

A SUBRECURSIVE INACCESSIBLE ORDINAL

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INTRODUCTION

The purpose of the article is to prove the minimal subrecursive inaccessibility of the ordinal τ introduced by Wainer[6]. We call an ordinal α *subrecursive inaccessible* (or *s-inaccessible*) if the slow-growing hierarchy $\{G_\gamma | \gamma \leq \alpha\}$ of number-theoretic functions *catches up with* the fast-growing hierarchy $\{F_\gamma | \gamma \leq \alpha\}$ i.e., there exists $p < \omega$ such that for all $x > p$,

$$G_\alpha(x) < F_\alpha(x) \leq G_\alpha(x+1).$$

In the article, we will complete the proof of the result of [6] that τ is a minimal s-inaccessible, by showing

- (I) Collapsing theorem (Section 2), and
- (II) (3)-built-upness of τ (Section 3).

We will use the result of [4] (the strong normalization theorem) when we will show (I) and (II).

It is known from the results of Girard[3] (cf.[6,Example 4]) that the set-theoretic ordinal height of τ is $\sup\{|ID_\nu| : \nu < \omega\}$ where ID_ν is the theory for ν -times iterated inductive definitions and $|ID_\nu|$ is its proof-theoretic ordinal. Hence (II) above indicates that Wainer's fundamental sequences for $|ID_\nu|$ ($\nu < \omega$) is natural in the sense of subrecursive hierarchy theory.

§1. SUBRECURSIVE INACCESSIBILITY

In this section we will define a tree-ordinal τ following [6] and show that τ is minimal s -inaccessible (Theorem 1.10 below) assuming the collapsing theorem and (3)-built-upness of τ which will be proved in Sections 2 and 3 respectively. In the following, the letters k, m, n, p, x denote non-negative integers.

DEFINITION 1.1.(1)(cf.[1]) The set Ω of the *tree-ordinals* consists of the infinitary terms generated inductively by:

- (i) $0 \in \Omega$.
- (ii) If $\alpha \in \Omega$, then $\alpha+1 \in \Omega$.
- (iii) If $\alpha_x \in \Omega$ for all $x < \omega$, then $(\alpha_x)_{x < \omega} \in \Omega$. (In this case we call $(\alpha_x)_{x < \omega}$ limit and write $\alpha[x]$ instead of α_x .)
- (2) For a given $p < \omega$, the subset $\Omega^{(p)\text{-bu}} \subseteq \Omega$ of (p) -*built-up* tree-ordinals consists of those $\alpha \in \Omega$ satisfying that:

$$\lambda[x] <_p \lambda[x+1] \quad \text{for all limit } \lambda \leq \alpha \text{ and } x < \omega,$$

where the relations $< (<_p)$ on Ω are the transitive closure of
 (i) $\beta < \beta+1$ ($\beta <_p \beta+1$) (ii) $\beta[x] < \beta$ for all $x < \omega$ ($\beta[p] < \beta$ resp.) if β is limit.

Then we define the fast-growing $\{F_\alpha\}_{\alpha \in \Omega}$ and slow-growing $\{G_\alpha\}_{\alpha \in \Omega}$ hierarchies as follows:

$$\begin{aligned} F_0(x) &= x+1, & G_0(x) &= 0, \\ F_{\alpha+1}(x) &= F_\alpha^x(F_\alpha(x)), & G_{\alpha+1}(x) &= G_\alpha(x)+1, \\ F_\lambda(x) &= F_{\lambda[x]}(x), & G_\lambda(x) &= G_{\lambda[x]}(x), \end{aligned}$$

where λ is limit and the superscript x denotes iteration x -times

of F_α (i.e., if $F: \omega \rightarrow \omega$ then $F^0(x) = x$, $F^{m+1}(x) = F(F^m(x))$).

PROPOSITION 1.2([5, Theorem 3.1]). *For some $p < \omega$, we assume $\alpha \in \Omega^{(p)-bu}$. Then the following holds:*

- (1) $F_\alpha(x) < F_\alpha(x+1)$ and $G_\alpha(x) \leq G_\alpha(x+1)$ for $p \leq x+1$.
- (2) If $\beta <_m \alpha$ for $p \leq m$, then $F_\beta(x) < F_\alpha(x)$ and $G_\beta(x) < G_\alpha(x)$ for $x > m$.

Proof. Induction on $\alpha \in \Omega$ similarly to [5, Theorem 3.1]. $\square$

LEMMA 1.3. *For $p < \omega$ and $\alpha \in \Omega^{(p)-bu}$, the following holds:*

- (1) For all $x > p$, $G_\alpha(x) < F_\alpha(x)$.
- (2) If α is s -inaccessible (see Intro. for the definition), then α is limit and G_α eventually dominates every F_β with $\beta < \alpha$ (i.e., for all but finitely many x , $F_\beta(x) < G_\alpha(x)$).

Proof. (1) Induction on α . (2) Clearly α cannot be 0. For any $\beta+1 \in \Omega^{(p)-bu}$ and $x > p$,

$$G_{\beta+1}(x) = G_\beta(x)+1 < F_\beta(x)+1 \leq F_\beta(x+1) \leq F_\beta^{x+1}(x) = F_{\beta+1}(x).$$

Hence α must be limit. Assume $\beta < \alpha$. Then $\beta+1 < \alpha$ since α is

limit, and then we can see that for some $m > p$, $\beta+1 <_m \alpha$. Hence $F_\beta(x+1) < F_\beta^{x+1}(x) = F_{\beta+1}(x) < F_\alpha(x) \leq G_\alpha(x+1)$, by 1.2. $\square$

PROPOSITION 1.4([7, p.215]). *Let $p < \omega$ and $\alpha \in \Omega^{(p)-bu}$ satisfy that $G_{\alpha[n+1]} = F_{\alpha[n]}$ for all $n < \omega$. Then α is s -inaccessible and, if $\alpha[0]$ is finite (i.e., $\alpha[0] = 0+1+\dots+1$), then no $\beta < \alpha$ is s -inaccessible.*

Proof. If $G_{\alpha[n+1]} = F_{\alpha[n]}$ for each n , then $F_\alpha(x) = F_{\alpha[x]}(x) =$

$G_{\alpha[x+1]}(x) \leq G_{\alpha[x+1]}(x+1) = G_{\alpha}(x+1)$ and so α is s -inaccessible. If $\alpha[0]$ is finite and $\beta < \alpha$ were s -inaccessible then $\alpha[0] < \beta$ since β is limit. So $\alpha[n] < \beta \leq \alpha[n+1]$ for some n . By 1.3, for sufficient large x , $G_{\alpha[n+1]}(x) = F_{\alpha[n]}(x) < G_{\beta}(x) \leq G_{\alpha[n+1]}(x)$ since $\beta \leq_x \alpha[n+1]$. $\square$

DEFINITION 1.5([6]). The sets Ω_n of *higher level tree-ordinals* are defined by induction similarly to the case of Ω :

- (i) $0 \in \Omega_n$.
- (ii) If $\alpha \in \Omega_n$, then $\alpha+1 \in \Omega_n$.
- (iii) If $\alpha_{\gamma} \in \Omega_n$ for all $\gamma \in \Omega_k$ ($k < n$), then $(\alpha_{\gamma})_{\gamma \in \Omega_k} \in \Omega_n$. (In this case, we call $(\alpha_{\gamma})_{\gamma \in \Omega_k}$ limit and write $\alpha[\gamma]$ instead of α_{γ} .)

In the following we identify Ω_0 with ω , and Ω_1 with Ω .

DEFINITION 1.6([6, Definition 5]). The level n fast-growing hierarchies of functions $\varphi_n: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n$ is defined by:

- (i) $\varphi_n(0, \beta) = \beta+1$,
- (ii) $\varphi_n(\alpha+1, \beta) = \varphi_n^{\beta}(\alpha, \varphi_n(\alpha, \beta))$,
- (iii) $\varphi_n(\lambda, \beta) = (\varphi_n(\lambda[\gamma], \beta))_{\gamma \in \Omega_k}$ for $\lambda = (\lambda[\gamma])_{\gamma \in \Omega_k}$ ($k < n$),
- (iv) $\varphi_n(\lambda, \beta) = \varphi_n(\lambda[\beta], \beta)$ for $\lambda = (\lambda[\gamma])_{\gamma \in \Omega_n}$,

where φ_n^{β} denotes the iteration β -times of φ_n (i.e., if $\psi: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n$, then $\psi^0(\alpha, \beta) = \beta$, $\psi^{\delta+1}(\alpha, \beta) = \psi(\alpha, \psi^{\delta}(\alpha, \beta))$, $\psi^{\lambda}(\alpha, \beta) = (\psi^{\lambda[\gamma]}(\alpha, \beta))_{\gamma \in \Omega_m}$ for $\lambda = (\lambda[\gamma])_{\gamma \in \Omega_m}$).

Note that, in the case $n = 0$, $\varphi_0(\alpha, \beta) = F_{\alpha}(\beta)$ for $\alpha \in \Omega_1$ and

$\beta \in \Omega_0 (= \omega)$. We define $\omega_k \in \Omega_n$ by $\omega_k = (\gamma)_{\gamma \in \Omega_k}$ (i.e., $\omega_k[\gamma] = \gamma$).

DEFINITION 1.7([6, Definition 7]). The sets $T_n (\subseteq \Omega_n)$ of *named tree-ordinals* are defined inductively by:

- (i) $0, 1, \omega_0, \dots, \omega_{n-1} \in T_n$.
- (ii) $T_k \subseteq T_n$ for $k < n$.
- (iii) If $\alpha \in T_{n+1}$ and $\beta, \gamma \in T_n$, then $\varphi_n^\gamma(\alpha, \beta) \in T_n$.

COLLAPSING THEOREM([6]). Let $x < \omega$, $\alpha \in T_2$ and $\beta \in T_1$. Then

$$G_{\varphi_1}(\alpha, \beta)(x) = F_{c\alpha}(G_\beta(x)),$$

where the function $c (= c_x)$ which collapses each T_{n+1} to T_n is

defined by: $c0 = 0$, $c1 = 1$, $c\omega_0 = x$, $c\omega_{k+1} = \omega_k$,

$c(\varphi_{k+1}^\gamma(\delta, \xi)) = \varphi_k^{c\gamma}(c\delta, c\xi)$, $c(\varphi_0^\gamma(\delta, \xi)) = \varphi_0^\gamma(\delta, \xi)$. Hence, in particular, if α is generated in T_2 without reference to ω_0

then, as $G_{\omega_0}(x) = x$, we have $G_{\varphi_1}(\alpha, \omega_0) = F_{c\alpha}$.

Proof. See Section 2. □

DEFINITION 1.8([6, Example 4]). We define $\tau = (\tau[x])_{x < \omega}$ by setting $\tau[0] = 3$, $\tau[n+1] = \varphi_1(\dots \varphi_n(\varphi_{n+1}(3, \omega_n), \omega_{n-1}), \dots, \omega_0)$.

THEOREM 1.9. τ is a minimal s -inaccessible tree-ordinal.

Proof. From Section 3, τ is (3)-built-up. Hence 1.4 and the collapsing theorem complete the proof. □

§2. THE COLLAPSING THEOREM

In this section we will prove the collapsing theorem used in Section 1 using the strong normalization theorem in [4]. First, we introduce term structures $\langle \bar{T}_n, NT_n, \cdot[\cdot], \longrightarrow \rangle$ by considering each element in T_n as a finitary term and each defining equation of φ_n (Definition 1.6) as a rewrite (or reduction) rule of the terms. Let $\bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \dots; \bar{\varphi}_0, \bar{\varphi}_1, \dots$ be formal symbols.

DEFINITION 2.1. The sets $\bar{T}_n$ of *terms* are defined inductively by:

- (i) $\bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \dots, \bar{\omega}_{n-1} \in \bar{T}_n$.
- (ii) $\bar{T}_k \subseteq \bar{T}_n$ for $k < n$.
- (iii) If $a \in \bar{T}_{n+1}$ and $b, c \in \bar{T}_n$, then $\bar{\varphi}_n^c(a, b) \in \bar{T}_n$.

Naturally, terms in $\bar{T}_n$ are interpreted as tree-ordinals by the function $\text{ord}: \bar{T}_n \longrightarrow T_n$ such that (i) $\text{ord}(\bar{0}) = 0$, $\text{ord}(\bar{1}) = 1$, $\text{ord}(\bar{\omega}_k) = \omega_k$, (ii) $\text{ord}(\bar{\varphi}_n^c(a, b)) = \varphi_n^{\text{ord}(c)}(\text{ord}(a), \text{ord}(b))$.

Abbreviations. $\bar{\varphi}_n(a, b) = \bar{\varphi}_n^{\bar{1}}(a, b)$, $b+1 = \bar{\varphi}_n(\bar{0}, b)$.

DEFINITION 2.2. The sets NT_n of *normal terms* in $\bar{T}_n$; $\text{dom}(a) \in \{\emptyset, \{\bar{0}\}, \bar{T}_0, \dots, \bar{T}_{n-1}\}$ and $a[z]$ for $a \in NT_n$, $z \in \text{dom}(a)$ are defined inductively by:

- (N1) $\bar{0} \in NT_n$; $\text{dom}(\bar{0}) = \emptyset$.
- (N2) $\bar{1} \in NT_n$; $\text{dom}(\bar{1}) = \{\bar{0}\}$, $\bar{1}[\bar{0}] = \bar{0}$.
- (N3) $\bar{\omega}_i \in NT_n$ ($i < n$); $\text{dom}(\bar{\omega}_i) = \bar{T}_i$, $\bar{\omega}_i[z] = z$.
- (N4) $NT_k \subseteq NT_n$ for $k < n$.

(N5) Let $a \in NT_{n+1}$, $b, c \in NT_n$ and $A = \bar{\varphi}_n^c(a, b)$. Then $A \in NT_n$ if one of the following holds:

- (i) $c = \bar{1}$ and $a = \bar{0}$ (i.e., $A = b+1$); $\text{dom}(A) = \{\bar{0}\}$, $A[z] = b$.
- (ii) $\text{dom}(c) = \bar{T}_k(k < n)$; $\text{dom}(A) := \text{dom}(c)$, $A[z] = \bar{\varphi}_n^{c[z]}(a, b)$.
- (iii) $c = \bar{1}$ and $\text{dom}(a) = \bar{T}_k(k < n)$; $\text{dom}(A) = \text{dom}(a)$,
 $A[z] = \bar{\varphi}_n(a[z], b)$.

A *term-rewriting system* (S) (see e.g. Dershowitz[2] as for the definition) is introduced so that, for every term in $\bar{T}_n$ which is not normal, some rewrite rule in (S) is applied to it:

Definition of the rewrite rules of (S): For normal a, b, c ;

- (R1) $\bar{\varphi}_n^{\bar{0}}(a, b) \longrightarrow b$, (R2) $\bar{\varphi}_n(\bar{1}, b) \longrightarrow \bar{\varphi}_n^b(\bar{0}, \bar{\varphi}_n(\bar{0}, b))$,
- (R3) $\bar{\varphi}_n(a+1, b) \longrightarrow \bar{\varphi}_n^b(a, \bar{\varphi}_n(a, b))$,
- (R4) $\bar{\varphi}_n^{c+1}(a, b) \longrightarrow \bar{\varphi}_n(a, \bar{\varphi}_n^c(a, b))$,
- (R5) $\bar{\varphi}_n(a, b) \longrightarrow \bar{\varphi}_n(a[b], b)$ if $\text{dom}(a) = \bar{T}_n$.

PROPOSITION 2.3. *For every $a \in \bar{T}_n$, $a \in NT_n$ if and only if there is no $b \in T$ such that $a \xrightarrow{1} b$ (where $a \xrightarrow{1} b$ means that b is obtained from a by a single application of some rule of (S)).*

Proof. Induction on the length of a . □

STRONG NORMALIZATION THEOREM([4, Theorem 1]). *Every term a in $\bar{T}_n$ is strongly normalizable (i.e., there is no infinite sequence such that $a \xrightarrow{1} a_1 \xrightarrow{1} a_2 \xrightarrow{1} \dots$).*

Proof. See [4, Theorem 1]. □

Now we introduce a function $\bar{c}$ which represents the function c (in the collapsing theorem) on the terms as follows: (for each fixed $x < \omega$) (i) $\bar{c}\bar{0} = \bar{0}$, $\bar{c}\bar{1} = \bar{1}$, $\bar{c}\bar{\omega}_0 = \bar{x}$, $\bar{c}\bar{\omega}_{k+1} = \bar{\omega}_k$,
(ii) $\bar{c}(\bar{\varphi}_{n+1}^\gamma(\delta, \xi)) = \bar{\varphi}_n^{\bar{c}\gamma}(\bar{c}\delta, \bar{c}\xi)$ and $\bar{c}(\bar{\varphi}_0^\gamma(\delta, \xi)) = \bar{\varphi}_0^\gamma(\delta, \xi)$,
where $\bar{x}$ is the numeral of x (i.e., if $x = 0$ then $\bar{x} = \bar{0}$; if $x = y+1$ then $\bar{x} = \bar{\varphi}_0(\bar{0}, \bar{y}) (= \bar{y}+1)$).

LEMMA 2.4. Let $a \in \bar{T}_n$. Then the following hold.

- (1) If $a = b+1$ for some b , then $\bar{c}(b) = \bar{c}b+1$.
- (2) If $a \in NT_n$ and $\text{dom}(a) = \bar{T}_0$, then $\bar{c}(a[\bar{x}]) = \bar{c}a$ and $\text{ord}(a[\bar{x}]) = \text{ord}(a)$ for $x < \omega$
- (3) If $a \in NT_n$ and $\text{dom}(a) = \bar{T}_k$ for some $k > 0$, then $\text{ord}(a[b]) = \text{ord}(a)[\text{ord}(b)]$ and $\text{ord}(\bar{c}(a[b])) = \text{ord}(\bar{c}a)[\text{ord}(\bar{c}b)]$ for $b \in \text{dom}(a)$.
- (4) If $a \xrightarrow{1} b$, then $\text{ord}(a) = \text{ord}(b)$ and $\text{ord}(\bar{c}a) = \text{ord}(\bar{c}b)$.

Proof. (1)-(4) Induction on the length of a . □

LEMMA 2.5. If $x < \omega$ and $a \in \bar{T}_1$, then $G_{\text{ord}(a)}(x) = \text{ord}(\bar{c}a)$.

Proof. From the strong normalization theorem, the proof is proceeded by transfinite induction on a over the well-founded ordering $<<$ (where $<<$ on $\bar{T}_n$ is defined as the transitive closure of (i) $b[z] << b$ for normal b with $z \in \text{dom}(b)$, (iii) $d << b$ for non-normal b with $b \xrightarrow{1} d$).

Case 1. $a = \bar{0}$. This case is trivial.

Case 2. $a \in NT_1$ and $\text{dom}(a) = \{\bar{0}\}$. Then $a = \bar{1}$ or $b+1$ for some $b \in \bar{T}_1$. If $a = \bar{1}$, the assertion is trivial. If $a = b+1$, then $G_{\text{ord}(a)}(x) = G_{\text{ord}(b)}(x)+1 = \text{ord}(\bar{c}b)+1 = \text{ord}(\bar{c}a)$ by I.H. (= induction hypothesis) and 2.4(1).

Case 3. $a \in NT_1$ and $\text{dom}(a) = \bar{T}_0$. By 2.4(2) and I.H., $G_{\text{ord}(a)}(x) = G_{\text{ord}(a[\bar{x}])}(x) = \text{ord}(\bar{c}(a[\bar{x}])) = \text{ord}(\bar{c}a)$.

Case 4. $a \xrightarrow{1} b$ for some b . By 2.4(4) and I.H., $G_{\text{ord}(a)}(x) = G_{\text{ord}(b)}(x) = \text{ord}(\bar{c}b) = \text{ord}(\bar{c}a)$. □

Proof of the collapsing theorem (in Section 1). For $a \in \bar{T}_2$ and $b \in \bar{T}_1$, we have $\bar{c}(\bar{\varphi}_1(a, b)) = \bar{\varphi}_0(\bar{c}a, \bar{c}b)$ and hence $\text{ord}(\bar{c}(\bar{\varphi}_1(a, b))) = \varphi_0(\text{ord}(\bar{c}a), \text{ord}(\bar{c}b))$. Thus,

$$\begin{aligned} G_{\varphi_1(\text{ord}(a), \text{ord}(b))}(x) &= G_{\text{ord}(\bar{\varphi}_1(a, b))}(x) \\ &= \text{ord}(\bar{c}(\bar{\varphi}_1(a, b))) \quad \text{by 2.5} \\ &= \varphi_0(\text{ord}(\bar{c}a), \text{ord}(\bar{c}b)) \\ &= F_{\text{ord}(\bar{c}a)}(\text{ord}(\bar{c}b)) \\ &= F_{\text{ord}(\bar{c}a)}(G_{\text{ord}(b)}(x)) \quad \text{by 2.5.} \end{aligned}$$

For given $\alpha \in T_2$ and $\beta \in T_1$, we choose a and b above such that (i) $\text{ord}(a) = \alpha$, $\text{ord}(\bar{c}a) = c\alpha$ and (ii) $\text{ord}(b) = \beta$ (we can choose such a and b since the elements of T_n are constructed by the same way as to the element in $\bar{T}_n$). This completes the proof. □

§3. (3)-BUILT-UPNESS OF τ

In this section we will prove that τ is (3)-built-up. This completes the proof of Theorem 1.9 (τ is minimal s-inaccessible). First, we remark that the following proposition holds:

PROPOSITION 3.1([4, Lemma 3.4]). Let $\alpha \in T_n$ and $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$. Then $\alpha[\gamma] \in T_n$ for every $\gamma \in T_m$.

Proof. For a given $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n$, there is a normal $a \in T_n$ such that $\text{ord}(a) = \alpha$ by 2.4(4) and the strong normalization theorem. We fix such an $a \in T_n$ with the minimal length. The proof of this proposition can be proceeded by induction on the length of this term a for α . $\square$

It follows from this proposition that we can use transfinite induction on the terms in T_n ($n < \omega$) over the ordering $<$ of T_n .

DEFINITION 3.2. The *step-down relations* $<_k$ ($k < \omega$) on $\cup_{n < \omega} T_n$ are defined inductively as follows:

$\alpha <_k \beta$ if $\beta \neq 0$ and one of the following holds;

- (i) $\alpha \leq_k \gamma$ if $\beta = \gamma + 1$,
- (ii) $\alpha \leq_k \beta[k]$ if $\beta = (\beta[x])_{x \in \Omega_0}$,
- (iii) $\alpha <_k \beta[\gamma]$ for all $\gamma \in T_m \setminus \{0\}$ if $\beta = (\beta[\gamma])_{\gamma \in \Omega_m}$ ($m > 0$).

where $\alpha \leq_k \delta$ means that $\alpha <_k \delta$ or $\alpha = \delta$.

Note that if $\alpha, \beta \in T_1$ then the relations $<_k$ defined above are the same as ones defined in Definition 1.1(2).

LEMMA 3.3. For $\alpha \in T_{n+1}$, $\beta \in T_n$ and $\gamma \in T_n \setminus \{0\}$, $\beta <_k \varphi_n^\gamma(\alpha, \beta)$.

Proof. The lemma follows immediately from the two claims: $\square$

CLAIM 1. Let $\alpha \in T_{n+1}$ and $\beta \in T_n$. If $\delta <_k \varphi_n(\alpha, \delta)$ for all $\delta \in T_n$, then $\beta <_k \varphi_n^\gamma(\alpha, \beta)$ for $\gamma \in T_n \setminus \{0\}$.

Proof of Claim 1. Transfinite induction on $\gamma \in T_n$.

Case 1. $\gamma = \eta + 1$. Then $\beta \leq_k \varphi_n^\eta(\alpha, \beta) <_k \varphi_n(\alpha, \varphi_n^\eta(\alpha, \beta)) = \varphi_n^\gamma(\alpha, \beta)$ by I.H.

Case 2. $\gamma = (\gamma[x])_{x \in \Omega_0}$. Then $\beta \leq_k \varphi_n^{\gamma[k]}(\alpha, \beta) = \varphi_n^\gamma(\alpha, \beta)[k]$ by I.H. Hence $\beta <_k \varphi_n^\gamma(\alpha, \beta)$.

Case 3. $\gamma = (\gamma[\delta])_{\delta \in \Omega_m}$ ($0 < m < n$). We can prove that $\gamma[\delta] \in T_n \setminus \{0\}$ for $\delta \in T_m \setminus \{0\}$ similarly to 3.1. Hence $\beta <_k \varphi_n^{\gamma[\delta]}(\alpha, \beta) = \varphi_n^\gamma(\alpha, \beta)[\delta]$ for $\delta \in T_m \setminus \{0\}$ by I.H. Therefore $\beta <_k \varphi_n^\gamma(\alpha, \beta)$. $\square$

CLAIM 2. Let $\alpha \in T_{n+1}$. Then $\beta <_k \varphi_n(\alpha, \beta)$ for all $\beta \in T_n$.

Proof of Claim 2. Transfinite induction on $\alpha \in T_{n+1}$.

Case 1. $\alpha = 0$. Then $\beta <_k \beta + 1 = \varphi_n(\alpha, \beta)$.

Case 2. $\alpha = \gamma + 1$. Then $\delta <_k \varphi_n(\gamma, \delta)$ for all $\delta \in T_n$ by I.H. Hence, by Claim 1, $\beta <_k \varphi_n(\gamma, \beta) \leq_k \varphi_n^\beta(\gamma, \varphi_n(\gamma, \beta)) = \varphi_n(\alpha, \beta)$.

Case 3. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$ ($m < n$). By I.H., $\beta <_k \varphi_n(\alpha[\gamma], \beta) = \varphi_n(\alpha, \beta)[\gamma]$ for $\gamma \in T_m$. Hence $\beta <_k \varphi_n(\alpha, \beta)$.

Case 4. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_n}$. By I.H., $\beta <_k \varphi_n(\alpha[\beta], \beta) = \varphi_n(\alpha, \beta)$. $\square$

LEMMA 3.4. Let $\alpha \in T_{n+1}$ and $\beta, \delta, \gamma \in T_n$. If $\gamma <_k \delta$, then $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^\delta(\alpha, \beta)$.

Proof. Transfinite induction on $\delta \in T_n$.

Case 1. $\delta = 0$. This case is trivial.

Case 2. $\delta = \eta + 1$. By I.H. and 3.3, $\varphi_n^\gamma(\alpha, \beta) \leq_k \varphi_n^\eta(\alpha, \beta) <_k$

$$\varphi_n(\alpha, \varphi_n^\eta(\alpha, \beta)) = \varphi_n^\delta(\alpha, \beta).$$

Case 3. $\delta = (\delta[x])_{x \in \Omega_0}$. By I.H., $\varphi_n^\gamma(\alpha, \beta) \leq_k \varphi_n^{\delta[k]}(\alpha, \beta) = \varphi_n^\delta(\alpha, \beta)[k]$. Hence $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^\delta(\alpha, \beta)$.

Case 4. $\delta = (\delta[\xi])_{\xi \in \Omega_m} (0 < m < n)$. Then $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^{\delta[\xi]}(\alpha, \beta) = \varphi_n^\delta(\alpha, \beta)[\xi]$ for $\xi \in T_m \setminus \{0\}$ by I.H. Hence $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^\delta(\alpha, \beta)$. $\square$

LEMMA 3.5. Let $\alpha, \gamma \in T_{n+1}$, $\beta \in T_n \setminus \{0\}$ and $n > 0$. If $\gamma <_k \alpha$, then $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha, \beta)$.

Proof. Transfinite induction on $\alpha \in T_n$.

Case 1. $\alpha = 0$. This case is trivial.

Case 2. $\alpha = \eta + 1$. By I.H. and 3.3, $\varphi_n(\gamma, \beta) \leq_k \varphi_n(\eta, \beta) <_k \varphi_n^\beta(\eta, \varphi_n(\eta, \beta)) = \varphi_n(\alpha, \beta)$ since $\beta \neq 0$.

Case 3. $\alpha = (\alpha[x])_{x \in \Omega_0}$. By I.H., $\varphi_n(\gamma, \beta) \leq_k \varphi_n(\alpha[k], \beta) = \varphi_n(\alpha, \beta)[k]$. Hence $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha, \beta)$.

Case 4. $\alpha = (\alpha[\xi])_{\xi \in \Omega_m} (0 < m < n)$. By I.H., $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha[\xi], \beta) = \varphi_n(\alpha, \beta)[\xi]$ for $\xi \in T_m \setminus \{0\}$. Hence $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha, \beta)$.

Case 5. $\alpha = (\alpha[\xi])_{\xi \in \Omega_n}$. By I.H., $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha[\beta], \beta) = \varphi_n(\alpha, \beta)$ for $\beta \in T_n \setminus \{0\}$. $\square$

THEOREM 3.6([4, Theorem 3]). (1) Let $\alpha \in T_n^+$ and $\alpha = (\alpha[\xi])_{\xi \in \Omega_m}$. If $\gamma, \delta \in T_m$ and $\gamma <_k \delta$, then $\alpha[\gamma] <_k \alpha[\delta]$ (where the sets $T_n^+ (\subseteq T_n)$ are defined inductively by:

- (i) $0, 1, \omega_0, \dots, \omega_{n-1} \in T_n^+$, (ii) $T_k^+ \subseteq T_n^+$ for $k < n$,
 - (iii) if $\alpha \in T_{n+1}^+$, $\gamma \in T_n^+$ and $\beta \in T_n^+ \setminus \{0\}$, then $\varphi_n^\gamma(\alpha, \beta) \in T_n^+$.
- (2) Each $\alpha \in T_1^+$ is (k)-built-up for all $k < \omega$.

Proof. (1) Similarly to the proof of 3.1, for a given $\alpha \in T_n^+$, we can take a normal term $a \in \bar{T}_n^+$ with the minimal length such that $\text{ord}(a) = \alpha$ (where the sets $\bar{T}_n^+$ ($\subseteq \bar{T}_n$) are defined inductively by:

(i) $\bar{0}, \bar{1}, \bar{\omega}_0, \dots, \bar{\omega}_{n-1} \in \bar{T}_n^+$, (ii) $\bar{T}_k^+ \subseteq \bar{T}_n^+$ for $k < n$,

(iii) if $a \in \bar{T}_{n+1}^+$, $c \in \bar{T}_n^+$ and $b \in \bar{T}_n^+ \setminus \{0\}$, then $\bar{\varphi}_n^c(a, b) \in \bar{T}_n^+$.

Hence we fix such an $a \in \bar{T}_n^+$. The proof of this theorem will be proceeded by the induction on the length of the term a . We have the following cases:

Case 1. $a = \bar{\omega}_m$. Then $\alpha = \omega_m$. We have $\alpha[\gamma] = \gamma <_k \delta = \alpha[\delta]$.

Case 2. $a = \bar{\varphi}_n(d, b)$ and $\text{dom}(d) = \bar{T}_m$. Then $\alpha = \varphi_n(\lambda, \beta)$ so that $\lambda = (\lambda[\xi])_{\xi \in \Omega_m} = \text{ord}(d)$ and $\beta = \text{ord}(b) \in T_n^+ \setminus \{0\}$ from the definition of T_n^+ above and $a \in \bar{T}_n^+$. Hence, by I.H. $\lambda[\gamma] <_k \lambda[\delta]$ and 3.5, $\varphi_n(\lambda, \beta)[\gamma] = \varphi_n(\lambda[\gamma], \beta) <_k \varphi_n(\lambda[\delta], \beta) = \varphi_n(\lambda, \beta)[\delta]$.

Case 3. $a = \bar{\varphi}_n^e(d, b)$ and $\text{dom}(e) = \bar{T}_m$. This case is treated similarly to Case 2, using 3.4. This completes the proof of (1).

(2) We can show that for each $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n^+$ and $\gamma \in T_m^+$, $\alpha[\gamma] \in T_n^+$ similarly to 3.1. Hence for each $\alpha \in T_1^+$ and limit $\lambda \leq \alpha$, we have $\lambda \in T_1^+$. Thus by (1), $\lambda[x] <_k \lambda[x+1]$ for all k , $x < \omega$ and limit $\lambda \leq \alpha \in T_1^+$. $\square$

We remark that (k)-built-upness does not hold for some element in T_1 since, if we put $\alpha = \varphi_1(\omega_0, 0)$, then $\alpha[x] = \varphi_1(x, 0) = 1$ for all $x < \omega$.

THEOREM 3.7([4, Corollary 3.1]). τ is (3)-built-up.

Proof. From the definition of τ (Definition 1.8), $\tau[x] \in T_1^+$

for every $x < \omega$. By 3.6(2), $\tau[x]$ is (3)-built-up. Hence it is sufficient to prove that $\tau[x] <_3 \tau[x+1]$. For this, we have

$$\begin{aligned} \tau[x] &= \varphi_1(\dots \varphi_x(3, \omega_{x-1}) \dots, \omega_0) <_3 \varphi_1(\dots \varphi_x(\omega_0, \omega_{x-1}) \dots, \omega_0) \\ &= \varphi_1(\dots \varphi_x(\omega_x, \omega_{x-1}) \dots, \omega_0) <_3 \varphi_1(\dots \varphi_x(\varphi_{x+1}(3, \omega_x), \omega_{x-1}) \dots, \omega_0) \\ &= \tau[x+1] \text{ from } 3 <_3 \omega_0 \text{ and 3.5, 3.3. This completes the proof. } \square \end{aligned}$$

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