\}^3] = - \frac{1}{\sqrt{n}} \frac{3J(\theta) + 2K(\theta)}{I(\theta) \sqrt{I(\theta)}} + o(n^{-\frac{1}{2}})$$

最尤推定量 $\hat{\theta}_{ML}$ に対して

$$E_{\theta} \{ \sqrt{n I(\theta)} (\hat{\theta}_{ML} - \theta) \} = - \frac{J(\theta) + K(\theta)}{2 \sqrt{n} I(\theta) \sqrt{I(\theta)}} + o(n^{-\frac{1}{2}})$$

$$V_{\theta} \{ \sqrt{n I(\theta)} (\hat{\theta}_{ML} - \theta) \} = 1 + o(n^{-\frac{1}{2}})$$

$$E_{\theta} \left[\{ \sqrt{n I(\theta)} (\hat{\theta}_{ML} - \theta) - E_{\theta}(\sqrt{n I(\theta)} (\hat{\theta}_{ML} - \theta)) \}^3 \right] \\ = - \frac{1}{\sqrt{n}} \frac{3J(\theta) + 2K(\theta)}{I(\theta) \sqrt{I(\theta)}} + o(n^{-\frac{1}{2}})$$

従って

$$Z_n = \left\{ 1 + \frac{1}{\sqrt{n}} \left(- \frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta))) \sqrt{I(\theta)}} + \frac{2J(\theta) + K(\theta)}{2I(\theta) \sqrt{I(\theta)}} u(g_{00}(\theta)) \right) \right\} \\ \times \sqrt{n I(\theta)} (\hat{\theta}_{ML} - \theta) + \frac{1}{\sqrt{n}} \left\{ \frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta))) \sqrt{I(\theta)}} u(g_{00}(\theta)) \right. \\ \left. - \frac{g_{10}(\theta)}{\phi(u(g_{00}(\theta)))} - \frac{3J(\theta) + K(\theta)}{6I(\theta) \sqrt{I(\theta)}} \{u(g_{00}(\theta))\}^2 + \frac{K(\theta)}{6I(\theta) \sqrt{I(\theta)}} \right\} + o_p(n^{-\frac{1}{2}})$$

2.7

$$\begin{aligned}\hat{\theta}_n &= \hat{\theta}_{ML} + \frac{1}{\sqrt{n}} \left(-\frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} + \frac{2J(\theta) + K(\theta)}{2I(\theta)\sqrt{I(\theta)}} u(g_{00}(\theta)) \right) (\hat{\theta}_{ML} - \theta) \\ &\quad - \frac{u(g_{00}(\theta))}{\sqrt{nI(\theta)}} + \frac{1}{n} \left\{ \frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} u(g_{00}(\theta)) - \frac{g_{10}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} \right. \\ &\quad \left. - \frac{3J(\theta) + K(\theta)}{6\{I(\theta)\}^2} \{u(g_{00}(\theta))\}^2 + \frac{K(\theta)}{6\{I(\theta)\}^2} \right\} + o_p(n^{-1}),\end{aligned}$$

2.8

$$\begin{aligned}\hat{\theta}_n &= \hat{\theta}_{ML} - \frac{1}{\sqrt{n}} \frac{u(g(\hat{\theta}_{ML}))}{\sqrt{I(\hat{\theta}_{ML})}} + \frac{1}{n} \left\{ \frac{g_{01}(\hat{\theta}_{ML}) u(g_{00}(\hat{\theta}_{ML}))}{\phi(u(g_{00}(\hat{\theta}_{ML}))) I(\hat{\theta}_{ML})} \right. \\ &\quad - \frac{g_{10}(\hat{\theta}_{ML})}{\phi(u(g(\hat{\theta}_{ML})))\sqrt{I(\hat{\theta}_{ML})}} - \frac{3J(\hat{\theta}_{ML}) + K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2} \{u(g_{00}(\hat{\theta}_{ML}))\}^2 \\ &\quad \left. + \frac{K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2} \right\} + o_p(n^{-1})\end{aligned}$$

定理 2.

$$\begin{aligned}\hat{\theta}_{ML} &- \frac{1}{\sqrt{n}} \frac{u(g(\hat{\theta}_{ML}))}{\sqrt{I(\hat{\theta}_{ML})}} + \frac{1}{n} \left\{ -\frac{g_{10}(\hat{\theta}_{ML})}{\phi(u(g_{00}(\hat{\theta}_{ML})))\sqrt{I(\hat{\theta}_{ML})}} + \frac{g_{01}(\hat{\theta}_{ML}) u(g_{00}(\hat{\theta}_{ML}))}{\phi(u(g_{00}(\hat{\theta}_{ML}))) I(\hat{\theta}_{ML})} \right. \\ &\quad \left. - \frac{3J(\hat{\theta}_{ML}) + K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2} \{u(g_{00}(\hat{\theta}_{ML}))\}^2 + \frac{K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2} \right\}\end{aligned}$$

(2, second order asymptotic efficient in $C(g(\pi^{\frac{1}{2}}; \theta), 2)$).

定理1より次の結果が得られる。

定理3. θ の推定量 $\hat{\theta}_n$ に対して, 連続関数 $h(\theta)$ が存在して, 任意の t に対して,

$$P_0(\sqrt{n}(\hat{\theta}_n - \theta) \leq t) = \Phi(\sqrt{I(\theta)}t)$$

$$+ \phi(\sqrt{I(\theta)}t) \left\{ \sqrt{2\pi} g_{10}(\theta) + \frac{t^2(2K(\theta) + 3J(\theta))}{6\sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})$$

が成り立つならば, $\hat{\theta}_n$ は, *second order two-sided asymptotically efficient in $\mathcal{S}(G(n^{-\frac{1}{2}}, \theta), 2)$ である.*

定理4. 任意の連続関数 $h(\theta)$ に対して,

$$\hat{\theta}_{ML} + \frac{1}{n} \left\{ \frac{\sqrt{2\pi} h(\hat{\theta}_{ML})}{\sqrt{I(\hat{\theta}_{ML})}} + \frac{K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2} \right\}$$

は, *second order two-sided asymptotically efficient in $\mathcal{S}(G(n^{-\frac{1}{2}}, \theta), 2)$ である.*

系1

$$\hat{\theta}_{ML} + \frac{1}{n} \frac{K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2}$$

は, *second order two-sided asymptotically efficient in $\mathcal{S}(G(n^{-\frac{1}{2}}, \theta), 2)$ である.*

M. Akahira & K. Takeuchi: Asymptotic Efficiency of Statistical Estimators (Springer)