

博 士 論 文

p -adic Galois representations and completed
cohomology of Shimura varieties

(p 進 Galois 表現と志村多様体の完備化コホモロジー)

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cohomology of Shimura varieties

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To my family.

Abstract

The p -adic Langlands program is a series of conjectures about relations between p -adic Banach representations of reductive groups and p -adic Galois representations. From now, sufficient results are obtained only in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ for the p -adic Langlands program. One of the main problems in the p -adic Langlands program is to reconstruct differential geometric lost data from Weil-Deligne representations in the classical local Langlands correspondence. In the case of crystalline (or semistable) representations, the lost data is highly related to triangulations of the associated (φ, Γ) -modules. We treat problems about crystalline representation valued in general p -adic reductive groups in the first three parts of this thesis. In the last part of this thesis, we concentrate locally analytic subspace of completed cohomology groups of unitary Shimura curves, which is considered to give a candidate of the locally analytic version of the p -adic Langlands correspondence for $\mathrm{GL}_2(K)$.

In this thesis, we study crystalline representations valued in p -adic reductive groups and locally analytic completed cohomology of Shimura varieties. In the first part, we show the existence of a crystalline lift of a semisimple G -valued mod p Galois representation with regular Hodge–Tate weights with a fixed abelianization. This result is based on the preprint [2]. In the second part, we introduce trianguline deformation spaces for GSp_{2n} and show Zariski density of the crystalline points on the rigid generic fiber of the framed deformation space of GSp_{2n} -valued mod p Galois representations for irreducible components with a regular crystalline point. This result is based on the preprint [1]. We also calculate the dimension of each locus of Emerton–Gee stacks for GSp_{2n} and show the more general Zariski density result for GSp_{2n} . In the last part, we show the existence of isomorphisms between locally analytic completed cohomology and cohomology of flag varieties with coefficients of locally analytic sections induced from Hodge–Tate period maps for Shimura varieties of Hodge type. This result is based on the preprint [3].

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CHAPTER 1

Introduction

1.1. Outline of this thesis

In this thesis, we study some topics around the p -adic Langlands program. Especially, we focus on

- (1) Crystalline representations valued in p -adic reductive groups,
- (2) Locally analytic completed cohomology of Shimura varieties.

In Chapter 2, for a finite extension $K/\mathbb{Q}_p$ and a split reductive group G over $\mathcal{O}_L$ for a finite extension L/K , let $\bar{\rho}: \text{Gal}_K \rightarrow G(\bar{k}_L)$ be a continuous quasi-semisimple mod p G -valued representation of the absolute Galois group Gal_K . Let $\bar{\rho}^{\text{ab}}$ be the abelianization of $\bar{\rho}$ and fix a crystalline lift ψ of $\bar{\rho}^{\text{ab}}$. We show the existence of a crystalline lift ρ of $\bar{\rho}$ with regular Hodge–Tate weights such that the abelianization of ρ coincides with ψ . We also show analogous results in the case that G is a quasi-split tame group and $\bar{\rho}: \text{Gal}_K \rightarrow {}^L G(\bar{\mathbb{F}}_p)$ is a semisimple mod p L -parameter. These theorems are generalizations of those of Lin and Böckle–Iyengar–Paškūnas.

In the first part of Chapter 3, we introduce trianguline deformation spaces $X_{\text{tri}}(\bar{\rho})$ for a GSp_{2n} -valued residual representation $\bar{\rho}: \text{Gal}_K \rightarrow \text{GSp}_{2n}(k)$, where k is a finite field of characteristic $p > 0$ and Gal_K is the absolute Galois group of a finite extension $K/\mathbb{Q}_p$, and study their properties. We show Zariski density of the crystalline points on the rigid generic fiber $\mathfrak{X}_{\bar{\rho}}$ of the framed deformation space of $\bar{\rho}$ under some conditions. In the last part of Chapter 3, we calculate the dimension of each locus of Emerton–Gee stacks for GSp_{2n} . Then we show Zariski density of the crystalline points on the rigid generic fiber $\mathfrak{X}_{\bar{\rho}}$ of the framed deformation space of a GSp_{2n} -valued mod p Galois representation $\bar{\rho}: \text{Gal}_K \rightarrow \text{GSp}_{2n}(k_L)$, under a more general condition than that in Chapter 3.

In Chapter 4, for Shimura varieties of Hodge type, we show that there are natural isomorphisms between locally analytic complete cohomology groups and cohomology groups for flag varieties with coefficient which is given by their perfectoid covers. This result is a generalization of that of Pan for modular curves and Qiu–Su for unitary Shimura curves.

1.2. Crystalline lifts of semisimple G -valued Galois representations with fixed determinant

Let p be a prime number and $K/\mathbb{Q}_p$ be a finite extension with its ring of integers $\mathcal{O}_K$, $\text{Gal}_K := \text{Gal}(\bar{K}/K)$ be the absolute Galois group of K , L/K be a sufficiently large finite extension with its ring of integers $\mathcal{O}_L$. Let $\mathfrak{m}_K, \mathfrak{m}_L$ be the maximal ideals of $\mathcal{O}_K, \mathcal{O}_L$, and k, k_L be the residue fields of $\mathcal{O}_K, \mathcal{O}_L$, respectively. Let $\mathbb{C}_p$ be the p -adic completion of $\bar{K}$.

For a reductive group G over $\mathcal{O}_L$, a continuous G -valued representation

$$\rho: \text{Gal}_K \rightarrow G(L)$$

is *Hodge–Tate* (resp. *crystalline*, *potentially crystalline*) if for any algebraic L -representation $\tau: G \rightarrow \text{GL}(V)$, the composite $\tau \circ \rho$ is a Hodge–Tate (resp. crystalline, potentially crystalline) representation. For a Hodge–Tate G -valued representation $\rho: \text{Gal}_K \rightarrow$

$G(L)$, we say that ρ has *regular Hodge–Tate weights* if for each $\sigma \in \text{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)$, the associated cocharacter $\mathcal{HT}(\rho)^\sigma$ of $G_{\mathbb{C}_p}$ is regular, that is, not killed by any root of $G_{\mathbb{C}_p}$ (see Definition 2.2.6). Let G^{ab} be the abelianization of G . The composite of ρ with $G(L) \rightarrow G^{\text{ab}}(L)$ is written as ρ^{ab} . We call it the *abelianization* of ρ .

A continuous representation

$$\rho: \text{Gal}_K \rightarrow G(\mathcal{O}_L)$$

is said to be *Hodge–Tate* (resp. *crystalline*, *potentially crystalline*) if the composite $\text{Gal}_K \xrightarrow{\rho} G(\mathcal{O}_L) \hookrightarrow G(L)$ is Hodge–Tate (resp. crystalline, potentially crystalline).

From here, let G be a connected split reductive group over $\mathcal{O}_L$ and let T be a maximal torus of G . We say $\bar{\rho}: \text{Gal}_K \rightarrow G(\bar{k}_L)$ is *quasi-semisimple* if there exists a maximal torus T' of $G_{\bar{k}_L} := G \otimes_{\mathcal{O}_K} \bar{k}_L$ such that $\bar{\rho}(I_K) \subset T'(\bar{k}_L)$ and $\bar{\rho}(\text{Gal}_K) \subset N_{G_{\bar{k}_L}}(T')(\bar{k}_L)$, where $I_K \subset \text{Gal}_K$ denotes the inertia subgroup and $N_{G_{\bar{k}_L}}(T')$ denotes the normalizer of T' in $G_{\bar{k}_L}$.

The first main theorem of this paper is as follows:

Theorem 1.2.1 (Theorem 2.3.6). *Let $\bar{\rho}: \text{Gal}_K \rightarrow G(k_L)$ be a quasi-semisimple G -valued representation such that $\bar{\rho}(I_K) \subset T(k_L)$ and $\bar{\rho}(\text{Gal}_K) \subset N_G(T)(k_L)$. Let $\bar{\rho}^{\text{ab}}: \text{Gal}_K \rightarrow G^{\text{ab}}(k_L)$ be the composite of $\bar{\rho}$ and the natural map $G(k_L) \rightarrow G^{\text{ab}}(k_L)$. Let $\psi: \text{Gal}_K \rightarrow G^{\text{ab}}(\mathcal{O}_L)$ be a crystalline lift of $\bar{\rho}^{\text{ab}}$. Then there exist a finite extension L'/L and a crystalline lift $\rho: \text{Gal}_K \rightarrow G(\mathcal{O}_{L'})$ of $\bar{\rho}$ with regular Hodge–Tate weights such that $\rho^{\text{ab}} = \psi$. Moreover, if we do not require that the crystalline lift ρ has regular Hodge–Tate weights, L'/L can be taken to be a finite unramified extension.*

Theorem 1.2.1 is a generalization of the results of Lin [58, Theorem 5, Theorem 6] and Böckle–Iyengar–Paškūnas [20, Proposition 2.6].

We also prove analogous results of Theorem 1.2.1 for potentially crystalline lifts of semisimple mod p L -parameters for quasi-split tame groups over K . Let G be a connected quasi-split reductive group over K . We assume that G is tame, that is, G splits over a tame Galois extension E of K . Write $\widehat{G}$ for the Langlands dual group of G and ${}^L G = \widehat{G} \rtimes \text{Gal}_K$ for the L -group of G .

The second main theorem of this paper is as follows:

Theorem 1.2.2 (Theorem 2.4.5). *Let $\bar{\rho}: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{F}}_p)$ be a semisimple mod p L -parameter in the sense of [60, Definition 3.2.5]. Fix a maximal torus S of G such that $\bar{\rho}$ factors through ${}^L S(\overline{\mathbb{F}}_p)$ (see Theorem 2.4.3). Let $i: G^{\text{ab}} \rightarrow S$ be a right inverse of the composite $S \hookrightarrow G \twoheadrightarrow G^{\text{ab}}$. For $A = \overline{\mathbb{F}}_p^*$ or $\overline{\mathbb{Z}}_p^*$ and an L -parameter $\tau: \text{Gal}_K \rightarrow {}^L S(A)$, write τ^{ab} for the composite of τ and the map ${}^L S(A) \rightarrow {}^L G^{\text{ab}}(A)$ induced by the map i and we set $\bar{\psi} := \bar{\rho}^{\text{ab}}$. Let $\psi: \text{Gal}_K \rightarrow {}^L G^{\text{ab}}(\overline{\mathbb{Z}}_p)$ be a potentially crystalline lift of $\bar{\psi}$. Then $\bar{\rho}$ admits a potentially crystalline lift $\rho: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{Z}}_p)$ which has regular Hodge–Tate weights and satisfies $\rho^{\text{ab}} = \psi$.*

The existence of crystalline lifts with fixed determinant in the case of $G = \text{GL}_m$ is proved by Böckle–Iyengar–Paškūnas [20, Proposition 2.6, Proposition 2.7]. In the paper, they first proved the case where $\bar{\rho}: \text{Gal}_K \rightarrow \text{GL}_m(\overline{\mathbb{F}}_p)$ is absolutely irreducible and the general case is deduced from the former case using the geometry of the Emerton–Gee stacks. Also, Lin [58, Theorem 5, Theorem 6] showed the existence of crystalline lifts with regular Hodge–Tate weights of quasi-semisimple mod p G -valued representations when

G is a split reductive group over $\mathcal{O}_K$ (see [60, Theorem 5.3.1] for semisimple mod p L -parameters for quasi-split tame groups). In this paper, we combine the methods of [20] and [58], [60] to prove the existence of crystalline lifts of $\bar{\rho}$ with fixed abelianization.

The strategies of the proofs of Theorem 1.2.1 and Theorem 1.2.2 are based on Lin's strategies in [58] and [60], respectively, where the existence of crystalline lifts of semisimple mod p G -valued representations is shown. In order to control the abelianizations of crystalline lifts, we use the calculations of abelianizations of Galois representations via the reciprocity map of local class field theory as in the work of Böckle–Iyengar–Paškūnas [20].

1.3. Zariski density of crystalline points on GSp_{2n} -valued local deformation rings

In Chapter 3, we will show the following theorem:

Theorem 1.3.1 (Theorem 3.14.1). *Let $\mathfrak{X}_{\mathrm{reg-crys}}$ be the subset of the points which is crystalline with regular Hodge-Tate weights in $\mathfrak{X}_{\bar{\rho}}^{\square}$. If $p \geq 3$, then $\mathfrak{X}_{\mathrm{reg-crys}}$ is Zariski dense in $\mathfrak{X}_{\bar{\rho}}^{\square}$.*

We note that the special case of Theorem 1.3.1 when $p = 2$ will be shown. The main ingredient of the argument under the condition of $p = 2$ is in Section 3.11. For a fixed representation $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GL}_m(k_L)$, Emerton–Gee [39] has shown the existence of a crystalline lift $\rho: \mathrm{Gal}_K \rightarrow \mathrm{GL}_m(L)$ for general $\bar{\rho}$.

In the case of GL_m -valued representations, Zariski density of crystalline point on the rigid analytic space $\mathfrak{X}_{\bar{\rho}}$ that is the rigid generic fiber of the framed deformation space of $\bar{\rho}$ has been considered. This problem was solved by Chenevier [25], Nakamura [68], Iyengar [48] and Böckle–Iyengar–Paškūnas [20] in the general case.

1.3.1. Outline of the proof. Fix $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(k_L)$ from here. Theorem 1.3.1 is an analog in the case of the universal framed deformation rings for G -valued representations $\bar{\rho}$ studied by Balaji [4] for $G = \mathrm{GSp}_{2n}$.

As a weaker form of Theorem 1.3.1, we will show the following proposition in advance:

Proposition 1.3.2 (Proposition 3.10.2). *The Zariski closure $\overline{\mathfrak{X}}_{\mathrm{reg-crys}}$ is a union of irreducible components of $\mathfrak{X}_{\bar{\rho}}$.*

In [68], Nakamura has constructed the local eigenvarieties for calculating dimension of a smooth point of $\overline{\mathfrak{X}}_{\mathrm{reg-crys}}$ in each irreducible component $\mathfrak{Z} \subset \mathfrak{X}_{\bar{\rho}}$ and proved the assertion analogous to the case of GL_m . However, we can use the method of [18] showing the accumulation property around the benign points on the trianguline deformation spaces instead. We also need to show the dimension is the same as that of $\mathfrak{X}_{\bar{\rho}}$.

We introduce (φ, Γ) -modules with GSp_{2n} -structures (more generally, (φ, Γ) -modules with G -structures) over Robba rings $\mathcal{R}_L$ and construct trianguline deformation spaces $X_{\mathrm{tri}}(\bar{\rho})$ for GSp_{2n} and morphisms $X_{\mathrm{tri}}(\bar{\rho}) \rightarrow \mathfrak{X}_{\bar{\rho}}$.

If G is a split algebraic group and T is a split maximal torus of G , let $W(G, T) := N_G(T)/C_G(T)$ be the Weyl group. Fix an algebraic embedding $r: G \rightarrow \mathrm{GL}_m$ of regular type (Definition 3.5.5). Let $r_*(D_{\mathrm{crys}}(D)) := (D_{\mathrm{crys}}(D))(r)$.

The correspondence of triangulations of D and refinements of $D_{\mathrm{crys}}(D)$ for crystalline (φ, Γ) -modules D is generalized as follows:

Proposition 1.3.3 (Proposition 3.7.1). *Let D be a crystalline frameable (φ, Γ) -module with G -structure over $\mathcal{R}_L$. Assume the Frobenius action on $r_*(D_{\mathrm{crys}}(D))$ has m distinct eigenvalues in $K_0 \otimes_{\mathbb{Q}_p} L$. Then there is a bijection from the set of triangulations of*

D to that of refinements of $D_{\text{crys}}(D)$. Moreover, if there is a refinement of $D_{\text{crys}}(D)$, there exists exactly $\#W(G, T)$ refinements of $D_{\text{crys}}(D)$.

By Proposition 1.3.3 and the accumulation property of $X_{\text{tri}}(\bar{\rho})$, replacing L by its finite extension, we may take a unique irreducible component $\mathfrak{Y}_w$ in $X_{\text{tri}}(\bar{\rho})$ which maps in $\mathfrak{Z}$ for each element w of the Weyl group of GSp_{2n} . Let $\mathcal{P}_w$ be the triangulation of D associated to w and X_D , (resp. $X_{D, \mathcal{P}_w}$) be the (trianguline) deformation functor with the tangent space TX_D (resp. $TX_{D, \mathcal{P}_w}$). Then the maximality of the dimension of each $\mathfrak{Z}$ is reduced to prove the following surjectivity:

Corollary 1.3.4 (Corollary 3.8.14). *Let D be a crystalline (φ, Γ) -module with GSp_{2n} -structure over $\mathcal{R}_L$ such that $\tilde{D}$ is φ -generic with regular Hodge-Tate type and assume that there exists a noncritical refinement of $D_{\text{crys}}(D)$. Then the map*

$$\bigoplus_{w \in W(\text{GSp}_{2n}, T)} TX_{D, \mathcal{P}_w} \rightarrow TX_D$$

is surjective.

In the GL_m case, Corollary 1.3.4 is solved by Chenevier [26, Theorem 3.19] for $K = \mathbb{Q}_p$ and noncritical D , Nakamura [67, Theorem 2.62] for general K and noncritical D and Hellmann–Margerin–Schraen [45, Theorem 3.11] for general conditions. In [26] and [67], the problem is shown by studying the subspace of crystalline deformations and induction on m . In [45], it is shown by studying the deformation functors of trianguline almost de Rham $L \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$ -representations and their local models constructed in [18]. However, it seems that the direct generalizations of these methods do not work in the case of GSp_{2n} . We reprove the GL_m -analog [45, Theorem 3.11] and show Corollary 1.3.4 explicitly calculating a basis corresponds to each $w \in W(\text{GSp}_{2n}, T)$ in section 3.8.

Let $\bar{\rho}: \text{Gal}_K \rightarrow \text{GSp}_{2n}(k_L)$ be a continuous Galois representation and $R_{\bar{\rho}}^{\square}$ be the universal framed deformation ring constructed by Balaji [4]. Write $\mathcal{X}_{\bar{\rho}}^{\square} := \text{Spf}(R_{\bar{\rho}}^{\square})^{\text{rig}}$ for the rigid generic fiber of $\text{Spf}(R_{\bar{\rho}}^{\square})^{\text{rig}}$ as constructed in [51, Definition 7.1.3].

In Chapter 2, the author showed the theorem for connected split reductive G -valued semisimple representations. Zariski density of crystalline points on an irreducible component of $\mathcal{X}_{\bar{\rho}}^{\square}$ which has a crystalline point are shown in Chapter 3 using the geometry of the trianguline deformation space of GSp_{2n} . Therefore, when $\bar{\rho}$ is semisimple, the assertion of Theorem 1.3.1 follows by the results of the two chapters.

In the general case, we also plan to show that each component of $\mathcal{X}_{\bar{\rho}}^{\square}$ has a crystalline point. By the result of Paškūnas–Quast [70], it suffices to show that any representation ρ has a crystalline lift with fixed abelianizations.

To prove Theorem 1.3.1 for reducible GSp_{2n} -valued representations $\bar{\rho}$, we calculate the dimension of each locus of the Emerton–Gee stacks $\mathcal{X}_{K, \text{GSp}_{2n}, \text{red}}$ for GSp_{2n} constructed in [59].

Theorem 1.3.5 (Theorem 3.13.1). *Let $\bar{\alpha}: \text{Gal}_K \rightarrow \text{GL}_a(\overline{\mathbb{F}}_p)$ be an irreducible Galois representation. The locus of $\bar{x} \in \mathcal{X}_{K, \text{GSp}_{2n}, \text{red}}$ such that $\dim_{\overline{\mathbb{F}}_p} \text{Hom}_{\text{Gal}_K}(\bar{\alpha}, i^{\text{std}}(\bar{x})) \geq r$ is of dimension at most*

$$d_{2n, r} := [K: \mathbb{Q}_p]n^2 - \left(\frac{r^2}{2} - \frac{r}{2} + 1 \right).$$

The proof of Theorem 1.3.5 is based on the argument of [61, §10].

1.4. Locally analytic completed cohomology of Shimura varieties of Hodge type

Let p be a prime number and $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. In this paper, we relate the locally analytic completed cohomology of Shimura varieties of Hodge type to certain cohomology of flag varieties. We also study τ -locally analytic completed cohomology of unitary Shimura curves as the special case.

1.4.1. Main results for Shimura varieties of Hodge type. Let (G, X) be a Shimura datum of Hodge type. For neat open subgroups $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}^{\infty, p})$, let $\mathcal{S}h_{K_p K^p}$ be the attached Shimura variety and $\mathcal{S}h := \mathcal{S}h_{K_p K^p}^{\text{tor}}$ be its toroidal compactification under a fixed cone decomposition, which are seen to be rigid spaces over $\mathbb{C}_p$. For a sufficiently small K^p , let $\mathcal{S}h_{K^p}^{\text{tor}}$ be the perfectoid Shimura variety constructed in [54, Proposition 6.1] and $\pi_{\text{HT}}^{\text{tor}}: \mathcal{S}h_{K^p}^{\text{tor}} \rightarrow \mathcal{F}\ell$ be the Hodge-Tate period map to a flag variety $\mathcal{F}\ell$ (cf. [83, Theorem IV.1.1]).

In this paper, we will study locally analytic subspaces of completed cohomology groups of Shimura varieties of Hodge type

$$\begin{aligned} \tilde{H}^i(K^p, E) &:= \varprojlim_s \varinjlim_{K_p} H_{\text{ét}}^i(\mathcal{S}h_{K_p K^p}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E, \\ \tilde{H}_c^i(K^p, E) &:= \varprojlim_s \varinjlim_{K_p} H_{\text{ét}, c}^i(\mathcal{S}h_{K_p K^p}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E \end{aligned}$$

where K_p runs over compact open subsets of $G(\mathbb{Q}_p)$. Then we relate them to certain cohomology groups of flag varieties.

Let $D := \mathcal{S}h_{K_p K^p}^{\text{tor}} \setminus \mathcal{S}h_{K_p K^p}$ be the normal crossing divisor in $\mathcal{S}h_{K_p K^p}^{\text{tor}}$. We can write $D = \bigcup_{a \in I} D_{K_p, a}$ for a finite set I such that each $D_{K_p, a}$ is irreducible and $D_{K_p, I'} := \bigcap_{a \in I'} D_{K_p, a}$ is a smooth divisor for each subset $I' \subset I$. Let $\iota_{K_p, I'}: D_{K_p, I'} \hookrightarrow \mathcal{S}h_{K_p K^p}^{\text{tor}}$ denote the closed immersion. Write $\hat{\mathcal{O}}_{I'}$ for the pro-Kummer étale sheaf of completed (bounded) functions on $D_{K_p, I'}$. We let

$$\hat{\mathcal{I}}_{\mathcal{S}h} := \ker \left(\hat{\mathcal{O}}_{\mathcal{S}h} \rightarrow \bigoplus_{a \in I} \iota_{K_p, a, *} \hat{\mathcal{O}}_a \right).$$

Let $\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}$ be the analytic site on the perfectoid Shimura variety $\mathcal{S}h_{K^p}^{\text{tor}}$. Let $\mathcal{O}_{K^p} := \pi_{\text{HT}, *}^{\text{tor}}(\hat{\mathcal{O}}_{\mathcal{S}h}|_{\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}})$ and $\mathcal{I}_{K^p} := \pi_{\text{HT}, *}^{\text{tor}}(\hat{\mathcal{I}}_{\mathcal{S}h}|_{\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}})$ be sheaves of topological algebras on the flag variety $\mathcal{F}\ell$ in analytic topology. Let $\mathcal{O}_{K^p}^{\text{la}} \subset \mathcal{O}_{K^p}$ and $\mathcal{I}_{K^p}^{\text{la}} \subset \mathcal{I}_{K^p}$ be subsheaves of locally analytic sections.

Theorem 1.4.1 (Theorem 4.5.9). *For any $i \geq 0$ and a neat compact open subgroup $K^p \subset G(\mathbb{A}^{\infty, p})$, there are natural $G(\mathbb{Q}_p)$ -equivariant isomorphisms*

$$\begin{aligned} (\tilde{H}^i(K^p, \mathbb{Q}_p) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\text{la}} &\cong H^i(\mathcal{F}\ell_{\mu}, \mathcal{O}_{K^p}^{\text{la}}), \\ (\tilde{H}_c^i(K^p, \mathbb{Q}_p) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\text{la}} &\cong H^i(\mathcal{F}\ell_{\mu}, \mathcal{I}_{K^p}^{\text{la}}) \end{aligned}$$

Remark 1.4.2. Theorem 1.4.1 has been proved in the case of the modular curve in [69, Theorem 4.4.6] and in the case of unitary Shimura curves in [73, §3.5].

1.4.2. Main results for unitary Shimura curves. Next, suppose that (G, X) is a Shimura datum which gives a unitary Shimura curve defined over a totally real number field $F \neq \mathbb{Q}$. Let $\wp$ be a fixed place on F over p . Let $E/\mathbb{Q}_p$ be a finite extension such that

each set $\Sigma_{\wp_i} := \text{Hom}_{\mathbb{Q}_p}(F_{\wp_i}, E)$ where $\wp_i$ is a place of F over p has $[F_{\wp_i} : \mathbb{Q}_p]$ elements. Fix a place $\wp = \wp_1$ of F over p and an embedding $\tau: F_{\wp} \hookrightarrow E$ in $\Sigma_{\wp}$. We can write $G(\mathbb{Q}_p) = \text{GL}_2(F_{\wp}) \times G(\mathbb{Q}_p^{\wp})$ where $G(\mathbb{Q}_p^{\wp})$ be a p -adic Lie group. We let

$$K = K_{\wp} K_p^{\wp} K^p \subset \text{GL}_2(F_{\wp}) \times G(\mathbb{Q}_p^{\wp}) \times G(\mathbb{A}^{\infty, p})$$

be a compact open subgroup.

Similarly we define completed cohomology of unitary Shimura curves as

$$\tilde{H}^i(K^{\wp}, E) := \varprojlim_s \varinjlim_{K_{\wp}} H_{\text{ét}}^i(\mathcal{S}h_{K_{\wp} K^{\wp}}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E,$$

on which $\text{GL}_2(F_{\wp})$ acts.

We can also consider completed cohomology $\tilde{H}^i(K^p, E)$ of K^p -level. Write $\mathcal{S}h_{K^p}$ for the infinite level perfectoid unitary Shimura curve and $\pi_{\text{HT}}: \mathcal{S}h_{K^p} \rightarrow \mathbb{P}_{\mathbb{C}_p}^1$ be the Hodge-Tate period map. Let $\mathcal{O}_{K^p, E} := \pi_{\text{HT},*} \mathcal{O}_{\mathcal{S}h_{K^p}, E}$ and $\mathcal{O}_{K^p, E}^{\tau-\text{la}}$ be the τ -locally analytic subsheaf of $\mathcal{O}_{K^p, E}$. We can similarly define $\mathcal{O}_{K^{\wp}, E}^{\tau-\text{la}}$.

Theorem 1.4.3 (Theorem 4.5.12). *There is natural $G(\mathbb{Q}_p)$ -equivariant isomorphisms*

$$(\tilde{H}^1(K^{\wp}, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\tau-\text{la}} \cong H^1(\mathcal{F}\ell_{\mu}, \mathcal{O}_{K^p, E}^{\tau-\text{la}})^{K_p^{\wp}} \cong H^1(\mathcal{F}\ell_{\mu}, \mathcal{O}_{K^{\wp}, E}^{\tau-\text{la}}).$$

Remark 1.4.4. We note that Theorem 1.4.3 is proved in [73, §3.5] in more general settings, but we give another proof of this theorem. There are analogous results [69, Theorem 4.4.6] for the modular curve, [77, Theorem 6.2.6] for analytic cohomology of general infinite level Shimura varieties.

1.4.3. Outline of the proof. Let μ be the Hodge cocharacter associated with a fixed element $h: \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}} \rightarrow G(\mathbb{R})$ in X . A parabolic Lie subalgebra $\mathfrak{p}_{\mu} \subset \mathfrak{g} := \text{Lie}(G_{\mathbb{Q}_p})$ is naturally attached to μ .

Our argument uses the embeddings of perfectoid Shimura varieties of Hodge type to perfectoid Siegel modular varieties. Since the anticanonical tower of Siegel modular varieties has Tate's normalized traces by [83, Theorem III.2.36], we will verify that the following proposition in the first part of Section 4.5.

Let $(\text{GSp}_{2g}(\mathbb{Q}_p), D)$ be a Siegel Shimura datum and fix a closed immersion of Shimura data of Hodge type $(G, X) \hookrightarrow (\text{GSp}_{2g}(\mathbb{Q}_p), D)$. Fix a sufficiently small compact open subgroup $K'_p K^{p'} \subset \text{GSp}_{2g}(\mathbb{A}^{\infty})$ such that $K'_p K^{p'} \cap G(\mathbb{A}^{\infty}) = K_p K^p$. Let $\mathcal{Y} := \mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}$ be a toroidal compactification of the associated Siegel modular variety as a rigid space over $\mathbb{C}_p$. For a sufficiently small open subspace $U' \subset \mathcal{Y}$, we can attach an open affinoid perfectoid subspace $\mathcal{S}h_{K^p}^{\text{tor}}(U') \subset \mathcal{S}h_{K^p}^{\text{tor}}$.

We will compare local sections on perfectoid Siegel modular varieties and on perfectoid Shimura varieties of Hodge type in Section 4.5. In the section, we use the detailed argument in geometric Sen theory in [76].

We will also show a relation between completed cohomology of Shimura curves of K^p -level and those of $K^{\wp}$ -level, making use of the proof of [77, Theorem 6.2.6] and certain spectral sequences proved in the paper (Corollary 4.4.5).

1.5. Notation

For a group G and a subset $S \subset G$, we write $N_G(S)$ for the normalizer of S in G and $C_G(S)$ for the centralizer of S in G . For a finite extension L/F of fields, let $N_{L/F}: L^* \rightarrow F^*$ denote the norm map.

Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$ and K_0 be the maximal unramified subextension of $\mathbb{Q}_p$ in K . We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be the p -adic completion of $\overline{\mathbb{Q}_p}$. We write ζ_m for a m -th roots of unity. Let $K_n := K(\zeta_{p^n})$ and $K_\infty := \bigcup_n K_n$. Let k be the residue field of K , k' be the residue field K_∞ and $K'_0 := W(k')$. We write $\Gamma := \text{Gal}(K_\infty/K)$. Let L be a sufficiently large finite extension of K such that K splits in L and k_L be the residue field of L and $\Sigma := \text{Hom}_{\mathbb{Q}_p}(K, L)$. Let Gal_K be the absolute Galois group of K . Let W_K be the Weil group of K . For a finite Galois extension E/K , let $W_{E/K} := W_K/\overline{W_E^{\text{der}}}$, where $\overline{W_E^{\text{der}}}$ is the closure of the derived subgroup W_E^{der} of W_E , be the relative Weil group as defined in [87]. Let $\epsilon: \text{Gal}_K \rightarrow \mathbb{Z}_p^*$ be the p -adic cyclotomic character. Let I_K be the inertia subgroup of Gal_K . We fix a uniformizer $\varpi_K \in \mathcal{O}_K$ and normalize the reciprocity isomorphism $\text{rec}_K: K^* \rightarrow W_K^{\text{ab}}$ of local class field theory such that ϖ_K is mapped to the geometric Frobenius automorphism. Let B_m, U_m, T_m be respectively the standard upper Borel subgroup, its unipotent radical and the split maximal torus of GL_m for $m \geq 1$.

If G is a group scheme over a ring A , let $\text{Rep}_A(G)$ denote the full subcategory of the category of G_A -modules whose objects are finite free A -modules (cf. [49, §1.2.7]). We write $I_m = \text{diag}(1, \dots, 1)$ for the identity matrix of size m . Let $\mathfrak{S}_m$ be the symmetric group of degree m for $m \in \mathbb{Z}_{\geq 1}$. We have the natural isomorphism $\mathfrak{S}_m \cong W(\text{GL}_m, T_m)$. For an $(m \times m)$ -matrix $(a_{ij})_{i,j} \in M_m(A)$, we write ${}^t(a_{ij})_{i,j}$ for the transpose of $(a_{ij})_{i,j}$. For a scheme or a rigid space X and its point $x \in X$, let $k(x)$ be the residue field of x . We define the sign of Hodge-Tate weights of de Rham representations such that the Hodge-Tate weight of the p -adic cyclotomic character is $+1$.

Let $\mathcal{C}_L$ (resp. $\mathcal{C}_{\mathcal{O}_L}$) be the category of local artinian L -algebra (resp. local artinian $\mathcal{O}_L$ -algebra) with residue field isomorphic to L (resp. k_L), and for an A in $\text{ob}(\mathcal{C}_L)$ (resp. $\text{ob}(\mathcal{C}_{\mathcal{O}_L})$) let $\mathfrak{m}_A$ denote its maximal ideal and $\phi_A: A \rightarrow L$ (resp. $\phi_A: A \rightarrow k_L$) denote the natural reduction map. For a functor $F: \mathcal{C}_L \rightarrow \text{Set}$, let TF denote the tangent space $\ker(F(L[\epsilon]) \rightarrow F(L))$ of F .

For a scheme or a rigid space X , we write $\text{Bun}(X)$ for the category of locally free sheaves on the structure sheaf $\mathcal{O}_X$. For an $\mathcal{O}_X$ -algebra $\mathcal{A}_X$, we define $\text{Bun}(\mathcal{A}_X)$ as the category of locally free $\mathcal{A}_X$ -modules. For a point $x \in X$, let k_x be the residue field at the point x .

For a commutative ring R , let $R[\epsilon]$ be the R -algebra naturally isomorphic to $R[T]/(T^2)$ via $\epsilon \mapsto T$. We write $M[\epsilon] := M \otimes_R R[\epsilon]$ for the $R[\epsilon]$ -module base changed from M by the inclusion $R \rightarrow R[\epsilon]$. For a group scheme G over a commutative ring R , let $\text{Lie}_R(G) := \ker(G(R[\epsilon]) \rightarrow G(R))$ be the Lie module over R . Throughout the thesis, a “torsor” means a torsor in the étale topology over a scheme or a rigid space.

For an algebraic group G over a complete discrete valuation ring R , write G^{der} for the derived subgroup of G in [32, Theorem 5.3.1], which is the same as the classical definition $G^{\text{der}} = [G, G]$ in the case that R is a field. Let $G^{\text{ab}} := G/G^{\text{der}}$, which is representable by a group scheme by [49, Part I, 5.6 (9)]. Write $X^*(G) := \text{Hom}_R(G, \mathbb{G}_{m,R})$ for the character group of G . For a torus T over $\text{Spec } R$, we also write $X_*(T) := \text{Hom}_R(\mathbb{G}_{m,R}, T)$ for the cocharacter group of T .

Let A be a commutative topological R -algebra. We equip $G(A)$ with the topology due to Grothendieck (cf. [31, Proposition 2.1]). For a continuous representation $\rho: \text{Gal}_K \rightarrow G(A)$, let $\rho^{\text{ab}}: \text{Gal}_K \rightarrow G^{\text{ab}}(A)$ be the composite of ρ with the abelianization map $G(A) \rightarrow G^{\text{ab}}(A)$.

For a perfect $\mathbb{F}_p$ -algebra R , let $W(R)$ be the ring of Witt vectors of R . Let $\text{Mat}_{a \times b}$ be the algebraic group of $(a \times b)$ -matrices, which is isomorphic to $\mathbb{G}_a^{ab}$.

For a ring A , write $M_n(A)$ for the ring of $n \times n$ -matrix and $A^\times$ for the subgroup of invertible elements in A . For a matrix M , let M^t be the transpose of M . Let $\widehat{\mathbb{Z}} := \prod_{\ell:\text{prime}} \mathbb{Z}_\ell$ and $\widehat{\mathbb{Z}}^p := \prod_{\ell \neq p} \mathbb{Z}_\ell$. Let $\mathbb{A}$ be the ring of adeles and $\mathbb{A}^S$ be the ring of adeles outside of S for a finite set S of places on $\mathbb{Q}$.

For a finite extension $L/\mathbb{Q}_p$, let $\mathcal{O}_L$ be the ring of integers in L and ϖ_L be a fixed uniformizer in $\mathcal{O}_L$. Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. For a global field F and a place v on F , let F_v be the completion of F at v . In this case, write $\mathcal{O}_v := \mathcal{O}_{F_v}$ and $\varpi_v := \varpi_{F_v}$.

For a field extension L/K and a scheme X over $\text{Spec } K$, let $X_L := X \times_{\text{Spec } K} \text{Spec } L$ be the base change. Let GL_n , SL_n be the general linear group and the special linear group respectively. They are algebraic groups over $\text{Spec } \mathbb{Z}$. Let Set , (resp. (Sch/S) , $\text{Rep}_F(G)$) be the category of sets (resp. schemes over S for a scheme S , finite dimensional algebraic F -representations of G for a field F and an algebraic group G over F). For a representation W of a group G , let $W^\vee$ be the contragredient of W .

CHAPTER 2

Crystalline lifts of semisimple G -valued Galois representations with fixed determinant

2.1. Introduction

In this chapter, we show the existence of crystalline lifts of semisimple Galois representations valued in connected split reductive groups over rings of p -adic integers, whose determinant coincide with a fixed one.

In Section 2.2, we explain an argument for lifting of maximal split tori-valued representations of inertia groups to representations of whole Galois groups valued in normalizers of tori. In Section 2.3, we show the existence of crystalline lifts of semisimple G -valued Galois representations with fixed determinant. In Section 2.4, we show the existence of potentially crystalline lifts of semisimple L -parameters with fixed determinant.

2.2. Crystalline extensions to normalizers of tori

In this section, let G be a connected split reductive group over $\mathcal{O}_L$ and T be a maximal torus of G , i.e., T be a fiberwise maximal and split $\mathcal{O}_L$ -torus (cf. [32, Definition 3.2.1]).

We recall the notation of [58, §4]. Let $\Phi_K \in \text{Gal}_K$ be a lift of a topological generator of $\text{Gal}_K / I_K \cong \widehat{\mathbb{Z}}$. Let $W(G, T)$ be the Weyl group scheme of (G, T) . For the ease of notation, write $W(G, T)$ for the group $W(G, T)(\mathcal{O}_L) \cong W(G, T)(k_L)$, which is a subgroup of the finite group $W(G, T)(\bar{L})$.

Let $M_{T, \text{crys}, L}$ denote the set of continuous $T(\mathcal{O}_L)$ -valued representations $I_K \rightarrow T(\mathcal{O}_L)$ of the inertia group I_K , which can be extended to a crystalline representation $\text{Gal}_{K'} \rightarrow T(\mathcal{O}_L)$ for some finite unramified extension K'/K inside $\overline{\mathbb{Q}_p}$. Let M_{T, k_L} denote the set of continuous mod ϖ_L representations $I_K \rightarrow T(k_L)$ of I_K .

Both $M_{T, \text{crys}, L}$ and M_{T, k_L} are naturally equipped with the structures of $\mathbb{Z}[W(G, T)]$ -modules defined by $wv := (\sigma \mapsto wv(\sigma)w^{-1})$ for $w \in W(G, T)$ and $v \in M_{T, \text{crys}, L}$ (or $v \in M_{T, k_L}$). Similarly, they are naturally equipped with the structures of $\mathbb{Z}[\text{Gal}_K / I_K]$ -modules defined by $\alpha v := (\sigma \mapsto v(\alpha^{-1}\sigma\alpha))$ for $\alpha \in \text{Gal}_K$ and $v \in M_{T, \text{crys}, L}$ (or $v \in M_{T, k_L}$).

- Definition 2.2.1.**
- (1) A continuous G -valued representation $\bar{\rho}: \text{Gal}_K \rightarrow G(\bar{k}_L)$ is said to be *G -irreducible* if it does not factor through any $P(\bar{k}_L) \subset G(\bar{k}_L)$ where P is a parabolic subgroup of $G_{\bar{k}_L}$.
 - (2) A subgroup $\Gamma \subset G(\bar{k}_L)$ is said to be *G -completely reducible* if for any parabolic subgroup P of $G_{\bar{k}_L}$ such that $\Gamma \subset P(\bar{k}_L)$, there exists a Levi subgroup $L \subset P$ such that $\Gamma \subset L(\bar{k}_L)$. We also say that a continuous G -valued representation $\bar{\rho}: \text{Gal}_K \rightarrow G(\bar{k}_L)$ is *G -completely reducible* if the image of $\bar{\rho}$ is G -completely reducible.
 - (3) We say $\bar{\rho}: \text{Gal}_K \rightarrow G(\bar{k}_L)$ is *quasi-semisimple* if there exists a maximal torus T' of $G_{\bar{k}_L}$ such that $\bar{\rho}(I_K) \subset T'(\bar{k}_L)$ and $\bar{\rho}(\text{Gal}_K) \subset N_{G_{\bar{k}_L}}(T')(\bar{k}_L)$.

Note that a G -irreducible continuous G -valued representation $\text{Gal}_K \rightarrow G(\bar{k}_L)$ is G -completely reducible.

Theorem 2.2.2. *If a continuous G -valued representation $\text{Gal}_K \rightarrow G(\bar{k}_L)$ is G -completely reducible, then $\bar{\rho}$ is quasi-semisimple.*

PROOF. See [58, Theorem 4]. $\square$

For showing the main theorem, we will reduce to the case where $\bar{\rho}: \text{Gal}_K \rightarrow G(\bar{k}_L)$ is G -irreducible. By Theorem 2.2.2, it suffices to consider the existence of a lift $\rho: \text{Gal}_K \rightarrow N_G(T)(\mathcal{O}_L)$ of $\bar{\rho}$ satisfying $\rho(\Phi_K) = w^{-1}$ for an element $w \in N_G(T)(\mathcal{O}_L)$.

In the following, we generalize Lin's argument in [58, §4] to the case that $\rho(\Phi_K)$ is not of finite order and the homomorphism $\mathbb{Z} \rightarrow G(\mathcal{O}_L)$, $1 \mapsto \rho(\Phi_K)$ can be topologically extended to a homomorphism $\widehat{\mathbb{Z}} \rightarrow G(\mathcal{O}_L)$.

Definition 2.2.3. For a Hausdorff topological group G , we say that a homomorphism $\mathbb{Z} \rightarrow G$, $1 \mapsto g$ is *extendable* if it can be extended to a (unique) continuous homomorphism $\widehat{\mathbb{Z}} \rightarrow G$.

Lemma 2.2.4. *Let ℓ be a prime number, m be a positive integer, G be a locally profinite group, and $g \in G$.*

- (1) *If $\lim_{k \rightarrow \infty} g^{\ell^k} = e$, the homomorphism $\mathbb{Z} \rightarrow G$, $1 \mapsto g$ is extendable.*
- (2) *The homomorphism $1 \mapsto g$ is extendable if and only if the homomorphism $1 \mapsto g^m$ is extendable.*

PROOF. (1) The homomorphism $\mathbb{Z} \rightarrow G$, $1 \mapsto g$ can be extended to $\mathbb{Z}_\ell \rightarrow G$ because there exists a system $\{U_i\}_{i \geq 1}$ of neighborhoods of the unity $e \in G$, where each U_i is a profinite open subgroup of G . Then (1) follows by taking the composite of the projection $\widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}_\ell$ with a map $\mathbb{Z}_\ell \rightarrow G$.

(2) The “only if” part is obvious. If $1 \mapsto g^m$ is extendable, there exists a continuous homomorphism $\widehat{f}: m\widehat{\mathbb{Z}} \rightarrow G$ which extends $m\mathbb{Z} \rightarrow G$, that is the restriction of $f: \mathbb{Z} \rightarrow G$, $1 \mapsto g$. For each $a \in [0, m-1]$, the map

$$\widehat{f}'_a: a + m\widehat{\mathbb{Z}} \rightarrow G, \quad a + mb \mapsto f(a)\widehat{f}'(mb)$$

is continuous. Since $\{a + m\widehat{\mathbb{Z}} \mid a \in [0, m-1]\}$ is an open covering of $\widehat{\mathbb{Z}}$, the map $\bigsqcup_{a \in [0, m-1]} \widehat{f}'_a: \widehat{\mathbb{Z}} \rightarrow G$ is a continuous homomorphism which extends f . $\square$

We prove the following lemma, which is proved by Lin [58, Lemma 4] when w has finite order.

Lemma 2.2.5. *Let $\zeta: N_G(T) \rightarrow W(G, T)$ be the quotient map.*

- (1) *Let $v: I_K \rightarrow T(\mathcal{O}_L)$ be a continuous representation such that $v \in M_{T, \text{crys}}$. For an element $w \in N_G(T)(\mathcal{O}_L)$, the representation v extends to a continuous representation $\rho: \text{Gal}_K \rightarrow N_G(T)(\mathcal{O}_L)$ satisfying $\rho(\Phi_K) = w^{-1}$ and $\rho|_{I_K} = v$ if and only if*

$$v \in \ker(M_{T, \text{crys}, L} \xrightarrow{\zeta(w) \otimes 1 - 1 \otimes \Phi_K} M_{T, \text{crys}, L}).$$

- (2) *Let $\bar{v}: I_K \rightarrow T(k_L)$ be a continuous representation such that $\bar{v} \in M_{T, k_L}$. For an element $\bar{w} \in N_G(T)(k_L)$, the representation $\bar{v}$ extends to a continuous representation $\bar{\rho}: \text{Gal}_K \rightarrow N_G(T)(k_L)$ satisfying $\bar{\rho}(\Phi_K) = \bar{w}^{-1}$ and $\bar{\rho}|_{I_K} = \bar{v}$ if and only if*

$$\bar{v} \in \ker(M_{T, k_L} \xrightarrow{\zeta(\bar{w}) \otimes 1 - 1 \otimes \Phi_K} M_{T, k_L}).$$

PROOF. (1) Assume that $v \in \ker(\zeta(w) \otimes 1 - 1 \otimes \Phi_K: M_{T, \text{crys}, L} \rightarrow M_{T, \text{crys}, L})$. By [58, Lemma 4], the representation v extends to a representation $v': W_K \cong I_K \rtimes \mathbb{Z} \rightarrow N_G(T)(\mathcal{O}_L)$ such that $v'(\Phi_K) = w^{-1}$ and $v'|_{I_K} = v$. So it suffices to show that v' extends to a continuous representation of the absolute Galois group Gal_K , which is homeomorphic to the profinite set $I_K \times \widehat{\mathbb{Z}}$.

Since the centralizer $C_G(T)$ is the maximal torus T and the Weyl group $W(G, T)$ is finite, there exists a positive integer m such that $w^m \in T(\mathcal{O}_L)$. Fix an isomorphism $i: T \cong \mathbb{G}_{m, \mathcal{O}_L}^d$. Then there is also a positive integer m' such that the j -th component of $i(w^{mm'}) \in (\mathcal{O}_L^*)^d$ is in $1 + \mathfrak{m}_L$ for every $1 \leq j \leq d$. Then $\lim_{k \rightarrow \infty} (w^{mm'})^{p^k} = 1$ in $T(\mathcal{O}_L)$. The limit also exists and equals to 1 since $T(\mathcal{O}_L)$ is a closed subset of $T(\mathcal{O}_L)$. So $1 \mapsto w^{mm'}$ and then $1 \mapsto w$ is extendable by Lemma 2.2.4. The “only if” part is obvious.

(2) See [58, Lemma 4 (2)]. $\square$

We recall the notion of labeled Hodge–Tate cocharacters following [58, §6 Definition 3].

Definition 2.2.6. Let G be a reductive group over L .

- (1) For a Hodge–Tate G -valued representation $\rho: \text{Gal}_K \rightarrow G(L)$, there is a cocharacter $\mathcal{HT}(\rho)^\sigma: \mathbb{G}_{m, \mathbb{C}_p} \rightarrow G_{\mathbb{C}_p}$ associated with ρ for each $\sigma \in \Sigma_L = \text{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)$ by Tannakian theory (cf. [58, Definition 3]). The *labeled Hodge–Tate cocharacter* associated with ρ is defined by

$$\mathcal{HT}(\rho) := (\mathcal{HT}(\rho)^\sigma)_{\sigma \in \Sigma_L} \in \prod_{\sigma \in \Sigma_L} X_*(G_{\mathbb{C}_p}).$$

- (2) A cocharacter of $G_{\mathbb{C}_p}$ is said to be *regular* if it is not fixed by the action of each root of $G_{\mathbb{C}_p}$ with respect to a maximal torus of $G_{\mathbb{C}_p}$ which contains the image of the cocharacter. For a Hodge–Tate G -valued representation $\rho: \text{Gal}_K \rightarrow G(L)$, we say that ρ has *regular* Hodge–Tate weights if for each $\sigma \in \Sigma_L$, the cocharacter $\mathcal{HT}(\rho)^\sigma$ of $G_{\mathbb{C}_p}$ is regular.

We note that the terminology “colabeled Hodge–Tate cocharacter” is used in [58, §5.1] instead of “labeled Hodge–Tate cocharacter” in this thesis.

2.3. Crystalline lifts of quasi-semisimple representations with fixed abelianizations

We prove the first main theorem in this section. In addition to Lin’s argument [58, Proposition 2, Theorem 6], we use the calculation of the norms of Galois characters due to Böckle–Iyengar–Paškūnas [20, Lemma 2.1] in order to construct a crystalline lift with a fixed abelianization.

We use the notations in Section 2.2. In particular, let G be a connected split reductive group over $\mathcal{O}_L$ and $T \subset G$ be a maximal split $\mathcal{O}_L$ -torus. We call the homomorphism $K^* \rightarrow \mathcal{O}_K^*$ defined by $\varpi_K \mapsto 1$, $\mathcal{O}_K^* \ni a \mapsto a$ the *fundamental character (of K)*, and the composite of the fundamental character with the reduction map $\mathcal{O}_K^* \twoheadrightarrow k^*$ the *mod ϖ_K fundamental character (of K)*. Write $\omega_K: K^* \rightarrow \mathcal{O}_K^*$ for the fundamental character of K . Note that the fundamental character is the continuous character of K^* associated with the Lubin–Tate character, which is a crystalline character $\text{Gal}_K \rightarrow \mathcal{O}_K^*$. Let $\text{unr}(c): K^* \rightarrow \mathcal{O}_L^*$ be the unramified character which sends the uniformizer ϖ_K to an element $c \in \mathcal{O}_L^*$. We recall basic properties of crystalline characters of Gal_K .

Lemma 2.3.1. *The continuous character $K^* \rightarrow \mathcal{O}_L^*$ associated with a crystalline character $\text{Gal}_K \rightarrow \mathcal{O}_L^*$ can be written as the twist of $\text{unr}(c)$ for an element $c \in \mathcal{O}_L^*$ with a homomorphism $x \mapsto \prod_{\sigma: K \rightarrow L} \sigma(\omega_K(x))^{a_\sigma}$ where $a_\sigma \in \mathbb{Z}$ for each $\sigma \in \text{Hom}_{\mathbb{Q}_p}(K, L)$.*

PROOF. Since a homomorphism $\text{Res}_{K/\mathbb{Q}_p} \mathbb{G}_{m,K} \rightarrow \text{Res}_{L/\mathbb{Q}_p} \mathbb{G}_{m,L}$ of $\mathbb{Q}_p$ -tori comes from $\prod_{\sigma: K \rightarrow L} \sigma^{a_\sigma}$ for $a_\sigma \in \mathbb{Z}$, the lemma follows from [30, Proposition B.4]. $\square$

Lemma 2.3.2. *Let $f := [k_L : k]$ and K_f be an unramified extension of K of degree f . Let G be a connected split reductive group over K and $T \subset G$ be a K -split maximal torus. Let $M_{T, \text{crys}, L}^0 \subset M_{T, \text{crys}, L}$ be the $\mathbb{Z}[W(G, T)] \otimes \mathbb{Z}[\text{Gal}_K / I_K]$ -submodule which consists of continuous $T(\mathcal{O}_L)$ -valued representations of I_K that can be extended to a continuous representation $\text{Gal}_{K_f} \rightarrow T(\mathcal{O}_L)$. Then the natural reduction map $M_{T, \text{crys}, L}^0 \rightarrow M_{T, k_L}$ is surjective.*

PROOF. The group M_{T, k_L} is generated by the mod ϖ_{K_f} fundamental character. Since it lifts to the fundamental character of K_f which extends to a crystalline character of Gal_{K_f} (i.e. the Lubin–Tate character of Gal_{K_f}), the assertion follows. $\square$

We will construct a regular crystalline lift from a crystalline lift by twisting a crystalline lift which has sufficiently generic Hodge–Tate weights and the trivial abelianization. Such a lift is constructed by local class field theory and the following lemma:

Lemma 2.3.3. *Consider the map*

$$w - 1: X_*(T) \rightarrow X_*(T), \quad \chi \mapsto \text{ad}(w)(\chi) \cdot \chi^{-1}$$

for every non-trivial element $w \in W(G, T)$. Let $X_^{\text{ab}}: X_*(T) \rightarrow X_*(G^{\text{ab}})$ be the homomorphism of the cocharacter groups induced by the composite $T \hookrightarrow G \twoheadrightarrow G^{\text{ab}}$. Then the property*

$$(R_w): \text{the rank of } \ker(X_*^{\text{ab}}) / (\ker(X_*^{\text{ab}}) \cap \ker(w - 1)) \text{ is positive}$$

is satisfied for every non-trivial element $w \in W(G, T)$.

PROOF. Let Z be the center of G and $G^{\text{ad}} := G/Z$ be the adjoint group of G . Let $X_*(G^{\text{ad}})$ be the image of $X_*(T)$ by the natural map $X_*(T) \rightarrow \text{Hom}(\mathbb{G}_m, G^{\text{ad}})$. There exists a natural isomorphism

$$X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (X_*(Z) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (X_*(G^{\text{ad}}) \otimes_{\mathbb{Z}} \mathbb{Q})$$

since the composite $Z \hookrightarrow G \twoheadrightarrow G^{\text{ab}}$ is an isogeny.

The center and the adjoint group of G^{ad} is trivial and the Weyl group of G is naturally identified to that of G^{ad} . Therefore, the lemma follows from the case where the center of G is trivial. In this case, each non-trivial element $w \in W(G, T)$ of the Weyl group acts non-trivially on $X_*(T)$. Thus the lemma follows. $\square$

Let $(X^*(T), R, X_*(T), R^\vee)$ be the root datum of (G, T) . The following result is well-known.

Lemma 2.3.4. *Let $\alpha \in X^*(T)$ be an algebraic character of T . Then α is the restriction of an algebraic character of G if and only if $\langle \alpha, \beta^\vee \rangle = 0$ for any $\beta^\vee \in R^\vee$.*

PROOF. See [49, Part II, 1.18 (3)]. $\square$

Lemma 2.3.5. *The composite $T \hookrightarrow G \rightarrow G^{\text{ab}}$ has a right inverse, that is, there exists a homomorphism $G^{\text{ab}} \rightarrow T$ such that the composite $G^{\text{ab}} \rightarrow T \rightarrow G^{\text{ab}}$ is the identity.*

PROOF. Since the composite $T \hookrightarrow G \rightarrow G^{\text{ab}}$ is surjective, the induced map $X^*(G^{\text{ab}}) \rightarrow X^*(T)$ is injective. Let $M := \text{im}(X^*(G^{\text{ab}}) \rightarrow X^*(T))$. It suffices to show that the quotient $X^*(T)/M$ is torsion-free.

The set M consists of those elements $\alpha \in X^*(T)$ such that $\langle \alpha, \beta^\vee \rangle = 0$ for any $\beta^\vee \in R^\vee$ by Lemma 2.3.4.

Let $\alpha \in X^*(T)$ which satisfies $n\alpha \in M$ for an integer n . Then we have $\langle n\alpha, \beta^\vee \rangle = 0$ for any $\beta^\vee \in R^\vee$. Since $\langle n\alpha, \beta^\vee \rangle = n\langle \alpha, \beta^\vee \rangle$, we have $\langle \alpha, \beta^\vee \rangle = 0$ for any $\beta^\vee \in R^\vee$. Hence $\alpha \in M$. This shows that $X^*(T)/M$ is torsion-free, and the assertion follows. $\square$

We prove the first main theorem of this paper.

Theorem 2.3.6. *Let G be a connected split reductive group over $\mathcal{O}_L$ and T be a split maximal torus of G over $\mathcal{O}_L$.*

Let $\bar{\rho}: \text{Gal}_K \rightarrow G(k_L)$ be a quasi-semisimple G -valued representation such that $\bar{\rho}(I_K) \subset T(k_L)$ and $\bar{\rho}(\text{Gal}_K) \subset N_G(T)(k_L)$. Let $\bar{\rho}^{\text{ab}}: \text{Gal}_K \rightarrow G^{\text{ab}}(k_L)$ be the abelianization of $\bar{\rho}$ and $\psi: \text{Gal}_K \rightarrow G^{\text{ab}}(\mathcal{O}_L)$ be a crystalline lift of $\bar{\rho}^{\text{ab}}$. Then there exist a finite extension L'/L and a crystalline lift $\rho: \text{Gal}_K \rightarrow G(\mathcal{O}_{L'})$ with regular Hodge–Tate weights such that $\rho^{\text{ab}} = \psi$.

Moreover, if we do not require that the crystalline lift ρ has regular Hodge–Tate weights, L'/L can be taken to be a finite unramified extension.

PROOF. **Step 1.**

First, we introduce the notations of [58, Proposition 2] needed in the proof of the theorem. We naturally consider $M_{T, \text{crys}, L}$ and M_{T, k_L} as $\mathbb{Z}[W(G, T)] \otimes \mathbb{Z}[\text{Gal}_K / I_K]$ -modules. For an element $\tau \in W(G, T)$, write

$$M_{T, \tau, \text{crys}, L} := \ker(M_{T, \text{crys}, L} \xrightarrow{\tau \otimes 1 - 1 \otimes \Phi_K} M_{T, \text{crys}, L}),$$

$$M_{T, \tau, k_L} := \ker(M_{T, k_L} \xrightarrow{\tau \otimes 1 - 1 \otimes \Phi_K} M_{T, k_L})$$

for subgroups of $M_{T, \text{crys}, L}$ and M_{T, k_L} .

From here, let $\tau \in W(G, T)$ be the image of $\bar{\rho}(\Phi_K^{-1})$ by the quotient map $\zeta: N_G(T)(k_L) \rightarrow W(G, T)$. Then we have $\bar{\rho}|_{I_K} \in M_{T, \tau, k_L}$.

Let $f := [k_L : k]$ and K_f be an unramified extension of K of degree f . We may assume that L is sufficiently large so that $\tau^f = 1$ by replacing L with a finite unramified extension. We take a finite unramified extension L'/L such that L' contains all embeddings $K_f \hookrightarrow \overline{\mathbb{Q}}_p$ for $f \in \mathbb{Z}_{\geq 1}$. We put

$$\Xi := \sum_{i=0}^{f-1} \tau^i \otimes \Phi_K^{f-1-i}.$$

Let $M_{T, \text{crys}, L}^0 \subset M_{T, \text{crys}, L}$ be the $\mathbb{Z}[W(G, T)] \otimes \mathbb{Z}[\text{Gal}_K / I_K]$ -submodule which consists of continuous $T(\mathcal{O}_L)$ -valued representations of I_K that can be extended to a continuous representation $\text{Gal}_{K_f} \rightarrow T(\mathcal{O}_L)$. We put $M_{T, \tau, \text{crys}, L}^0 := M_{T, \text{crys}, L}^0 \cap M_{T, \tau, \text{crys}, L}$. Note that the natural reduction map $M_{T, \text{crys}, L}^0 \rightarrow M_{T, k_L}$ is surjective by Lemma 2.3.2.

Since $(\tau \otimes 1)^f = (1 \otimes \Phi_K)^f = \text{id}$ on both $M_{T, \text{crys}, L}$ and M_{T, k_L} , we have an injection $\Xi M_{T, \text{crys}}^0 \hookrightarrow M_{T, \tau, \text{crys}, L}^0$. So we have the following maps:

$$\Xi M_{T, \text{crys}, L}^0 \hookrightarrow M_{T, \tau, \text{crys}, L}^0 \hookrightarrow M_{T, \tau, \text{crys}, L} \rightarrow M_{T, \tau, k_L}.$$

It follows that the composite of the maps is the same as the composite of the natural surjection $\Xi M_{T, \text{crys}, L}^0 \rightarrow \Xi M_{T, k_L}$ and the natural isomorphism

$$\Xi M_{T, k_L} \cong M_{T, \tau, k_L}$$

in the proof of [58, Proposition 2]. Thus the composite $\Xi M_{T,\text{crys},L}^0 \rightarrow M_{T,\tau,k_L}$ is surjective.

Step 2.

Next, we prove the existence of crystalline lifts with fixed abelianizations. Fix isomorphisms $\mathbb{G}_m^{d_0} \times \mathbb{G}_m^{d_1} \cong G^{\text{ab}} \times T' \cong T$ for a split torus T' by Lemma 2.3.5.

For $A = k_L$ or $\mathcal{O}_L$ and a continuous representation $r: H \rightarrow T(A)$ of a locally profinite group H , we write $r_i: H \rightarrow \mathbb{G}_m(A) = A^*$ for the i -th component of r via the isomorphism $\mathbb{G}_m^{d_0} \times \mathbb{G}_m^{d_1} \cong T$ for each $i \in [1, d_0 + d_1]$.

Let $\bar{\rho}' \in M_{T,k_L}$ be a continuous representation such that $\Xi \bar{\rho}'$ is mapped to $\bar{\rho}|_{I_K}$ by the natural isomorphism $\Xi M_{T,k_L} \cong M_{T,\tau,k_L}$ in Step 1. We construct an element $\rho' \in M_{T,\text{crys},L'}^0$ as follows:

- For $d_0 + 1 \leq i \leq d_0 + d_1$, as $T(\mathcal{O}_L)$ -valued representations, take $\rho'_i \in M_{\mathbb{G}_m,\text{crys},L}^0$ to be mapped to $\bar{\rho}'_i$ by the surjection $M_{\mathbb{G}_m,\text{crys},L}^0 \rightarrow M_{\mathbb{G}_m,k_L}$ (see Lemma 2.3.2).
- For $1 \leq i \leq d_0$, let $\psi_i: \text{Gal}_K \rightarrow \mathcal{O}_L^*$ be the i -th component of ψ . The character $\psi'_i: K^* \rightarrow \mathcal{O}_L^*$ associated with ψ_i may be written as the twist of an unramified character $\text{unr}(c_i)$ with $x \mapsto \prod_{\sigma: K \rightarrow L} \sigma(\omega_K(x))^{a_{\sigma,i}}$, where $c_i \in \mathcal{O}_L^*$, $a_{\sigma,i} \in \mathbb{Z}$ for each $\sigma \in \text{Hom}_{\mathbb{Q}_p}(K, L)$ by Lemma 2.3.1. We may assume $K_f \neq K$. Then there exist distinct integers $b_{s,i}$ ($s \in \text{Hom}_{\mathbb{Q}_p}(K_f, L')$) such that the character

$$\theta: \mathcal{O}_{K_f}^* \rightarrow \mathcal{O}_{L'}^*, \quad x \mapsto \prod_{s \in \text{Hom}_{\mathbb{Q}_p}(K_f, L')} s(x)^{b_{s,i}}$$

satisfies the following conditions by [20, Lemma 2.5].

- The integers $b_{s,i}$ ($s \in \text{Hom}_{\mathbb{Q}_p}(K_f, L')$) are pairwise distinct.
- The character θ lifts the character $\mathcal{O}_{K_f}^* \rightarrow k_L^*$ associated with the composite of $I_{K_f} \xrightarrow{\cong} I_K$ and $\bar{\rho}'_i$.
- $\theta(x) = \psi'_i(x)$ for all $x \in \mathcal{O}_K^*$.

Let $\rho'_i: I_K \rightarrow \mathcal{O}_{L'}^*$ be the composite of the following maps

$$I_K = I_{K_f} \hookrightarrow W_{K_f} \twoheadrightarrow W_{K_f}^{\text{ab}} \xrightarrow[\cong]{\text{rec}_{K_f}^{-1}} K_f^* \xrightarrow{\prod_{s: K_f \rightarrow L'} s(\omega_{K_f}(-))^{b_{s,i}}} L'^*,$$

Then the continuous $T(\mathcal{O}_{L'})$ -valued representation $\rho' \in M_{T,\text{crys},L'}^0$ constructed as above satisfies the following properties:

- The image of ρ' by the surjection $M_{T,\text{crys},L'}^0 \rightarrow M_{T,k_{L'}}$ is $\bar{\rho}'$.
- The abelianization $\rho'^{\text{ab}}: I_K \rightarrow G^{\text{ab}}(\mathcal{O}_{L'})$ is extendable to a crystalline representation $\psi': \text{Gal}_{K_f} \rightarrow G^{\text{ab}}(\mathcal{O}_{L'})$ such that $\psi|_{W_K}$ is the same as the composite

$$W_K \twoheadrightarrow W_K^{\text{ab}} \xrightarrow[\cong]{\text{rec}_K^{-1}} K^* \hookrightarrow K_f^* \xrightarrow[\cong]{\text{rec}_{K_f}} W_{K_f}^{\text{ab}} \xrightarrow{\psi'|_{W_{K_f}}} G^{\text{ab}}(\mathcal{O}_{L'}).$$

Let R' and $\Xi R'$ be the $T(\mathcal{O}_{L'})$ -valued crystalline representations $\text{Gal}_{K_f} \rightarrow T(\mathcal{O}_{L'})$ extended from ρ' and $\Xi \rho'$ respectively as above. By the calculation in the proof of [20,

Lemma 2.1], we have

$$\begin{aligned}
(\Xi R')^{\text{ab}}(\text{rec}_{K_f}(y)) &= \prod_{\sigma \in \text{Gal}(K_f/K)} R'^{\text{ab}}(\sigma \text{rec}_{K_f}(y) \sigma^{-1}) \\
&= \prod_{\sigma \in \text{Gal}(K_f/K)} R'^{\text{ab}}(\text{rec}_{K_f}(\sigma(y))) \\
&= R'^{\text{ab}}(\text{rec}_{K_f}(\text{Nm}_{K_f/K}(y))) \\
&= \psi|_{W_K}(\text{rec}_K(\text{Nm}_{K_f/K}(y))) \\
&= \psi|_{W_{K_f}}(\text{rec}_{K_f}(y))
\end{aligned}$$

for any $y \in K_f^*$. Here, the first equality follows from the fact that each τ^i acts trivially on the abelianization ρ'^{ab} , the fourth equality follows from the second bullet of the properties of the representation $\rho' \in M_{T, \text{crys}, L'}^0$, and the last equality follows from $\text{Nm}_{K_f/K} \circ \text{rec}_{K_f}^{-1} = \text{rec}_K^{-1} \circ \lambda$ by local class field theory, where $\lambda: W_{K_f}^{\text{ab}} \rightarrow W_K^{\text{ab}}$ is induced by the inclusion $W_{K_f} \hookrightarrow W_K$. So we have $(\Xi R')^{\text{ab}}|_{W_{K_f}} = \psi|_{W_{K_f}}$, and by continuity $(\Xi R')^{\text{ab}} = \psi|_{\text{Gal}_{K_f}}$. Moreover, it follows that the image of $\Xi \rho'$ by the surjection $\Xi M_{T, \text{crys}, L'}^0 \rightarrow M_{T, \tau, k_{L'}}$ is $\Xi \bar{\rho}' = \bar{\rho}|_{I_K}$ by the construction.

Let $w \in N_G(T)(\mathcal{O}_{L'})$ be an element which is a lift of the element $\bar{\rho}(\Phi_K)^{-1}$. We may assume that the image of w by the composite $N_G(T)(\mathcal{O}_{L'}) \hookrightarrow G(\mathcal{O}_{L'}) \rightarrow G^{\text{ab}}(\mathcal{O}_{L'})$ is $\psi(\Phi_K)^{-1}$ by the assumption of the existence of a right inverse of the map $T \rightarrow G^{\text{ab}}$.

Since $\Xi M_{T, \text{crys}, L'}^0 \subset M_{T, \tau, \text{crys}, L'}^0$, there exists a continuous representation $\rho: \text{Gal}_K \rightarrow N_G(T)(\mathcal{O}_{L'}) \subset G(\mathcal{O}_{L'})$ such that $\rho|_{\text{Gal}_{K_f}} = \Xi R'$ and $\rho^{\text{ab}} = \psi$ by Lemma 2.2.5 (1) where w is taken as above. Then ρ is a crystalline lift of $\bar{\rho}$ satisfying $\rho^{\text{ab}} = \psi$. But the Hodge–Tate weights of ρ are not necessarily regular.

Step 3.

Finally, we show the existence of a crystalline lift with fixed abelianization which has regular Hodge–Tate weights. We prove it by an argument of Lin [58, Lemma 12, Theorem 6].

Let $i: T \hookrightarrow G$ be the embedding and fix embeddings $K_f \subset L' \subset \mathbb{C}_p$. For constructing such a crystalline lift with regular Hodge–Tate weights, we first show the existence of a crystalline representation $t: \text{Gal}_{K_f} \rightarrow T(\mathcal{O}_{L'})$ satisfying the following properties for each $\sigma \in \text{Hom}_{\mathbb{Q}_p}(L, \mathbb{C}_p)$:

- $\mathcal{HT}(i(t))^\sigma$ is regular if $\sigma|_{K_f}$ coincides with the fixed embedding $K_f \hookrightarrow \mathbb{C}_p$.
- $\mathcal{HT}(i(t))^\sigma$ is the trivial cocharacter if $\sigma|_{K_f}$ does not coincide with the fixed embedding $K_f \hookrightarrow \mathbb{C}_p$.
- t^{ab} is trivial.

The existence of such t is reduced to show $\ker(X_*^{\text{ab}}) \not\subset \bigcup_{w \in W(G, T) \setminus \{e\}} \ker(w - 1)$ by considering powers of the fundamental character of K_f for each coordinate of $T(\mathcal{O}_{L'}) \cong (\mathcal{O}_{L'}^*)^{d_0 + d_1}$ (see the proof of [58, Lemma 12]). In fact, let

$$x \in \ker(X_*^{\text{ab}}) \setminus \left(\bigcup_{w \in W(G, T) \setminus \{e\}} \ker(w - 1) \right).$$

Then the composite

$$W_{K_f} \twoheadrightarrow W_{K_f}^{\text{ab}} \xrightarrow[\cong]{\text{rec}_{K_f}^{-1}} K_f^* \hookrightarrow L^* \cong \mathbb{G}_m(L') \xrightarrow{x(L')} T(L')$$

extends to a required crystalline representation $t: \text{Gal}_{K_f} \rightarrow T(L')$. The assertion follows from Lemma 2.3.3.

For such t , let $v_0 \in M_{T, \text{crys}, L'}^0$ be the composite of the following maps

$$I_K = I_{K_f} \xrightarrow{t|_{I_{K_f}}} T(\mathcal{O}_{L'}).$$

Then we have $v_0^{\text{ab}} = 1$. For a sufficiently large positive integer M , the representation $v' := v \cdot (\Xi v_0)^{MN}$, where $v := \rho|_{I_K}$ and N is the cardinality of $k_{L'}^*$, satisfies $v'^{\text{ab}} = (v \cdot (\Xi v_0)^{MN})^{\text{ab}} = v^{\text{ab}}$ and $\overline{v'} = \overline{v}$. Since $v, \Xi v_0 \in \Xi M_{T, \text{crys}, L'}^0 \subset M_{T, \tau, \text{crys}, L'}^0$, the representation v' extends to a continuous representation $\rho'': \text{Gal}_K \rightarrow G(\mathcal{O}_{L'})$, which is a crystalline lift of $\overline{\rho}$ by $K_f \subset L'$ and [58, Corollary 2]. We have $\rho''^{\text{ab}}|_{I_K} = \rho^{\text{ab}}|_{I_K}$ by the construction. So replacing L' with a finite extension, there is an unramified representation $u: \text{Gal}_K \rightarrow Z(\mathcal{O}_{L'})$ whose composite with the map $Z(\mathcal{O}_{L'}) \rightarrow G^{\text{ab}}(\mathcal{O}_{L'})$ is $(\rho''^{\text{ab}})^{-1} \cdot \rho^{\text{ab}}$ since $Z \rightarrow G^{\text{ab}}$ is an isogeny of algebraic tori. Then $\rho_{\text{reg}} := u\rho''$ is a crystalline lift of $\overline{\rho}$ which satisfies $\rho_{\text{reg}}^{\text{ab}} = \rho^{\text{ab}} = \psi$ by the construction. In addition, the representation ρ_{reg} has regular Hodge–Tate weights since for each $\sigma \in \text{Hom}_{\mathbb{Q}_p}(L', \mathbb{C}_p)$, there exists an integer $i \in [0, f-1]$ and an element $\sigma' \in \text{Hom}_{K_f}(L', \mathbb{C}_p)$ such that $\sigma = \sigma' \circ \Phi_K^{f-1-i}$ and

$$\mathcal{HT}(\Xi v_0)^\sigma = \tau^i \mathcal{HT}(v_0)^{\sigma'} \tau^{-i}$$

by the calculation in the proof of [58, Theorem 6]. $\square$

2.4. Potentially crystalline lifts of semisimple L -parameters with fixed abelianizations

In this section, we also prove the existence of potentially crystalline lifts with a fixed potentially crystalline abelianization of semisimple mod p L -parameters for connected quasi-split tame groups.

First, we recall the definitions of L -groups and L -parameters following [60, §2, §3]. Let $K/\mathbb{Q}_p$ be a finite extension and G be a connected quasi-split reductive group over K . Fix a K -pinning $(B, T, \{X_\alpha\})$ of G . The pinned group $(G, B, T, \{X_\alpha\})$ has a dual pinned group $(\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$ defined over $\mathbb{Z}$. Let $E \subset \overline{K}$ be a splitting field of G , which is a finite Galois extension over K . The Galois action on the based root datum induces a Galois action $\text{Gal}(E/K) \rightarrow \text{Aut}(\widehat{G}, \widehat{B}, \widehat{T}, \{Y_\alpha\})$. Write ${}^L G := \widehat{G} \rtimes \text{Gal}_K$ (see [21] and [88, §1.1] for more details). Write ${}^L \{*\} \cong \text{Gal}_K$ for the L -group of the trivial group.

Definition 2.4.1 (cf. [60, Definition 3.1.1, Definition 3.1.3]). Let G, H be connected reductive groups over K and $\widehat{G}, \widehat{H}$ be their Langlands dual groups.

- (1) For a finite ring A , a group homomorphism

$$\widehat{G}(A) \rtimes \text{Gal}(E/K) \rightarrow \widehat{H}(A) \rtimes \text{Gal}(E/K)$$

is said to be an L -homomorphism (with A -coefficients) if each element $g \times \sigma \in \widehat{G}(A) \rtimes \text{Gal}(E/K)$ is sent to $h \times \sigma \in \widehat{H}(A) \rtimes \text{Gal}(E/K)$.

- (2) For a finite ring A , a group homomorphism

$$\widehat{G}(A) \rtimes W_{E/K} \rightarrow \widehat{H}(A) \rtimes W_{E/K}$$

is said to be an L -homomorphism (with A -coefficients) if each element $g \times \sigma \in \widehat{G}(A) \rtimes \text{Gal}(E/K)$ is sent to $h \times \sigma \in \widehat{H}(A) \rtimes \text{Gal}(E/K)$.

- (3) For a finite ring A , a group homomorphism

$$\widehat{G}(A) \rtimes \text{Gal}_K = {}^L G(A) \rightarrow {}^L H(A) = \widehat{H}(A) \rtimes \text{Gal}_K$$

is said to be an L -homomorphism (with A -coefficients) if each element $g \times \sigma \in \widehat{G}(A) \rtimes \text{Gal}_K$ is sent to $h \times \sigma \in \widehat{H}(A) \rtimes \text{Gal}_K$, and there exists an open subgroup $\Gamma \subset \text{Gal}_K$ such that $1 \times \sigma \in \widehat{G}(A) \rtimes \text{Gal}_K$ is sent to $1 \times \sigma \in \widehat{H}(A) \rtimes \text{Gal}_K$ for every $\sigma \in \Gamma$.

- (4) An L -parameter (of Gal_K (resp. $\text{Gal}(E/K)$, $W_{E/K}$) for G with A -coefficients) is an L -homomorphism

$$\text{Gal}_K \simeq {}^L \{*\}(A) \rightarrow {}^L G(A),$$

$$(\text{resp. } \text{Gal}(E/K) \rightarrow \widehat{G}(A) \rtimes \text{Gal}(E/K), W_{E/K} \rightarrow \widehat{G}(A) \rtimes W_{E/K}),$$

defined up to $\widehat{G}(A)$ -conjugacy. Note that an L -parameter can be considered as an ${}^L G(A)$ -valued Galois representation $\text{Gal}_K \rightarrow {}^L G(A)$.

- (5) For a profinite ring R , an L -parameter (for G with R -coefficients) is a compatible system of L -parameters $\text{Gal}_K \rightarrow {}^L G(A)$, where A is a finite quotient of R .
- (6) For $A = \overline{k}_L$ or $\overline{\mathbb{Z}}_p$, an L -parameter (for G with A -coefficients) is the composite of an L -parameter $\text{Gal}_K \rightarrow {}^L G(A')$ for a finitely generated $\mathbb{Z}_p$ -algebra $A' \subset A$ and the inclusion ${}^L G(A') \hookrightarrow {}^L G(A)$.

Definition 2.4.2 (cf. [60, Definition 5.1.1]). Let $\Lambda \subset \overline{\mathbb{Z}}_p$ be a discrete valuation ring. For a connected quasi-split reductive group G over K which splits over a finite Galois extension E/K , an L -parameter $\rho: \text{Gal}_K \rightarrow \widehat{G}(\Lambda) \rtimes \text{Gal}(E/K)$ is said to be *crystalline* (resp. *potentially crystalline*) if for some closed embeddings of algebraic groups $\widehat{G} \rtimes \text{Gal}_{E/K} \hookrightarrow \text{GL}_d$ for some $d \geq 1$, the composite $\text{Gal}_K \rightarrow \widehat{G}(\Lambda) \rtimes \text{Gal}(E/K) \rightarrow \text{GL}_d(\Lambda)$ is crystalline (resp. *potentially crystalline*).

An L -parameter $\bar{\rho}: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{F}}_p)$ is said to be *semisimple* if $\bar{\rho}$ is semisimple in the sense of [60, Definition 3.2.5].

The following Langlands–Shelstad factorizations of semisimple L -parameters are shown in [60].

Theorem 2.4.3 ([60, Theorem 3.4.1]). *For a semisimple L -parameter $\bar{\rho}: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{F}}_p)$, there exists a tame K -torus S of G and a canonical L -homomorphism ${}^L j: {}^L S_{\overline{\mathbb{F}}_p} \rightarrow {}^L G_{\overline{\mathbb{F}}_p}$ such that $\bar{\rho}$ factors through ${}^L j$.*

For a semisimple L -parameter $\bar{\rho}: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{F}}_p)$, we fix a tame K -torus $S \subset G$ satisfying the condition of Theorem 2.4.3 in the assertion of Theorem 2.4.5.

For $A = \overline{\mathbb{Z}}_p$ or $\overline{\mathbb{F}}_p$, note that the module

$$H_{\text{cont}}^1(W_{E/K}, X^*(T_E) \otimes_{\mathbb{Z}} A^*) \quad (\text{resp. } H_{\text{cont}}^1(\text{Gal}_K, X^*(T_E) \otimes_{\mathbb{Z}} A^*))$$

is naturally bijective to the set of $\widehat{G}(A)$ -conjugacy classes of L -parameters

$$W_{E/K} \rightarrow \widehat{G}(A) \rtimes W_{E/K} \quad (\text{resp. } \text{Gal}_K \rightarrow {}^L G(A) \rtimes \text{Gal}_K).$$

We also use the p -adic local Langlands correspondence for tori with divisible coefficients.

Theorem 2.4.4 ([13]). *For a K -torus T with splitting field E , which is a finite Galois extension over K , and a divisible abelian topological group D , there exists a natural isomorphism*

$$H_{\text{cont}}^1(W_{E/K}, X^*(T_E) \otimes_{\mathbb{Z}} D) \xrightarrow{\sim} \text{Hom}_{\text{cont}}(T(K), D).$$

Write $T_E := T \times_K E$. Then the following natural maps

$$\begin{aligned} H_{\text{cont}}^1(\text{Gal}_K, X^*(T_E) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p^*) &\rightarrow H_{\text{cont}}^1(W_{E/K}, X^*(T_E) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p^*), \\ H_{\text{cont}}^1(\text{Gal}_K, X^*(T_E) \otimes_{\mathbb{Z}} \overline{\mathbb{Z}}_p^*) &\rightarrow H_{\text{cont}}^1(W_{E/K}, X^*(T_E) \otimes_{\mathbb{Z}} \overline{\mathbb{Z}}_p^*) \end{aligned}$$

are also isomorphisms.

PROOF. For the first isomorphism, see [13], [60, Theorem 4.3.1]. Let A be a finite discrete ring. Then the set of L -homomorphisms $W_{E/K} \rightarrow \widehat{T}(A) \rtimes W_{E/K}$ is naturally bijective to the set of L -homomorphisms $W_K \rightarrow \widehat{T}(A) \rtimes W_K$ since W_E acts trivially on $\widehat{T}$.

By [60, Lemma 3.1.2 (2), (4)], an L -homomorphism $W_K \rightarrow \widehat{T}(A) \rtimes W_K$ naturally corresponds to an L -parameter $\text{Gal}_K \rightarrow {}^L G(A)$. So we have an isomorphism

$$(2.4.1) \quad H_{\text{cont}}^1(\text{Gal}_K, X^*(T_E) \otimes_{\mathbb{Z}} A^*) \xrightarrow{\sim} H_{\text{cont}}^1(W_{E/K}, X^*(T_E) \otimes_{\mathbb{Z}} A^*)$$

By taking limits of the isomorphisms (2.4.1) for finite discrete coefficients A , we have an isomorphism

$$(2.4.2) \quad H_{\text{cont}}^1(\text{Gal}_K, X^*(T_E) \otimes_{\mathbb{Z}} \mathcal{O}_E^*) \xrightarrow{\sim} H_{\text{cont}}^1(W_{E/K}, X^*(T_E) \otimes_{\mathbb{Z}} \mathcal{O}_E^*)$$

for each finite extension $E/\mathbb{Q}_p$.

Finally, to obtain the last isomorphism of the theorem by taking colimits of (2.4.2), it suffices to show that the image of each continuous representation $\text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{Z}}_p)$ (resp. $W_{E/K} \rightarrow \widehat{G}(\overline{\mathbb{Z}}_p) \rtimes W_{E/K}$) is included in ${}^L G(\mathcal{O}_L)$ (resp. $\widehat{G}(\mathcal{O}_L) \rtimes W_{E/K}$) for a finite extension $\mathcal{O}_L/\mathbb{Z}_p$. The assertions are shown by the following Baire category argument.

For the continuous representation $\rho: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{Z}}_p)$, we have

$$\rho(\text{Gal}_K) = \bigcup_{\mathcal{O}_L/\mathbb{Z}_p} (\rho(\text{Gal}_K) \cap {}^L G(\mathcal{O}_L)),$$

where $\mathcal{O}_L/\mathbb{Z}_p$ runs over finite extensions. By Baire category theorem, there exists a finite extension $\mathcal{O}_L/\mathbb{Z}_p$ such that the closure $\overline{\rho(\text{Gal}_K) \cap {}^L G(\mathcal{O}_L)}$ contains an open subset of ${}^L G(\overline{\mathbb{Z}}_p)$, so that the index

$$|\rho(\text{Gal}_K): \rho(\text{Gal}_K) \cap \overline{\rho(\text{Gal}_K) \cap {}^L G(\mathcal{O}_L)}|$$

is finite. Thus there exists a finite extension $\mathcal{O}_{L'}/\mathcal{O}_L$ such that $\rho(\text{Gal}_K) \subset {}^L G(\mathcal{O}_{L'})$.

The case of $W_{E/K} \rightarrow \widehat{G}(\overline{\mathbb{Z}}_p) \rtimes W_{E/K}$ is similar by the above argument replacing Gal_K with the image of $I_K \hookrightarrow W_K \rightarrow W_{E/K}$. $\square$

For a K -torus S and an L -parameter $\bar{\rho}: \text{Gal}_K \rightarrow {}^L S(\overline{\mathbb{F}}_p)$, let $[\bar{\rho}]: \text{Gal}_K \rightarrow {}^L S(\overline{\mathbb{Z}}_p)$ be the L -parameter induced by the isomorphisms in Theorem 2.4.4 for $D = \overline{\mathbb{F}}_p^*$ and the Teichmüller character $\overline{\mathbb{F}}_p^* \hookrightarrow \overline{\mathbb{Z}}_p^* (\subset W(\overline{\mathbb{F}}_p)^*)$ from $\bar{\rho}$.

Theorem 2.4.5. *Let G be a connected quasi-split tame group over K and $\bar{\rho}: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{F}}_p)$ be a semisimple mod p L -parameter. Fix a tame K -torus S of G such that $\bar{\rho}$ factors through ${}^L S(\overline{\mathbb{F}}_p)$. Let $i: G^{\text{ab}} \rightarrow S$ be a right inverse of the composite $S \hookrightarrow G \twoheadrightarrow G^{\text{ab}}$.*

For $A = \overline{\mathbb{F}}_p^*$ or $\overline{\mathbb{Z}}_p^*$ and an L -parameter $\tau: \text{Gal}_K \rightarrow {}^L S(A)$, write τ^{ab} for the composite of τ and the map ${}^L S(A) \rightarrow {}^L G^{\text{ab}}(A)$ induced by the map i and we set $\overline{\psi} := \overline{\rho}^{\text{ab}}$.

Let $\psi: \text{Gal}_K \rightarrow {}^L G^{\text{ab}}(\overline{\mathbb{Z}}_p)$ be a potentially crystalline lift of $\overline{\psi}$. Then $\overline{\rho}$ admits a potentially crystalline lift $\rho: \text{Gal}_K \rightarrow {}^L G(\overline{\mathbb{Z}}_p)$ which has regular Hodge–Tate weights and satisfies $\rho^{\text{ab}} = \psi$.

PROOF. Let E be a finite Galois extension of K such that the image of $\overline{\rho}$ is contained in ${}^L G(E)$, S splits over E , $\psi|_{\text{Gal}_E}$ is crystalline, the character $\text{Gal}_K \rightarrow k^*$ associated with the mod ϖ_K fundamental character is trivial on Gal_E . We fix an embedding $E \hookrightarrow \overline{K}$ and a uniformizer $\varpi_E \in E$.

For a K -torus T which splits over E , let $\text{Nm}_{E/K}: T(E) \rightarrow T(K)$ be the norm map induced by $\text{Nm}_{E/K}: \text{Res}_{E/K} T_E \rightarrow T$, where $T_E := S \times_K E$. By Theorem 2.4.4 and [60, Proposition 4.3.3], we have the following commutative diagram

$$\begin{array}{ccccccc} H_K^1(G^{\text{ab}}) & \xrightarrow{\cong} & \text{Hom}_{\text{cont}}(G^{\text{ab}}(K), \overline{\mathbb{Z}}_p^*) & \longrightarrow & \text{Hom}_{\text{cont}}(S(K), \overline{\mathbb{Z}}_p^*) & \xleftarrow{\cong} & H_K^1(S) \\ \downarrow -|_{\text{Gal}_E} & & \downarrow -\circ \text{Nm}_{E/K} & & \downarrow -\circ \text{Nm}_{E/K} & & \downarrow -|_{\text{Gal}_E} \\ H_E^1(G^{\text{ab}}) & \xrightarrow{\cong} & \text{Hom}_{\text{cont}}(G^{\text{ab}}(E), \overline{\mathbb{Z}}_p^*) & \longrightarrow & \text{Hom}_{\text{cont}}(S(E), \overline{\mathbb{Z}}_p^*) & \xleftarrow{\cong} & H_E^1(S) \end{array}$$

where $H_K^1(-) := H_{\text{cont}}^1(\text{Gal}_K, X^*(- \times_K \overline{K}) \otimes_{\mathbb{Z}} \overline{\mathbb{Z}}_p^*)$.

By the existence of the right inverse of the map $S \rightarrow G^{\text{ab}}$, we have $S \cong G^{\text{ab}} \times S'$ for some K -torus S' . Let $\overline{\rho}_{S'}: \text{Gal}_K \rightarrow {}^L S'(\overline{\mathbb{F}}_p)$ be the ${}^L S'(\overline{\mathbb{F}}_p)$ -component of $\overline{\rho}$. Then $[\overline{\rho}_{S'}]: \text{Gal}_K \rightarrow {}^L S'(\overline{\mathbb{Z}}_p)$ is a potentially crystalline L -parameter with its reduction $\overline{\rho}_{S'}$ since its image is finite. Let

$$\rho' := (\psi, [\overline{\rho}_{S'}]) \in H_K^1(G^{\text{ab}}) \oplus H_K^1(S') \cong H_K^1(S).$$

It follows that ρ' is a potentially crystalline lift of $\overline{\rho}$ and we have $\rho'^{\text{ab}} = \psi$ by the construction.

Note that $\text{Gal}(E/K)$ acts on $X_*(\widehat{S}_E) = X^*(S_E)$. For a character $x \in X^*(S_E)$, let $\widetilde{\chi}_x: S(E) \rightarrow \overline{\mathbb{Z}}_p^*$ be the composite of a character $x(E): S(E) \rightarrow E^*$, the fundamental character $E^* \rightarrow \mathcal{O}_E^*$, and a fixed inclusion $\mathcal{O}_E^* \hookrightarrow \overline{\mathbb{Z}}_p^*$.

By Step 3 of the proof of Theorem 2.3.6, there exists a cocharacter $x \in X_*(\widehat{S}_E)$ which is regular, i.e., not killed by any roots of $\widehat{S}_E$, and the trivial abelianization.

It follows that for any $\sigma \in \text{Gal}(E/K)$, the cocharacter $\sigma \cdot x \in X_*(\widehat{S}_E)$ is also regular, and the L -parameter corresponding to $\chi_x := \widetilde{\chi}_x|_{S(K)}$ has regular Hodge–Tate weights by [60, Lemma 5.2.6].

Then $\rho' \rho''^M$, where $\rho'' \in H_K^1(S)$ is the potentially crystalline representation corresponding to χ_x , has regular Hodge–Tate weights for a sufficiently large integer M . The L -parameter $\rho' \rho''^M [\overline{\rho}]^{-M}$ is what we want. $\square$

CHAPTER 3

Zariski density of crystalline points on GSp_{2n} -valued local deformation rings

3.1. Introduction

In this chapter, we prove that the subspaces of crystalline points in universal framed deformation spaces of GSp_{2n} -valued mod p Galois representations are Zariski dense, under some conditions.

The content of the first half of this chapter is based on the paper [1]. In Section 3.3, we recall the notation of (φ, Γ) -modules over Robba rings and their fundamental properties. In Section 3.4, we extend the argument of (framed) (φ, Γ) -modules to G -valued ones, following Lee [56] for $G = \mathrm{GSp}_{2n}$ and Lin [59] in the general case. In Section 3.5, we introduce the notion of (φ, Γ) -modules with G -structures over general bases. In Section 3.6, we introduce trianguline deformation spaces for GSp_{2n} . In Section 3.7, we show the correspondence between triangulations of $D_{\mathrm{rig}}(\rho)$ and refinements of $D_{\mathrm{crys}}(\rho)$ for G -valued crystalline representations ρ . In Section 3.8, we explicitly calculate the tangent spaces of deformation functors of $D_{\mathrm{rig}}(\rho)$. In Section 3.9 we prove locally irreducibility and accumulation property of the trianguline deformation spaces for GSp_{2n} , which generalizes the result of Breuil–Hellmann–Schraen [18]. In Section 3.10, we prove the main theorem for irreducible components with a regular crystalline point.

The goal in the latter part of this chapter is to weaken the assumption for the theorem stated in the first part. In Section 3.11 we find sufficient conditions for the existence of crystalline lifts from partial lifts of GSp_{2n} -valued mod p Galois representations for $p = 2$. In Section 3.13, we calculate the dimension of the Emerton–Gee stacks for GSp_{2n} following the method of Lin [61]. In Section 3.14, we prove the main theorem of this chapter.

3.2. Notation

In the following two chapters, we adopt a notation of general symplectic groups GSp_{2n} as follows: Let GSp_{2n} be the split reductive group over $\mathbb{Z}$ defined by

$$\mathrm{GSp}_{2n}(R) = \{g \in \mathrm{GL}_{2n}(R) \mid {}^t g J g = \mathrm{sim}(g) J \text{ for some } \mathrm{sim}(g) \in R^*\}$$

for any commutative ring R , where J is defined by $\mathbf{x} = (x_{ij})_{i,j}$ in GL_{2n} such that $x_{ij} = \mathrm{sgn}(j - i)$ if $i + j = 2n + 1$ and otherwise $x_{ij} = 0$. There is the natural morphism $r_{\mathrm{std}}: \mathrm{GSp}_{2n} \rightarrow \mathrm{GL}_{2n}$ of the group scheme over $\mathbb{Z}$ given by the definition. We define the standard Borel subgroup as $B = \mathrm{GSp}_{2n} \cap B_{2n}$ and the standard maximal split torus $T = \mathrm{GSp}_{2n} \cap T_{2n}$ of GSp_{2n} , embedding GSp_{2n} into GL_{2n} by r_{std} . We fix the isomorphism $T \cong \mathbb{G}_m^{n+1}$ such that $(t_1, \dots, t_{n+1}) \in \mathbb{G}_m^{n+1} \cong T$ is sent to the element

$$(t_1, \dots, t_n, t_n^{-1} t_{n+1}, \dots, t_1^{-1} t_{n+1}) \in T_{2n}$$

by the embedding $T \rightarrow T_{2n}$.

3.3. (φ, Γ) -modules over Robba rings

We introduce the *sheaf of relative Robba rings* $\mathcal{R}_X$ for a rigid analytic spaces X over $\mathbb{Q}_p$. See also [53, §2, §6] and [46, §2]. Let X denote a rigid space over $\mathbb{Q}_p$. We write $\mathbb{U}_{K'}$ for the unit open disc over a p -adic field K' and $\mathbb{U}_{K',r} \subset \mathbb{U}_{K'}$ denotes the admissible open subspace of points of absolute value $\geq r$ for some $r \in p^{\mathbb{Q}} \cap [0, 1)$. For an r as above, we write $\mathcal{R}_X^r$ for the sheaf

$$X \supset U \mapsto \Gamma(U \times \mathbb{U}_{r,K'_0}, \mathcal{O}_{U \times \mathbb{U}_{r,K'_0}})$$

and we write $\mathcal{R}_X$ for the sheaf $\bigcup_{r>0} \mathcal{R}_X^r$. This sheaf of rings is endowed with a continuous $\mathcal{O}_X$ -linear ring homomorphism $\varphi: \mathcal{R}_X \rightarrow \mathcal{R}_X$ and a continuous $\mathcal{O}_X$ -linear action of the group Γ (See [43, §6.1]).

If $X = \mathrm{Sp}(A)$ for an affinoid algebra over $\mathbb{Q}_p$, we write $\mathcal{R}_A := \Gamma(\mathcal{R}_{\mathrm{Sp}(A)}, \mathcal{O}_{\mathcal{R}_{\mathrm{Sp}(A)}})$ for the Robba ring with coefficient A . The ring $\mathcal{R}_L$ is not noetherian, but a Bézout domain (cf. e.g. [8, Proposition 1.1.1]). Let $M \subset \mathcal{R}_L^n$ be a $\mathcal{R}_L$ -submodule, the module

$$M^{\mathrm{sat}} := \{m \in \mathcal{R}_L^n \mid \exists f \in \mathcal{R}_L \setminus \{0\}, fm \in M\} = (M \otimes_{\mathcal{R}_L} \mathrm{Frac}(\mathcal{R}_L)) \cap \mathcal{R}_L^n$$

is called the *saturation* of M and we say that M is *saturated* if $M^{\mathrm{sat}} = M$ or, which is the same, if $\mathcal{R}_L^n/M$ is torsion free (cf. [6, §2.2.3]).

Let X be a scheme or a rigid space over $\mathbb{Q}_p$. A K -filtered φ -module over X or a *filtered φ -module over X* if the p -adic field K is understood is a coherent $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$ -module $\mathcal{F}$ that is locally on X finite free together with an $\mathrm{id} \otimes \varphi$ -linear automorphism $\Phi: \mathcal{F} \rightarrow \mathcal{F}$ and a decreasing filtration on $\mathcal{F}_K := \mathcal{F} \otimes_{K_0} K$ by $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K$ -submodules that are locally on X direct summands as $\mathcal{O}_X$ -modules.

A (φ, Γ) -module over $\mathcal{R}_X$ consists of an $\mathcal{R}_X$ -module D that is Zariski locally on X finite free over $\mathcal{R}_X$ together with a φ -linear isomorphism $\phi: D \rightarrow D$ and a semi-linear Γ -action commuting with ϕ . We write $\Phi F_K(X)$ (resp. $\Phi F_{K,m}(X)$) for the category of K -filtered φ -modules (resp. K -filtered φ -modules of rank m) over X . We also write $\Phi \Gamma(\mathcal{R}_X)$ (resp. $\Phi \Gamma_m(\mathcal{R}_X)$) for the category of (φ, Γ) -modules (resp. (φ, Γ) -modules of rank m) over $\mathcal{R}_X$.

Let X be a rigid space defined over L . We associate to a K -filtered φ -module $\mathcal{F}$ over X of rank 1, a (φ, Γ) -module $\mathcal{R}_X(\mathcal{F})$ over X as follows. First we assume X is an affinoid $\mathrm{Sp}(A)$. By [53, Lemma 6.2.3], there exists a uniquely determined

$$(a, (k_\tau)_\tau) \in \Gamma(X, \mathcal{O}_X^*) \times \mathbb{Z}^\Sigma$$

such that

$$\mathcal{F} \cong \mathcal{F}(a; (k_\tau)_\tau) := \mathcal{F}_a \otimes_{K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_X} \mathcal{F}((k_\tau)_\tau)$$

such that $\mathcal{F}_a$ is a unique free rank one $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$ -module on which $\Phi^{[K_0: \mathbb{Q}_p]}$ act via the multiplication with $a \otimes 1$, the graded module of $\mathcal{F}_{a,K}$ is concentrated on degree 0, the underlying φ -module structure of $\mathcal{F}((k_\tau)_\tau)$ is trivial, and

$$(\mathrm{gr}^i(\mathcal{F}((k_\tau)_\tau)) \otimes_{\mathcal{O}_X \otimes_{\mathbb{Q}_p} K, \mathrm{id} \otimes \tau} \mathcal{O}_X \cong \begin{cases} 0 & (i \neq -k_\tau) \\ \mathcal{O}_X & (i = -k_\tau) \end{cases}$$

for all $\tau \in \Sigma = \mathrm{Hom}_{\mathbb{Q}_p}(K, L)$. If $\mathcal{F} = \mathcal{F}_a$ for some $a \in \Gamma(X, \mathcal{O}_X^*)$, we define

$$\mathcal{R}_X(\mathcal{F}_a) := \mathcal{F}_a \otimes_{\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0} \mathcal{R}_X$$

where φ acts diagonally and Γ acts on the second factor. Let $\delta: K^* \rightarrow \Gamma(X, \mathcal{O}_X^*)^*$ be a continuous character. We can write $\delta = \delta_1 \delta_2$ such that $\delta_1|_{\mathcal{O}_K^*} = 1$ and δ_2 extends to a

character of Gal_K . Then we set

$$\mathcal{R}_X(\delta) := \mathcal{R}_X(\mathcal{F}_{\delta_1(\varpi_K)}) \otimes_{\mathcal{R}_X} \mathcal{F}_{\text{rig}}(\delta_2).$$

The construction of $\mathcal{R}_X(\delta)$ is well-defined and the multiplication of characters $\delta: K^* \rightarrow \Gamma(X, \mathcal{O}_X)$ corresponds to the tensor product of (φ, Γ) -modules.

For an embedding $\tau \in \Sigma$, we also write $\tau: K^* \rightarrow L^*$ for the restriction of τ to K^* . It follows that $\mathcal{R}_L(\tau)$ is a (φ, Γ) -submodule of the trivial one $\mathcal{R}_L$ by [53, Example 6.2.6, (2)]. Since $\mathcal{R}_L$ is a Bézout domain, there exists an element $t_\tau \in \mathcal{R}_L$ generating the submodule $\mathcal{R}_L(\tau)$. Using the notations, we write

$$\mathcal{R}_X(\mathcal{F}((k_\tau)_\tau)) := \left(\prod_{\tau \in \Sigma} t_\tau^{k_\tau} \right) \cdot \mathcal{R}_X \subset \mathcal{R}_X \left[\frac{1}{t} \right]$$

with actions of φ and Γ inherited from $\mathcal{R}_X \left[\frac{1}{t} \right]$. Finally we define

$$\mathcal{R}_X(\mathcal{F}) := \mathcal{R}_X(\mathcal{F}_a) \otimes_{\mathcal{R}_X} \mathcal{R}_X(\mathcal{F}((k_\tau)_\tau)).$$

For a general X , the constructions $\delta \mapsto \mathcal{R}_X(\delta)$ and $\mathcal{F} \mapsto \mathcal{R}_X(\mathcal{F})$ are valid by gluing locally on X . In the case $X = \text{Sp}(L)$, we note that the construction is compatible with the $\delta \mapsto \mathcal{R}_X(\delta)$ via the functors D_{crys} and V_{crys} of Fontaine.

The functor $\text{Hom}_{\text{cont}}(\mathcal{O}_K^*, \mathbb{G}_m(-))$ from the category of rigid spaces over $\mathbb{Q}_p$ to the category of abelian groups is representable by the rigid generic fiber of $\text{Spf}(\mathbb{Z}_p[[\mathcal{O}_K^*]])$. Let $\mathcal{W}$ denote this rigid space, it is called the *weight space* of K . We define $\mathcal{T}$ as the functor that associate a rigid space X with the abelian group $\text{Hom}_{\text{cont}}(K^*, \mathbb{G}_m(X))$. It is also representable by a rigid space which we denote by $\mathcal{T}$. There is a natural restriction $\mathcal{T} \rightarrow \mathcal{W}$ and $\mathcal{T}$ is identified with $\mathbb{G}_m \times \mathcal{W}$ by $\delta \rightarrow (\delta(\varpi_K), \delta|_{\mathcal{O}_K^*})$. Note that the set of isomorphism classes of (φ, Γ) -modules over $\mathcal{R}_L$ is naturally bijective to $\mathcal{T}(L)$ by [53, Theorem 6.2.14, (i)]. For $(k_\tau)_\tau \in \prod_{\tau: K \hookrightarrow L} \mathbb{Z}$, let $\delta((k_\tau)_\tau): K^* \rightarrow L^*$ denote the character $z \mapsto \prod_{\tau} \tau(z)^{k_\tau}$.

Let Δ denote the maximal p -torsion subgroup of Γ . For a (φ, Γ) -module D , we define a Herr complex $C^\bullet(D)$ as

$$(D^\Delta \xrightarrow{(\varphi-1, \gamma'-1)} D^\Delta \oplus D^\Delta \xrightarrow{(1-\gamma') \oplus (\varphi-1)} D^\Delta)$$

where γ' is a topological generator of Γ/Δ and the first term is of degree 0. Write $H^i(D)$ for i -th cohomology of the complex $C^\bullet(D)$ for $i \in [0, 2]$. We also write ϵ for the composition $\epsilon \circ \text{rec}_K: K^* \rightarrow L^*$ of the p -adic cyclotomic character ϵ and rec_K , which is the same as $N_{K/\mathbb{Q}_p} \cdot |N_{K/\mathbb{Q}_p}| = \delta((1, \dots, 1)) \cdot |\delta((1, \dots, 1))|$.

Lemma 3.3.1. *For a $\delta \in \mathcal{T}(L)$, then*

- (1) $H^0(\mathcal{R}_L(\delta)) \neq 0$ if and only if $\delta = \delta((-k_\tau)_\tau)$ for some $(k_\tau)_\tau \in \prod_{\tau: K \hookrightarrow L} \mathbb{Z}_{\geq 0}$.
- (2) $H^2(\mathcal{R}_L(\delta)) \neq 0$ if and only if $\delta = \epsilon \cdot \delta((k_\tau)_\tau)$ for some $(k_\tau)_\tau \in \prod_{\tau: K \hookrightarrow L} \mathbb{Z}_{\geq 0}$.
- (3) $\dim_L(H^1(\mathcal{R}_L(\delta))) = [K: \mathbb{Q}_p]$ if and only if $\delta \neq \delta((-k_\tau)_\tau)$ and $\delta \neq \epsilon \cdot \delta((k_\tau)_\tau)$ for all $(k_\tau)_\tau \in \prod_{\tau} \mathbb{Z}_{\geq 0}$.

PROOF. See [53, Proposition 6.2.8]. □

Corollary 3.3.2. *Let $\delta: K^* \rightarrow L^*$ be a continuous character. Then a non-zero (φ, Γ) -submodule of $\mathcal{R}_L(\delta)$ is of the form $(\prod_{\tau \in \Sigma} t_\tau^{k_\tau}) \mathcal{R}_L(\delta)$ where $k_\tau \geq 0$.*

PROOF. See [53, Corollary 6.2.9]. □

We write $\mathcal{T}_{\mathrm{reg}} \subset \mathcal{T}$ for the set of *regular characters*, which is the set of $\delta \in \mathcal{T}$ such that $\delta \neq \delta((-k_\tau)_\tau)$ and $\delta \neq \epsilon \cdot \delta((k_\tau)_\tau)$ for all $(k_\tau)_\tau \in \prod_\tau \mathbb{Z}_{\geq 0}$.

For an integer $d > 0$, let $\mathcal{T}_{\mathrm{reg}}^d \subset \mathcal{T}^d$ denote the set of *regular parameters* $(\delta_1, \dots, \delta_d) \in \mathcal{T}^d$ such that $\delta_i/\delta_j \in \mathcal{T}_{\mathrm{reg}}$ for $i < j$. Note that $\mathcal{T}_{\mathrm{reg}}^d \neq (\mathcal{T}_{\mathrm{reg}})^d$.

A character $\delta \in \mathcal{T}(A)$ is *algebraic of weight $(k_\tau)_\tau$* if $\delta = \delta((k_\tau)_\tau)$ for $(k_\tau)_\tau \in \mathbb{Z}^\Sigma$. We say that $\delta \in \mathcal{T}(A)$ is *locally algebraic of weight $(k_\tau)_\tau$* for some $(k_\tau)_\tau \in \mathbb{Z}^\Sigma$ if $\delta \otimes \delta((-k_\tau)_\tau)$ becomes trivial after restricting to some open subgroup of $\mathcal{O}_K^*$.

A point $\underline{\delta} = (\delta_1, \dots, \delta_d) \in \mathcal{T}^d(A)$ is *algebraic* (resp. *locally algebraic*) if the i -th factor δ_i of $\underline{\delta}$ is algebraic (resp. locally algebraic) for $i \in [1, d]$. The (locally) algebraicity of elements of $\mathcal{W}^d(A)$ is similarly defined.

Lemma 3.3.3. *Let $\delta = (\delta_1, \dots, \delta_d) \in \mathcal{T}_{\mathrm{reg}}^d(X)$ for a rigid space X over L . If D is a successive extension of $\mathcal{R}_X(\delta_i)$ which are ordered by δ , then $H^1(D)$ is a locally free $\mathcal{O}_X$ -module of rank $d[K: \mathbb{Q}_p]$.*

PROOF. See [46, Proposition 2.3]. □

3.4. Framed (φ, Γ) -modules with G -structures

3.4.1. Framed (φ, Γ) -modules with G -structures over Robba rings. Let A be an affinoid algebra over $\mathbb{Q}_p$ and $\mathcal{R}_A$ be the Robba ring with coefficient A . We introduce frameable (φ, Γ) -modules with G -structures over $\mathcal{R}_A$ defined in [59, §9.2]. Let G be an algebraic group over K . We fix a faithful algebraic embedding $G \rightarrow \mathrm{GL}(V)$ over K and choose a trivialization $b_0: V \xrightarrow{\cong} \mathbb{G}_a^m$. Write $V_{(-)}$ for $V \otimes_K (-)$. Let T^{b_0} be the trivial G -torsor over $\mathcal{R}_A$ defined by

$$(3.4.1) \quad R \mapsto \{\text{trivializations } b: V_R \xrightarrow{\cong} R^{\oplus m} \mid b = g \circ b_0 \text{ for some } g \in G(R)\}.$$

Write $\varphi^*V_{\mathcal{R}_A}$ for $\mathcal{R}_A \otimes_{\varphi, \mathcal{R}_A} V_{\mathcal{R}_A}$, where φ acts on the first factor, and $\varphi^*b_0: \varphi^*V_{\mathcal{R}_A} \rightarrow \mathcal{R}_A^{\oplus m}$ for the trivialization which sends $1 \otimes b_0^{-1}(e_i)$ to e_i , where $\{e_1, \dots, e_m\}$ is the standard basis for $\mathbb{G}_a^m$. Similarly as above, write $T^{\varphi^*b_0}$ for trivial G -torsor over $\mathcal{R}_A$ defined by

$$(3.4.2) \quad R \mapsto \{\text{trivializations } b: \varphi^*V_R \xrightarrow{\cong} R^{\oplus m} \mid b = g \circ \varphi^*b_0 \text{ for some } g \in G(R)\}.$$

The cases of $T^{\gamma^*b_0}$ for $\gamma \in \Gamma$ are the same as above. Let $f: V_{\mathcal{R}_A} \rightarrow V_{\mathcal{R}_A}$ be a φ -semilinear morphism. Under the condition of $b_0 \circ (\varphi^*f) \in T^{\varphi^*b_0}$, the association $b \mapsto b \circ (\varphi^*f)$ defines a morphism $T^{b_0} \rightarrow T^{\varphi^*b_0}$ of G -torsor. Let f also denote the last G -torsor morphism and f_{b_0} denote the morphism $G_{\mathcal{R}_A} \rightarrow G_{\mathcal{R}_A}$ of trivial G -torsors induced by f and the fixed trivializations b_0, φ^*b_0 . Let I_m be the identity matrix $(e_1|e_2|\dots|e_m) \in \mathrm{GL}(\mathbb{G}_a^m)$ and we put $[f]_{b_0} := f_{b_0}(I_m)$. Then we have the following lemma.

Lemma 3.4.1. *For a φ -semilinear map $f: V_{\mathcal{R}_A} \rightarrow V_{\mathcal{R}_A}$, a γ -semilinear map $g: V_{\mathcal{R}_A} \rightarrow V_{\mathcal{R}_A}$ for $\gamma \in \Gamma$, and $h \in G(\mathcal{R}_A)$,*

$$\begin{aligned} [f \circ g]_{b_0} &= [f]_{b_0} \varphi([g]_{b_0}), \\ [g \circ f]_{b_0} &= [g]_{b_0} \gamma([f]_{b_0}), \\ [f]_{h \circ b_0} &= h[f]_{b_0} \varphi(h)^{-1}. \end{aligned}$$

PROOF. See [59, Lemma 9.2.1, Lemma 9.2.2]. We note that g (resp. f) does not need to be γ -semilinear (resp. ϕ -semilinear) in the first (resp. second) equation. □

Definition 3.4.2. We define *framed (φ, Γ) -modules with G -structures* over $\mathcal{R}_A$ as data $([\phi], [\gamma] \mid \gamma \in \Gamma)$ of coordinate matrices of $G(\mathcal{R}_A) \subset \mathrm{GL}_m(\mathcal{R}_A)$ such that $[\phi]\varphi([\gamma]) = [\gamma]\gamma([\phi])$ for any $\gamma \in \Gamma$ and

$$\Gamma \rightarrow G(\mathcal{R}_A), \quad \gamma \mapsto [\gamma]$$

are continuous homomorphism.

3.4.2. Framed (φ, Γ) -modules with G -structures over Fontaine rings. Let A be a $\mathbb{Z}_p$ -algebra. We introduce frameable (φ, Γ) -modules with G -structures over the Fontaine ring $\mathbb{A}_A$, which is defined in [59, §9.2]. We note that the argument of this section is essentially the same as in Section 3.4.1, in which the case over Robba rings. Let G be an algebraic group over K . We fix a faithful algebraic embedding $G \rightarrow \mathrm{GL}(V)$ over K and choose a trivialization $b_0: V \xrightarrow{\cong} \mathbb{G}_a^m$. Write $V_{(-)}$ for $V \otimes_K (-)$. Let T^{b_0} be the trivial G -torsor over $\mathbb{A}_A$ defined by

$$(3.4.3) \quad R \mapsto \{\text{trivializations } b: V_R \xrightarrow{\cong} R^{\oplus m} \mid b = g \circ b_0 \text{ for some } g \in G(R)\},$$

where R runs over $\mathbb{A}_A$ -algebras. Write $\varphi^*V_{\mathbb{A}_A}$ for $\mathbb{A}_A \otimes_{\varphi, \mathbb{A}_A} V_{\mathbb{A}_A}$, where φ acts on the first factor, and $\varphi^*b_0: \varphi^*V_{\mathbb{A}_A} \rightarrow \mathbb{A}_A^{\oplus m}$ for the trivialization which sends $1 \otimes b_0^{-1}(e_i)$ to e_i , where $\{e_1, \dots, e_m\}$ is the standard basis for $\mathbb{G}_a^m$. Similarly as above, write $T^{\varphi^*b_0}$ for trivial G -torsor over $\mathbb{A}_A$ defined by

$$(3.4.4) \quad R \mapsto \{\text{trivializations } b: \varphi^*V_R \xrightarrow{\cong} R^{\oplus m} \mid b = g \circ \varphi^*b_0 \text{ for some } g \in G(R)\}.$$

The cases of $T^{\gamma^*b_0}$ for $\gamma \in \Gamma$ are the same as above. Let $f: V_{\mathbb{A}_A} \rightarrow V_{\mathbb{A}_A}$ be a φ -semilinear morphism. Under the condition of $b_0 \circ (\varphi^*f) \in T^{\varphi^*b_0}$, the association $b \mapsto b \circ (\varphi^*f)$ defines a morphism $T^{b_0} \rightarrow T^{\varphi^*b_0}$ of G -torsor. Let f also denote the last G -torsor morphism and f_{b_0} denote the morphism $G_{\mathbb{A}_A} \rightarrow G_{\mathbb{A}_A}$ of trivial G -torsors induced by f and the fixed trivializations b_0, φ^*b_0 . Let I_m be the identity matrix $(e_1|e_2|\dots|e_m) \in \mathrm{GL}(\mathbb{G}_a^m)$ and we put $[f]_{b_0} := f_{b_0}(I_m)$. Then we have the following lemma.

Lemma 3.4.3 ([59, Lemma 9.2.1, Lemma 9.2.2]). *For a φ -semilinear map $f: V_{\mathbb{A}_A} \rightarrow V_{\mathbb{A}_A}$, a γ -semilinear map $g: V_{\mathbb{A}_A} \rightarrow V_{\mathbb{A}_A}$ for $\gamma \in \Gamma$ and $h \in G(\mathbb{A}_A)$,*

$$\begin{aligned} [f \circ g]_{b_0} &= [f]_{b_0} \varphi([g]_{b_0}), \\ [g \circ f]_{b_0} &= [g]_{b_0} \gamma([f]_{b_0}), \\ [f]_{h \circ b_0} &= h[f]_{b_0} \varphi(h)^{-1}. \end{aligned}$$

Let G be an algebraic group over K .

Definition 3.4.4. We define *framed (φ, Γ) -modules with G -structures over $\mathbb{A}_A$* or *framed (φ, Γ) -modules with G -structures* when A is obvious as data $([\phi], [\gamma] \mid \gamma \in \Gamma)$ of coordinate matrices of $G(\mathbb{A}_A) \subset \mathrm{GL}(\mathbb{A}_A^{\oplus m})$ such that $[\phi]\varphi([\gamma]) = [\gamma]\gamma([\phi])$ for any $\gamma \in \Gamma$ and

$$\Gamma \rightarrow G(\mathbb{A}_A), \quad \gamma \mapsto [\gamma]$$

are continuous homomorphism. A G -torsor with semilinear (φ, Γ) -structure which is isomorphic to a framed (φ, Γ) -modules with G -structures is called *frameable (φ, Γ) -modules with G -structures*.

By the associations (3.4.3), (3.4.4) and their subsequent argument, it is also possible that framed (φ, Γ) -modules with G -structures are constructed as geometric torsors naturally included in (φ, Γ) -modules with $\mathrm{GL}(V)$ -structures. [59, Lemma 9.2.3] shows that the two pairs $([\phi], [\gamma] \mid \gamma \in \Gamma)$ and $([\phi]', [\gamma]' \mid \gamma \in \Gamma)$ are equivalent (as frameable (φ, Γ) -modules

with G -structures) if and only if there exists $h \in G(\mathbb{A}_A)$ such that $[\phi] = h[\phi]'\varphi(h)^{-1}$ and $[\gamma] = h[\gamma]'\gamma(h)^{-1}$ for any $\gamma \in \Gamma$. Let $\Phi\Gamma_G(\mathbb{A}_A)$ denote the category of frameable (φ, Γ) -modules with G -structures over $\mathbb{A}_A$.

3.5. (φ, Γ) -modules with G -structures

In this section, we introduce the Tannakian formalism to define (φ, Γ) -modules with G -structures over general bases.

3.5.1. (φ, Γ) -modules with G -structures over general bases. Before defining general (φ, Γ) -modules with G -structures, we review Tannakian formalism over schemes or rigid spaces.

Lemma 3.5.1. *For a scheme or a rigid space X over K , the category of G -torsors over X is naturally equivalent to the category of exact tensor functors $\mathrm{Rep}_K(G) \rightarrow \mathrm{Bun}(X)$.*

PROOF. The case of schemes is [57, Theorem 2.5.2]. For rigid spaces, the proof of [85, Theorem 19.5.2] is applied in the case that $\mathrm{Rep}_K(\mathcal{G}) = \mathrm{Rep}_K(G)$ is defined over K . Since the set of quasi-compact admissible opens of X is naturally bijective to that of quasi-compact opens of the associated adic space X^{ad} , the Lemma follows. $\square$

Let $\Phi\Gamma(\mathcal{R}_A)$ (resp. the category of $\Phi\Gamma_m(\mathcal{R}_A)$) denote the category of finite projective (φ, Γ) -modules (resp. finite projective (φ, Γ) -modules of rank m) over $\mathcal{R}_A$.

Definition 3.5.2. Let X be a rigid space. We define a (φ, Γ) -module with G -structure over $\mathcal{R}_X$ as a exact tensor functor $\mathrm{Rep}_K(G) \rightarrow \Phi\Gamma(\mathcal{R}_X)$ whose underlying fiber functor $\mathrm{Rep}_K(G) \rightarrow \mathrm{Bun}(\mathcal{R}_X)$ is associated with a geometric étale G -torsor which is étale locally on X trivial by the equivalence of Lemma 3.5.1. For an affinoid algebra A over K , (φ, Γ) -modules with G -structures over $\mathcal{R}_A$ are similarly defined.

Morphisms between (φ, Γ) -modules with G -structures over a fixed $\mathcal{R}_X$ or $\mathcal{R}_A$ are natural transformations of the exact tensor functors.

We note that (φ, Γ) -modules with G -structures over $\mathcal{R}_A$ can be seen as geometric torsors over $\mathcal{R}_A$ with semilinear (φ, Γ) -actions. A (φ, Γ) -module with G -structure which is isomorphic to a framed (φ, Γ) -modules with G -structures is called a *frameable (φ, Γ) -modules with G -structures*. Let $\Phi\Gamma_G(\mathcal{R}_A)$ denote the category of frameable (φ, Γ) -modules with G -structures over $\mathcal{R}_A$, which is a full subcategory of the category of (φ, Γ) -modules with G -structures over $\mathcal{R}_A$.

[59, Lemma 9.2.3] shows that the two pairs $([\phi], [\gamma] \mid \gamma \in \Gamma)$ is isomorphic to $([\phi]', [\gamma]' \mid \gamma \in \Gamma)$ (as frameable (φ, Γ) -modules with G -structures) if and only if there exists $h \in G(\mathcal{R}_A)$ such that $[\phi] = h[\phi]'\varphi(h)^{-1}$ and $[\gamma] = h[\gamma]'\gamma(h)^{-1}$ for any $\gamma \in \Gamma$.

Let P be a parabolic subgroup of G . There exists the Levi decomposition $P = U \rtimes L$ where U is the unipotent radical and L is the Levi subgroup. Write $\mathrm{ad}: L \rightarrow \mathrm{GL}(U)$ for the adjoint action. We assume that U is abelian and fix an isomorphism $b: U \cong \mathbb{G}_a^m$. When $([\phi], [\gamma] \mid \gamma \in \Gamma) \in \Phi\Gamma_P(\mathcal{R})$, write $[\phi] = [\phi_U][\phi_L]$ and $[\gamma] = [\gamma_U][\gamma_L]$ for the factors of Levi decompositions. By abuse of notation, write $\mathrm{ad}_{[-]}$ for the matrix $b \circ \mathrm{ad}_{[-]} \circ b^{-1} \in \mathrm{GL}(\mathbb{G}_a^m)$.

Lemma 3.5.3. *We have*

$$(\mathrm{ad}_{[\phi_L]}, \mathrm{ad}_{[\gamma_L]} \mid \gamma \in \Gamma) \in \mathrm{ob}(\Phi\Gamma_{\mathrm{GL}(\mathbb{G}_a^m)}(\mathcal{R}_A)).$$

PROOF. It follows $\mathrm{ad}_{[\phi]} \varphi(\mathrm{ad}_{[\gamma]}) = \mathrm{ad}_{[\gamma]} \gamma(\mathrm{ad}_{[\phi]})$ by [59, Lemma 9.4.2]. Since the conjugate actions are continuous homomorphisms, $\mathrm{ad}_{[\gamma]}$ for $\gamma \in \Gamma$ induce a continuous homomorphism $\Gamma \rightarrow \mathrm{GL}(\mathcal{R}_A^{\oplus m})$. $\square$

For a framed (φ, Γ) -module with GL_m -structure $([\phi], [\gamma] \mid \gamma \in \Gamma)$, we define the *framed Herr complex* $C^\bullet(\mathrm{ad}_{[\phi_L]}, \mathrm{ad}_{[\gamma_L]} \mid \gamma \in \Gamma)$ as

$$(\mathcal{R}_A^{\oplus m})^\Delta \xrightarrow{(\varphi - [\phi]^{-1}, \gamma' - [\gamma']^{-1})} (\mathcal{R}_A^{\oplus m})^\Delta \oplus (\mathcal{R}_A^{\oplus m})^\Delta \xrightarrow{(\gamma' - \phi([\gamma'])^{-1}, \gamma'([\phi])^{-1} - \varphi)} (\mathcal{R}_A^{\oplus m})^\Delta$$

where γ' is a topological generator in Γ/Δ . This complex is naturally isomorphic to the Herr complex $C^\bullet(D)$ where D is a frameable (φ, Γ) -module with G -structure that corresponds to $([\phi], [\gamma] \mid \gamma \in \Gamma)$.

Lemma 3.5.4. *For a framed (φ, Γ) -module with L -structure $([\phi_L], [\gamma_L] \mid \gamma \in \Gamma)$, the A -module of the extension classes of $([\phi_L], [\gamma_L] \mid \gamma \in \Gamma)$ to a framed (φ, Γ) -module with P -structure is naturally isomorphic to the module $H^1(C^\bullet(\mathrm{ad}_{[\phi_L]}, \mathrm{ad}_{[\gamma_L]} \mid \gamma \in \Gamma))$.*

PROOF. We set

$$H^1 := H^1(C^\bullet(\mathrm{ad}_{[\phi_L]}, \mathrm{ad}_{[\gamma_L]} \mid \gamma \in \Gamma)),$$

$$Z^1 := \ker(C^1(\mathrm{ad}_{[\phi_L]}, \mathrm{ad}_{[\gamma_L]} \mid \gamma \in \Gamma)) \rightarrow C^2(\mathrm{ad}_{[\phi_L]}, \mathrm{ad}_{[\gamma_L]} \mid \gamma \in \Gamma)).$$

There is an injective homomorphism $L \rightarrow P$, which induces a framed (φ, Γ) -module with P -structure $([\phi^0], [\gamma^0] \mid \gamma \in \Gamma)$ from $([\phi_L], [\gamma_L] \mid \gamma \in \Gamma)$.

Each element $x \in Z^1$ naturally gives a (φ, Γ) -module with P -structure $([\phi_x], [\gamma_x] \mid \gamma \in \Gamma)$. We construct a natural map

$$Z^1 \rightarrow M, \quad x \mapsto (\text{The isomorphism class of } ([\phi^0][\phi_x], [\gamma^0][\gamma_x] \mid \gamma \in \Gamma)),$$

We can easily check that the above map induces a bijection $H^1 \cong M$. $\square$

3.5.2. Embeddings of regular type and spaces of Borel extensions. For the natural immersion $T \subset G$, fix an isomorphism $T \cong \mathbb{G}_m^d$. We write $\mathcal{T}_T$ for the parameter space $\mathcal{T}^d = \mathrm{Hom}_{\mathrm{cont}}(K^*, \mathbb{G}_m^d(-)) \cong \mathrm{Hom}_{\mathrm{cont}}(K^*, T(-))$. Similarly define $\mathcal{W}_T$ and the restriction $\mathcal{T}_T \rightarrow \mathcal{W}_T$ as in Section 3.3.

We say that an element $\underline{\delta}$ in $\mathcal{T}_T(A)$ or $\mathcal{W}_T(A)$ for $A \in \mathcal{C}_L$ is *algebraic* (resp. *locally algebraic*) if the corresponding object in $\mathcal{T}^d$ is algebraic (resp. locally algebraic).

Definition 3.5.5. We define *embeddings of (G, B, T) of regular type* as faithful algebraic embedding $i: G \rightarrow \mathrm{GL}_m$ such that the following conditions are satisfied:

- (1) i induces a morphism $(G, B, T) \rightarrow (\mathrm{GL}_m, B_m, T_m)$ and $G \cap B_m = B$.
- (2) The inverse image of $\mathcal{T}_{\mathrm{reg}}^m$ under the composite $\mathcal{T}_T \rightarrow \mathcal{T}_{T_m} \cong \mathcal{T}^m$, where the last isomorphism is induced by the fixed $T \cong \mathbb{G}_m^d$, is Zariski open and Zariski dense in $\mathcal{T}_T$.
- (3) There are lift homomorphisms $\iota: W(G, T) \rightarrow N_G(T)$ and $\iota': W(\mathrm{GL}_m, T_m) \rightarrow N_{\mathrm{GL}_m}(T_m)$ such that $(i \circ \iota)(W(G, T)) \subset \iota'(W(\mathrm{GL}_m, T_m))$. In this case, the embedding i induces a natural inclusion of Weyl groups $W(G, T) \hookrightarrow W(\mathrm{GL}_m, T_m)$ and we see $W(G, T)$ as the subgroup of $W(\mathrm{GL}_m, T_m)$.
- (4) For the sets $\Delta(B) := \Delta(G, B, T) = \{\alpha_1, \dots, \alpha_l\}$, $\Delta(B_m) := \Delta(\mathrm{GL}_m, B_m, T_m)$ of simple roots of (G, B, T) , $(\mathrm{GL}_m, B_m, T_m)$, there is a partition $\Delta(B_m) = \Delta_1 \sqcup \dots \sqcup \Delta_l$ such that the subgroup $W_{\Delta_i} \subset W(\mathrm{GL}_m, T_m)$ associated with $\Delta_i \subset \Delta(B_m)$ contains the nontrivial symmetry s_i in $W_{\{\alpha_i\}} \subset W(G, T)$.
- (5) There exists a sequence $\emptyset = I_0 \subset I_1 \subset \dots \subset I_s = \Delta(B)$ of subsets of $\Delta(B)$ such that the unipotent radical $\tilde{U}'_{i+1}$ of $\tilde{L}_{i+1} \cap \tilde{P}_i = \tilde{L}_i(\tilde{L}_{i+1} \cap \tilde{U}_i)$ is abelian for $i \in \{0, \dots, s-1\}$, where $\tilde{P}_i$ is the parabolic subgroup of GL_m associated with the subset $I_i \subset \Delta(B) \subset \Delta(B_m)$ and $\tilde{P}_i = \tilde{U}_i \rtimes \tilde{L}_i$ is the Levi decomposition.

A split reductive group (G, B, T) is called *of regular type* if it has an embedding of regular type.

Example 3.5.6. Let us consider the case $G = \mathrm{GSp}_{2n}$, B is the standard Borel subgroup and T is the maximal split torus. Recall that we have fixed an isomorphism $T \cong \mathbb{G}_m^{n+1}$ in Section 3.2. Then the standard representation $r_{\mathrm{std}}: \mathrm{GSp}_{2n} \rightarrow \mathrm{GL}_{2n}$ is regular type. The lifts $W(\mathrm{GSp}_{2n}, T) \rightarrow N_{\mathrm{GSp}_{2n}}(T)$ and $W(\mathrm{GL}_m, T_m) \rightarrow N_{\mathrm{GL}_m}(T_m)$ which send each element to a symplectic permutation matrix induces the injection of the Weyl groups $W(\mathrm{GSp}_{2n}, T) \hookrightarrow W(\mathrm{GL}_{2n}, T_{2n}) \cong \mathfrak{S}_{2n}$. As the subgroup of $\mathfrak{S}_{2n}$, $W(\mathrm{GSp}_{2n}, T)$ is generated by

$$\{(1, 2)(2n - 1, 2n), (2, 3)(2n - 2, 2n - 1), \dots, (n - 1, n)(n + 1, n + 2), (n, n + 1)\}.$$

Lemma 3.5.7. *Under the situation of Definition 3.5.5, we have $\tilde{P}_i \cap G = P_i$.*

PROOF. Since there exists compatible lift homomorphisms $\iota: W(G, T) \rightarrow N_G(T)$ and $\iota': W(\mathrm{GL}_m, T_m) \rightarrow N_{\mathrm{GL}_m}(T_m)$ by the assumption, the lemma follows from the Bruhat decomposition. $\square$

The following lemma is often used in the paper.

Lemma 3.5.8. *Let R be a commutative ring and G be a reductive group scheme over R . For any R -Levi subgroup of a parabolic R -subgroup of G , the natural map*

$$H^1(R, L) \rightarrow H^1(R, P)$$

is bijective.

PROOF. See [42, XXVI, Corollaire 2.3]. $\square$

Lemma 3.5.9. *Let us naturally see $\mathcal{R}_X$ as the scheme $\mathcal{R}_X \rightarrow X$. Let P be a parabolic subgroup and $P = U \rtimes L$ be the Levi decomposition. Then any extension of a geometric étale L -torsor over $\mathcal{R}_X$ which is Zariski (resp. étale) locally on X trivial to a geometric étale P -torsor over $\mathcal{R}_X$ is Zariski (resp. étale) locally on X trivial.*

PROOF. This follows from Lemma 3.5.8. $\square$

Let (G, B, T) be a split reductive group of regular type and fix an embedding $(G, B, T) \rightarrow (\mathrm{GL}_m, B_m, T_m)$ of regular type. We write $\mathcal{T}_{T, \mathrm{reg}}$ for the inverse image of $\mathcal{T}_{\mathrm{reg}}^m$ under the composite $\mathcal{T}_T \rightarrow \mathcal{T}_{T_m} \cong \mathcal{T}^m$. We set p_i to be the composite of the fixed isomorphism $T \cong \mathbb{G}_m^d$ and the i -th projection $\mathbb{G}_m^d \rightarrow \mathbb{G}_m$.

Let D be a (φ, Γ) -module with G -structure over $\mathcal{R}_X$. Let $\mathcal{P}: \mathrm{Rep}_K(B) \rightarrow \Phi\Gamma(\mathcal{R}_X)$ be a (φ, Γ) -module with B -structure. Write $\mathcal{P} \times^B T \times^{p_i} \mathbb{G}_m: \mathrm{Rep}_K(\mathbb{G}_m) \rightarrow \Phi\Gamma(\mathcal{R}_X)$ be the composite of $\mathcal{P}$ and the restriction $\mathrm{Rep}_K(\mathbb{G}_m) \rightarrow \mathrm{Rep}_K(B)$ induced by $B \rightarrow T \cong \mathbb{G}_m^d \xrightarrow{p_i} \mathbb{G}_m$, which can be naturally associated with a classical rank one (φ, Γ) -module $\mathcal{R}_X(\delta_i)$ for some $\delta_i \in \mathcal{T}(X)$. If the composite of $\mathcal{P}$ with the restriction $\mathrm{Rep}_K(G) \rightarrow \mathrm{Rep}_K(B)$ is identical to D , we call $\mathcal{P}$ a *triangulation (with the parameter $\delta = (\delta_1, \dots, \delta_d)$) of D* .

If $G = \mathrm{GSp}_{2n}$, triangulations of an object in $\Phi\Gamma_{\mathrm{GSp}_{2n}}(\mathcal{R}_A)$ for an affinoid $\mathbb{Q}_p$ -algebra A can be translated into those of the associated rank $2n$ (φ, Γ) -module which are preserved by the symplectic form.

A (φ, Γ) -module with G -structure D which admits such a triangulation $\mathcal{P}$ is said to be *trianguline* (with the parameter $\underline{\delta} = (\delta_1, \dots, \delta_d)$). When $G = \mathrm{GL}_m$ for $m \in \mathbb{Z}_{\geq 1}$, the notations are compatible with those of (φ, Γ) -modules of rank m . We consider the functor $\mathcal{S}$ that assigns to a rigid space X over L the isomorphism classes of quadruples

$(D, \mathcal{P}, \underline{\delta}, \nu = (\nu_1, \dots, \nu_d))$, where D is a (φ, Γ) -module with G -structure over $\mathcal{R}_X$, $\mathcal{P}$ is a triangulation of D , $\underline{\delta} \in \mathcal{T}_{T, \text{reg}}$ and

$$\nu_i: \mathcal{P} \times^B T \times^{p_i} \mathbb{G}_m \xrightarrow{\cong} \mathcal{R}_X(\delta_i)$$

is a trivialization.

Proposition 3.5.10. *For a split reductive group (G, B, T) and a fixed embedding $G \rightarrow \text{GL}_m$ of regular type, the functor $\mathcal{S}$ is representable by a rigid space and the map $\mathcal{S} \rightarrow \mathcal{T}_{T, \text{reg}}$ is smooth of relative dimension $[K: \mathbb{Q}_p](\dim B - \dim T)$.*

PROOF. Since the embedding $G \rightarrow \text{GL}_m$ is of regular type, there is a natural inclusion $\Delta(B) \subset \Delta(B_m)$ of the sets of simple roots and a sequence $\emptyset = I_0 \subset I_1 \subset \dots \subset I_s = \Delta(B)$ such that the unipotent radical $\tilde{U}'_{i+1}$ of $\tilde{L}_{i+1} \cap \tilde{P}_i = \tilde{L}_i(\tilde{L}_{i+1} \cap \tilde{U}_i)$ is abelian for $i \in [0, s-1]$. It also induce an injection of Weyl groups $W(G, T) \rightarrow W(\text{GL}_m, T_m)$. Write P_i for the parabolic subgroup of G associated with the subset $I_i \subset \Delta(B)$. We have $\tilde{P}_i \cap G = P_i$ by Lemma 3.5.7. Thus, the unipotent radical U'_{i+1} of $L_{i+1} \cap P_i = L_i(L_{i+1} \cap U_i)$ is also abelian and we have a natural immersion $U'_{i+1} \hookrightarrow \tilde{U}'_{i+1}$. Then we will successively consider the space of extensions of (φ, Γ) -modules with $L_i \cap B$ -structures to that of (φ, Γ) -modules with $L_{i+1} \cap B$ -structures for each $i \in [0, s-1]$.

For each $i \in [0, s-1]$, we write $\mathcal{S}_{i+1}$ for isomorphism classes of quadruples $(D, \mathcal{P}, \underline{\delta}, \nu = (\nu_1, \dots, \nu_d))$, where D is a $L_{i+1} \cap B$ -torsor over $\mathcal{R}_X$, $\mathcal{P}$ is a triangulation of D , $\underline{\delta} \in \mathcal{T}_{T, \text{reg}}$ and

$$\nu_i: \mathcal{P} \times^B T \times^{p_i} \mathbb{G}_m \xrightarrow{\cong} \mathcal{R}_X(\delta_i)$$

is a trivialization. Note that $\mathcal{S}_s = \mathcal{S}$.

We assume that $\mathcal{S}_i$ is representable by a rigid space X_i for some $i \in [1, s-1]$. Let $\widetilde{\text{ad}}'_{i+1}: \tilde{L}'_{i+1} \rightarrow \text{GL}(\tilde{U}'_{i+1})$ (resp. $\text{ad}'_{i+1}: L'_{i+1} \rightarrow \text{GL}(U'_{i+1})$) be the adjoint map where $\tilde{L}'_{i+1}$ (resp. L'_{i+1}) is the Levi subgroup of $\tilde{L}_{i+1} \cap \tilde{P}_i$ (resp. $L_{i+1} \cap P_i$). We also write $\tilde{D}_{i+1}$ (resp. D_{i+1}) for the induced (φ, Γ) -module with $\text{GL}(\tilde{U}'_{i+1})$ -structure (resp. (φ, Γ) -module with $\text{GL}(U'_{i+1})$ -structure) over $\mathcal{R}_{X_i}$ from the universal (φ, Γ) -module with $L_i \cap P_{i-1}$ -structure Y_i by $\widetilde{\text{ad}}'_{i+1}$ (resp. ad'_{i+1}). Note that $\tilde{U}'_{i+1}$ and U'_{i+1} are isomorphic to direct sums of finite copies of $\mathbb{G}_a$. Let r_i be an integer such that $U'_{i+1} \cong \mathbb{G}_a^{r_i}$.

The (φ, Γ) -module $\tilde{D}_{i+1}$ admits a triangulation, which is defined over K , with parameter in $\mathcal{T}_{\text{reg}}^d$. So considering the intersection of D_{i+1} and the triangulation of $\tilde{D}_{i+1}$, here D_{i+1} also admits a triangulation with parameter in $\mathcal{T}_{\text{reg}}^d$. It follows that the extension classes of the universal (φ, Γ) -module with $L_i \cap B$ -structure Y_i to (φ, Γ) -modules with $L_{i+1} \cap B$ -structures are classified by the locally free sheaf $H^1(D_{i+1})$ of rank $r_i[K: \mathbb{Q}_p]$ on X_i by Lemma 3.3.3, Lemma 3.5.4, and Lemma 3.5.9. According to the Tate-duality ([53, Theorem 4.4.5]), we find that

$$\mathcal{M}_{X_i} := \mathcal{E}xt_{\mathcal{R}_{X_i}}^1(\mathcal{R}_{X_i}, D_{i+1})$$

is also locally free $\mathcal{O}_{X_i}$ -module of rank $r_i[K: \mathbb{Q}_p]$. We may put

$$\mathcal{S}_{i+1} := \underline{\text{Spec}}_{X_i}(\text{Sym}^\bullet \mathcal{M}_{X_i}^\vee),$$

where Spec is the relative spectrum in the sense of [29, 2.2]. It follows that the morphism $\mathcal{S}^\square \rightarrow \mathcal{T}_{T, \text{reg}}$ is smooth of relative dimension $[K: \mathbb{Q}_p](\dim B - \dim T)$ from the construction. $\square$

3.5.3. (φ, Γ) -modules with GSp_{2n} -structures. In the last part of this section, we show some properties of (φ, Γ) -modules with G -structures in the case of $G = \mathrm{GSp}_{2n}$.

We will associate representations $\rho: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(A)$ with framed (φ, Γ) -module with G -structure over $\mathcal{R}_A$.

Lemma 3.5.11. *There is an equivalence of groupoids*

$$\{\rho: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(A)\} \xrightarrow{\sim} \left\{ (\rho', \chi, \alpha) \left| \begin{array}{l} \rho': \mathrm{Gal}_K \rightarrow \mathrm{GL}_{2n}(A), \chi: \mathrm{Gal}_K \rightarrow \mathrm{GL}_1(A), \\ \alpha: \rho' \cong (\rho')^\vee \otimes \chi, (\alpha^\vee \otimes \chi)^{-1} \circ \alpha = -1 \end{array} \right. \right\}.$$

PROOF. See [56, Lemma 4.1.2]. $\square$

Similarly, we obtain the equivalence for framed (φ, Γ) -modules with G -structures.

Lemma 3.5.12. *There is an equivalence of groupoids*

$$\Phi\Gamma_{\mathrm{GSp}_{2n}}(\mathcal{R}_A) \xrightarrow{\sim} \left\{ (\tilde{D}, \delta, \alpha) \left| \begin{array}{l} \tilde{D} \in \mathrm{ob}(\Phi\Gamma_{2n}(\mathcal{R}_A)), \delta \in \mathrm{ob}(\Phi\Gamma_1(\mathcal{R}_A)), \\ \alpha: \tilde{D} \cong (\tilde{D})^\vee \otimes \delta, (\alpha^\vee \otimes \delta)^{-1} \circ \alpha = -1 \end{array} \right. \right\}.$$

Then we construct a symplectic version of the functor D_{rig} (cf. [53, Theorem 2.2.17]) as the composition of the following functors:

$$(\rho: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(A)) \mapsto (\rho', \chi, \alpha) \xrightarrow{D_{\mathrm{rig}}} (D_{\mathrm{rig}}(\rho'), D_{\mathrm{rig}}(\chi), D_{\mathrm{rig}}(\alpha)),$$

where the first map is from Lemma 3.5.11. By the equivalence of groupoids in Lemma 3.5.12, $(D_{\mathrm{rig}}(\rho'), D_{\mathrm{rig}}(\chi), D_{\mathrm{rig}}(\alpha))$ determines functorially the equivalence class of frameable (φ, Γ) -modules with GSp_{2n} -structures over $\mathcal{R}_A$. Fix a quasi-inverse of the equivalence in Lemma 3.5.12 and let $D_{\mathrm{rig}}(\rho)$ denote the image of $(D_{\mathrm{rig}}(\rho'), D_{\mathrm{rig}}(\chi), D_{\mathrm{rig}}(\alpha))$ by the functor.

Lemma 3.5.13. *There exists a fully faithful and exact functor*

$$\{\rho: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(A)\} \rightarrow \Phi\Gamma_{\mathrm{GSp}_{2n}}(\mathcal{R}_A) \quad (\rho: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(A)) \mapsto D_{\mathrm{rig}}(\rho)$$

from the category of continuous homomorphisms $\rho: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(A)$ to the category of frameable (φ, Γ) -modules with GSp_{2n} -structures over $\mathcal{R}_A$. It also commutes with base change in A .

PROOF. Existence of a functor follows by the above argument. It follows that the functor is fully faithful, exact and commuting with base changes by [53, Theorem 2.2.17]. $\square$

Lemma 3.5.14. *For an object $D \in \mathrm{ob}(\Phi\Gamma_{\mathrm{GSp}_{2n}}(\mathcal{R}_A))$, which is sent to $(\tilde{D}, \delta, \alpha)$ by the equivalence of groupoids in Lemma 3.5.12, there is a natural bijection from the set of triangulations of D to the set of triangulations of $\tilde{D}$*

$$\mathcal{P} := (\tilde{D}_0 := 0 \subset \tilde{D}_1 \subset \cdots \subset \tilde{D}_{2n} =: \tilde{D})$$

such that α sends $\mathcal{P}$ to the triangulation of $\tilde{D}^\vee$

$$\mathcal{P}^\vee := (\tilde{D}_{2n}^\vee = 0 \subset \tilde{D}_{2n-1}^\vee \subset \cdots \subset \tilde{D}_0^\vee = \tilde{D}).$$

PROOF. This follows from the construction of the equivalence of groupoids in Lemma 3.5.12. $\square$

3.6. Trianguline deformation spaces for GSp_{2n}

In this section, we introduce trianguline deformation spaces for GSp_{2n} and show their fundamental properties.

Let $T \subset \mathrm{GSp}_{2n}$ denote the standard maximal split torus. We fix a representation $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(k_L)$. Then there exists the universal framed deformation ring $R_{\bar{\rho}}$ which pro-represents the functor from $A \in \mathcal{C}_{\mathcal{O}_L}$ to the set of continuous representations $\rho_A: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(A)$ such that $\bar{\rho}$ is the composition

$$\mathrm{Gal}_K \xrightarrow{\rho_A} \mathrm{GSp}_{2n}(A) \xrightarrow{\phi_A} \mathrm{GSp}_{2n}(k_L)$$

for example by [4, Theorem 1.3.6]. We call the local ring $R_{\bar{\rho}}$ the *universal framed deformation ring of $\bar{\rho}$* . We write $\mathfrak{X}_{\bar{\rho}}$ for the rigid analytic space over L , which is the rigid generic fiber of the universal framed deformation space $\mathrm{Spf}(R_{\bar{\rho}})$ of $\bar{\rho}$. We define $U_{\mathrm{tri}}(\bar{\rho})$ as the set of points $(x, \delta) \in \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_{T, \mathrm{reg}}$ such that for the representation $\rho_x: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(L')$, which is associated with a point $x \in \mathfrak{X}_{\bar{\rho}}$, its framed (φ, Γ) -module with GSp_{2n} -structure $D_{\mathrm{rig}}(\rho_x)$ is trianguline with the parameter $\delta \in \mathcal{T}_{T, \mathrm{reg}}$.

Definition 3.6.1. Let $X_{\mathrm{tri}}(\bar{\rho})$ denote the Zariski closure of $U_{\mathrm{tri}}(\bar{\rho})$ in $\mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_T$. We add the structure of the reduced closed analytic subvariety ([14, 9.5.3 Proposition 4]) of $\mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_T$ to $X_{\mathrm{tri}}(\bar{\rho})$. We call the rigid space $X_{\mathrm{tri}}(\bar{\rho})$ the *trianguline deformation space of $\bar{\rho}$* .

We write ω' for the morphism $X_{\mathrm{tri}}(\bar{\rho}) \rightarrow \mathcal{T}_T$, which is the composition of the inclusion $X_{\mathrm{tri}}(\bar{\rho}) \hookrightarrow \mathfrak{X}_{\bar{\rho}} \times \mathcal{T}_T$ and the second projection. We also write ω for the composition of ω' and the restriction $\mathcal{T}_T \rightarrow \mathcal{W}_T$.

Theorem 3.6.2. Assume that $\bar{\rho}$ has a crystalline lift.

(1) The rigid space $X_{\mathrm{tri}}(\bar{\rho})$ is equidimensional of dimension

$$2n^2 + n + 1 + [K: \mathbb{Q}_p](n^2 + n + 1).$$

(2) The subset $U_{\mathrm{tri}}(\bar{\rho})$ is Zariski open and Zariski dense in $X_{\mathrm{tri}}(\bar{\rho})$.

(3) The open subspace $U_{\mathrm{tri}}(\bar{\rho})$ is smooth over $\mathbb{Q}_p$ and the restriction of ω' to $U_{\mathrm{tri}}(\bar{\rho})$ is smooth morphism.

PROOF. This follows by the same argument of [16, Théorème 2.6]. We will explain the proof of (1) in detail.

Let $\mathcal{S}$ be the rigid space as in Proposition 3.5.10. For a positive integer $m \geq 1$, let $\mathcal{S}_m$ be the rigid space as in the proof of [16, Théorème 2.6]. Considering the pushout by the standard representation $r_{\mathrm{std}}: \mathrm{GSp}_{2n} \rightarrow \mathrm{GL}_{2n}$ and the symplectic similitude character $r_{\mathrm{sml}}: \mathrm{GSp}_{2n} \rightarrow \mathbb{G}_m$, we have the natural injection $s_{\mathrm{std}}: \mathcal{S} \rightarrow \mathcal{S}_{2n}$ and the natural map $s_{\mathrm{sml}}: \mathcal{S} \rightarrow \mathcal{S}_1$.

Let $\mathcal{S}_m^{\mathrm{adm}} \subset \mathcal{S}_m$ be the admissible subspace in the sense of [44, Theorem 1.2]. It is the maximal open adic subspace of $\mathcal{S}_m$ with the property that there exists a vector bundle $\mathcal{V}_m$ on $\mathcal{S}_m^{\mathrm{adm}}$ and a family of Galois representation $\mathcal{G}_K \rightarrow \mathrm{Aut}_{\mathcal{O}_{\mathcal{S}_m^{\mathrm{adm}}}}(\mathcal{V})$ such that $D_{\mathrm{rig}}(\mathcal{V})$ is the preimage of the universal (φ, Γ) -module of $\mathcal{S}_m$ by the map $\mathcal{S}_m^{\mathrm{adm}} \hookrightarrow \mathcal{S}_m$.

Let $\mathcal{S}^{\mathrm{adm}} := s_{\mathrm{std}}^{-1}(\mathcal{S}_{2n}^{\mathrm{adm}}) \cap s_{\mathrm{sml}}^{-1}(\mathcal{S}_1^{\mathrm{adm}})$ be the open subspace of $\mathcal{S}$. The subspace $\mathcal{S}^{\mathrm{adm}}$ is nonempty by considering the point associated with the tensor product of a crystalline lift of $\bar{\rho}$ from the assumption and a sufficiently generic crystalline similitude character.

There exist the natural GSp_{2n} -torsor $\mathcal{S}^{\square, \mathrm{adm}} \rightarrow \mathcal{S}^{\mathrm{adm}}$, which is constructed from $\mathcal{V}_{2n}$ and $\mathcal{V}_1$ by Lemma 3.5.12. Then the action of $\mathcal{G}_K$ on sections of $\mathcal{S}^{\square, \mathrm{adm}}$ is defined by a

continuous homomorphism

$$\tilde{\rho}: \mathcal{G}_K \rightarrow \mathrm{GSp}_{2n}(\Gamma(\mathcal{S}^{\square, \mathrm{adm}}, \mathcal{O}_{\mathcal{S}^{\square, \mathrm{adm}}}).$$

Because $\mathcal{G}_K$ is compact and topologically finitely generated, the subset of points of $\mathcal{S}^{\square, \mathrm{adm}}$ which factor via the inclusion

$$\mathrm{GSp}_{2n}(\Gamma(\mathcal{S}^{\square, \mathrm{adm}}, \mathcal{O}_{\mathcal{S}^{\square, \mathrm{adm}}}^+)) \subset \mathrm{GSp}_{2n}(\Gamma(\mathcal{S}^{\square, \mathrm{adm}}, \mathcal{O}_{\mathcal{S}^{\square, \mathrm{adm}}}))$$

is a nonempty open admissible subspace of $\mathcal{S}^{\square, \mathrm{adm}}$. Let $\mathcal{S}^{\square}(\bar{\rho}) \subset \mathcal{S}^{\square, \mathrm{adm}}$ be the admissible open subspace of points $x \in \mathcal{S}^{\square, \mathrm{adm}}$ such that $\tilde{\rho}_x$ reduces to $\bar{\rho}$ modulo the maximal ideal of the local ring at $x \in \mathcal{S}^{\square, \mathrm{adm}}$. This subspace is nonempty by the assumption of the existence of a crystalline lift of $\bar{\rho}$.

Let $X := X_{\mathrm{tri}}(\bar{\rho})$ and $\delta_1, \dots, \delta_{2n}: K^* \rightarrow \Gamma(X, \mathcal{O}_X)^*$ be the induced continuous cocharacters. Let D_X be the universal (φ, Γ) -module with GSp_{2n} -structure and $\tilde{D}_X := D_X \times^{\mathrm{GSp}_{2n}} \mathrm{GL}_{2n}$ be the pushout, which is seen to be a (φ, Γ) -module of rank $2n$.

By [53, Corollary 6.3.10], there exist

- A rigid space X' and a projective surjective morphism $f: X' \rightarrow X$.
- For $1 \leq i \leq 2n$, line bundles $\mathcal{L}_i$ on X' , $\mathcal{R}_{X'}$ -modules $\mathcal{M}_i$ which is killed by a power of t , and analytic closed subsets $Z_i \subset X'$ which does not contain any irreducible components of X' .
- An increasing filtration $\mathrm{Fil}_{\bullet}$ of (φ, Γ) -submodules of $\tilde{D}_X$ such that for $1 \leq i \leq 2n$ there are isomorphisms

$$0 \rightarrow \mathrm{gr}_i \tilde{D}_{X'} \rightarrow \mathcal{R}_{X'}(\delta'_i) \otimes \mathcal{L}_i \rightarrow \mathcal{M}_i \rightarrow 0,$$

where $\tilde{D}_{X'} := f^* \tilde{D}_X$ and each δ'_i is the composite of δ_i and the natural map $\Gamma(X, \mathcal{O}_X)^* \rightarrow \Gamma(X', \mathcal{O}_{X'})^*$.

Let $U \subset X'$ be the intersection of the complements of $\bigcup_i Z_i$ in X' and the preimage of $\mathcal{T}_{T, \mathrm{reg}} \subset \mathcal{T}_T$ via $X' \rightarrow X \rightarrow \mathcal{T}_T$. The subspace U is a Zariski dense in X' . Moreover, the pullback D_U of the (φ, Γ) -module with GSp_{2n} -structure D_X via the composite $U \hookrightarrow X' \xrightarrow{f} X$ has a natural induced triangulation by Lemma 3.5.14.

Then by the argument of the proof of [16, Théorème 2.6 (i)], we may construct a Zariski open and Zariski dense subspace $V \in X$ and a $\mathbb{G}_m^{n+1}$ -torsor $V^{\square} \rightarrow V$ such that $\dim V^{\square} = \dim \mathcal{S}^{\square}(\bar{\rho})$.

Let $\kappa: \mathcal{S}^{\square}(\bar{\rho}) \rightarrow \mathcal{T}_{T, \mathrm{reg}}$ be the composite

$$\mathcal{S}^{\square}(\bar{\rho}) \hookrightarrow \mathcal{S}^{\square, \mathrm{adm}} \rightarrow \mathcal{S}^{\square} \hookrightarrow \mathcal{S} \rightarrow \mathcal{T}_{T, \mathrm{reg}},$$

where the last map is the smooth projection in Proposition 3.5.10. The relative dimensions of the second and the last map are $2n^2 + n + 1$ and $[K: \mathbb{Q}_p]n^2$ by Proposition 3.5.10, respectively. Since $\mathcal{T}_T$ is smooth of dimension $(n+1) + [K: \mathbb{Q}_p](n+1)$, we have

$$\dim \mathcal{S}^{\square}(\bar{\rho}) = 2n^2 + 2n + 2 + [K: \mathbb{Q}_p](n^2 + n + 1).$$

Therefore, we have

$$\begin{aligned} \dim X_{\mathrm{tri}}(\bar{\rho}) &= \dim X = 2n^2 + 2n + 2 + [K: \mathbb{Q}_p](n^2 + n + 1) - (n+1) \\ &= 2n^2 + n + 1 + [K: \mathbb{Q}_p](n^2 + n + 1). \end{aligned}$$

We omit the proofs of (2) and (3). □

3.7. Triangulations and refinements

In this section, we show some properties of triangulations of D and refinements of $D_{\text{crys}}(D)$ for a (φ, Γ) -module with G -structure D .

First, we introduce the notation of crystalline (φ, Γ) -modules over Robba rings (cf. [45, §3.3]). There exists a left exact functor D_{crys} from the category of (φ, Γ) -modules over $\mathcal{R}_L$ to the category of K -filtered φ -modules over L , which is defined by $D_{\text{crys}}(D) := (D \left[\frac{1}{t} \right])^\Gamma$. We say that a (φ, Γ) -module D over $\mathcal{R}_L$ is *crystalline* if $\dim_{K_0} D_{\text{crys}}(D) = \text{rank}_{\mathcal{R}_L} D$. Note that for a crystalline representation $\rho: \text{Gal}_K \rightarrow \text{GL}(V)$, the module $D_{\text{rig}}(\rho)$ is crystalline and there is an isomorphism

$$D_{\text{crys}}(\rho) := (B_{\text{crys}} \otimes_{\mathbb{Q}_p} V)^{\text{Gal}_K} \cong D_{\text{crys}}(D_{\text{rig}}(\rho))$$

by [7, Théorème 3.6]. Let D be a crystalline (φ, Γ) -module. There exists a bijection between saturated sub- (φ, Γ) -modules D' and φ -stable sub- $L \otimes_{\mathbb{Q}_p} K_0$ -modules $\mathcal{F}$ of $D_{\text{crys}}(D)$ by the same argument of [6, §2.4.2]. The explicit construction of the bijection and its inverse are as follows:

$$D' \mapsto D_{\text{crys}}(D') = (D' \left[\frac{1}{t} \right])^\Gamma, \quad \mathcal{F} \mapsto (\mathcal{R}_L \left[\frac{1}{t} \right] \cdot \mathcal{F}) \cap D.$$

A *refinement* of a K -filtered φ -module $\mathcal{F}$ of rank m over L is a filtration $\mathcal{F}_\bullet$

$$\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_m := \mathcal{F}$$

of $\mathcal{F}$ such that each $\mathcal{F}_i$, which is free of rank i , is a $L \otimes_{\mathbb{Q}_p} K_0$ -submodule stable under φ . Consequently, there is a bijection

$$\mathcal{P} = (D_i)_{0 \leq i \leq m} \mapsto D_{\text{crys}}(\mathcal{P}) := (D_{\text{crys}}(D_i))_{0 \leq i \leq m}$$

between the set of (classical) triangulations of D and the set of refinements of $D_{\text{crys}}(D)$ for crystalline D .

Let (G, B, T) be a split reductive group of regular type over K and fix an algebraic embedding $r: G \rightarrow \text{GL}_m$ of regular type. We say that a (φ, Γ) -module with G -structure $\omega: \text{Rep}_K(G) \rightarrow \Phi\Gamma(\mathcal{R}_L)$ is *crystalline* (resp. *semistable*) if $\omega(V)$ is crystalline for some faithful representation $r: G \rightarrow \text{GL}(V)$, equivalently for any faithful representation $r: G \rightarrow \text{GL}(V)$ by [57, Theorem C.1.7]. Let D be a crystalline (resp. semistable) (φ, Γ) -module with G -structure $\text{Rep}_K(G) \rightarrow \Phi\Gamma(\mathcal{R}_L)$. Since D_{crys} is exact on its restriction to the category of crystalline (φ, Γ) -modules, the composition

$$\text{Rep}_K(G) \rightarrow \Phi\Gamma(\mathcal{R}_L) \xrightarrow{D_{\text{crys}}} \Phi F_K(L)$$

is also an exact tensor functor. Write $D_{\text{crys}}(D)$ for the composition functor and

$$r_*(D_{\text{crys}}(D)) := (D_{\text{crys}}(D))(r).$$

Though the underlying G -torsor of the $D_{\text{crys}}(D)$ is not in general a trivial torsor over $K_0 \otimes_{\mathbb{Q}_p} L$, it becomes trivial after replacing K and L by a sufficiently large finite extension. A *refinement* $\mathfrak{P}$ of $D_{\text{crys}}(D)$ is a sub- B -torsor of $D_{\text{crys}}(D)$ over $K_0 \otimes_{\mathbb{Q}_p} L$ which is invariant under the Frobenius action φ .

Proposition 3.7.1. *Let D be a crystalline frameable (φ, Γ) -module with G -structure over $\mathcal{R}_L$. Assume the Frobenius action $\varphi^{[K_0: \mathbb{Q}_p]}$ on $r_*(D_{\text{crys}}(D))$ has m distinct eigenvalues in L . Then there is a bijection from the set of triangulations of D to that of refinements of $D_{\text{crys}}(D)$. Moreover, if there is a refinement of $D_{\text{crys}}(D)$, there exists exactly $\#W(G, T)$ refinements of $D_{\text{crys}}(D)$.*

PROOF. Recall that a triangulation $\mathcal{P}$ of D can be seen as an exact tensor functor $\omega_{\mathcal{P}}: \mathrm{Rep}_K(B) \rightarrow \Phi\Gamma(\mathcal{R}_L)$ whose composition with the restriction $\mathrm{Rep}_K(G) \rightarrow \mathrm{Rep}_K(B)$ is identical to D . We construct a map from the set of triangulations of D to that of refinements of $D_{\mathrm{crys}}(D)$ as $\omega_{\mathcal{P}} \mapsto \omega_{\mathcal{P}} \circ D_{\mathrm{crys}}$.

We construct an inverse. By Lemma 3.5.8, we may assume that the underlying G -torsor of $D_{\mathrm{crys}}(D)$ is trivial. A refinement of $D_{\mathrm{crys}}(D)$ gives a (classical) refinement of $r_*(D_{\mathrm{crys}}(D))$ and it induces a (classical) triangulation of $r_*(D)$ by the classical correspondence between triangulations and refinements. It suffices to show that the triangulation is isomorphic to the form $r_*(\mathcal{P})$ for some unique triangulation $\mathcal{P}$ of D .

For a commutative K -algebra A , there is the map $G(A)/B(A) \rightarrow (G/B)(A)$ induced from the exact sequence of pointed sets

$$1 \rightarrow B(A) \rightarrow G(A) \rightarrow (G/B)(A) \rightarrow H^1(A, B)$$

by [24, §1.2.2], where G/B is the flag variety of (G, B, T) . Assume that A is a Bézout domain. Since the Picard group of A is trivial, it follows $H^1(A, B) \cong H^1(A, T) \cong 1$ by Lemma 3.5.8 and the map $G(A)/B(A) \rightarrow (G/B)(A)$ is bijective. In particular, we have $G(\mathcal{R}_L)/B(\mathcal{R}_L) = (G/B)(\mathcal{R}_L)$ and $G(\mathcal{R}_L[\frac{1}{t}])/B(\mathcal{R}_L[\frac{1}{t}]) = (G/B)(\mathcal{R}_L[\frac{1}{t}])$. The faithful representation r induces the proper morphism $G/B \rightarrow \mathrm{GL}_m/B_m$ of the flag varieties over K . It is a monomorphism of fppf-sheaves by the definition of r and also a monomorphism of schemes. So it is a closed immersion by [52, Tag 04XV]. The variety G/B is smooth and then reduced as well. Then, there exists a unique dotted arrow such that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{R}_L[\frac{1}{t}]) & \longrightarrow & G/B \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathrm{Spec}(\mathcal{R}_L) & \longrightarrow & \mathrm{GL}_m/B_m, \end{array}$$

is commutative because the RHS map is a closed immersion. Thus, the first part of the proposition is shown.

We now proceed to show the last part of the proposition. For an algebraic group H , the notation H^0 denotes the set of $K_0 \otimes_{\mathbb{Q}_p} L$ -valued points of H for the sake of brevity. We may assume that $D_{\mathrm{crys}}(D)$ is a trivial G -torsor over $K_0 \otimes_{\mathbb{Q}_p} L$ and fix an isomorphism of G -torsors $D_{\mathrm{crys}}(D) \cong G(K_0 \otimes_{\mathbb{Q}_p} L)$. We have a natural morphism of Bruhat decompositions

$$(3.7.1) \quad \bigsqcup_{w \in W(G, T)} B^0 w B^0 \rightarrow \bigsqcup_{w \in W(\mathrm{GL}_m, T_m)} B_m^0 w B_m^0$$

compatible with the embedding $G \rightarrow \mathrm{GL}_m$ and the inclusion of Weyl groups in Definition 3.5.5 (3). The set of triangulations of $r_*(D)$ is written by

$$\{h w B_m^0 \mid w \in W(\mathrm{GL}_m, T_m)\}$$

for an element $h \in \mathrm{GL}_m$. Therefore, if there exist a refinement gB for some $g \in G$, the set of refinements of $D_{\mathrm{crys}}(D)$ is exactly

$$\{g w B^0 \mid w \in W(G, T)\}$$

by the existence of the map (3.7.1). □

Remark 3.7.2. The correspondence between triangulations and refinements in Proposition 3.7.1 can be shown without the assumption on the Frobenius action.

Besides, in the case of $G = \mathrm{GSp}_{2n}$, after replacing K and L with sufficient large finite extensions, there exists a refinement of $D_{\mathrm{crys}}(D)$ under the condition of Proposition 3.7.1 due to triangulation of an element $M \in \mathrm{GSp}_{2n}(\overline{\mathbb{Q}}_p)$ by an element $U \in \mathrm{GSp}_{2n}(\overline{\mathbb{Q}}_p)$.

Let D be a (φ, Γ) -module over $\mathcal{R}_L$ of rank m . If D is crystalline, we write

$$\begin{aligned}\mathcal{F} &:= D_{\mathrm{crys}}(D), \\ \mathcal{F}_K &:= D_{\mathrm{crys}}(D)_K := D_{\mathrm{crys}}(D) \otimes_{K_0} K, \\ \mathcal{F}_{K,\tau} &:= \mathcal{F}_K \otimes_{K \otimes_{\mathbb{Q}_p} L, \tau} L, \text{ and} \\ \mathrm{Fil}^\bullet &:= (\mathrm{Fil}^i)_{i \in \mathbb{Z}} = (\mathrm{Fil}^i \mathcal{F}_K)_{i \in \mathbb{Z}}\end{aligned}$$

for the Hodge filtration on $\mathcal{F}_K$ and $\mathrm{gr}^i := \mathrm{Fil}^i / \mathrm{Fil}^{i+1}$. For a refinement

$$\mathcal{F}_\bullet := (\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_m = \mathcal{F})$$

of $\mathcal{F}$, we associate it with an ordering

$$(\varphi_1, \dots, \varphi_m) \in (K_0 \otimes_{\mathbb{Q}_p} L)^{\oplus m}$$

of the eigenvalues of $\Phi^{[K_0: \mathbb{Q}_p]}$ on $\mathcal{F}$ such that the formula

$$\det(T - \Phi^{[K_0: \mathbb{Q}_p]}|_{\mathcal{F}_i}) = \prod_{j=1}^i (T - \varphi_j),$$

holds and

$$\mathbf{k} := (k_{\tau,1}, \dots, k_{\tau,m})_\tau \in (\mathbb{Z}^{\oplus m})^\Sigma$$

such that the quotient $\mathcal{F}_{K,\tau,i}/\mathcal{F}_{K,\tau,i-1}$, where $\mathcal{F}_{K,\tau,i} := \mathcal{F}_i \otimes_{K_0 \otimes_{\mathbb{Q}_p} L, \tau} L$, of the filtered modules over L has the non-zero graded quotient in degree $-k_{\tau,i}$ for any $\tau \in \Sigma$ and $1 \leq i \leq m$, taking into account multiplicities. We call $\mathbf{k}$ the *Hodge-Tate type* of D . We say that a Hodge-Tate type $\mathbf{k}$ is *regular* if $k_{\tau,1}, \dots, k_{\tau,m}$ are distinct for all $\tau \in \Sigma$. We say that a Hodge-Tate type $\mathbf{k}$ (resp. a parameter $(z^{-k_{\tau,1}}, \dots, z^{-k_{\tau,m}})_{\tau \in \Sigma} \in (\mathcal{W}_{T_m}(L))^\Sigma$) is *strictly dominant* if all these sequences of integers $k_{\tau,1}, \dots, k_{\tau,n}$ for $\tau \in \Sigma$ are strictly decreasing.

We recall the notion of noncriticality of refinements in the regular crystalline case. Take $\mathbf{k}$ to be strictly dominant and let D be a rank m crystalline (φ, Γ) -module which has a regular Hodge-Tate type $(k_{\tau,1}, \dots, k_{\tau,m})_\tau \in (\mathbb{Z}^{\oplus m})^\Sigma$. Let $\mathcal{F}_\bullet$ be a refinement of $\mathcal{F} := D_{\mathrm{crys}}(D)$ and $\mathrm{Fil}^\bullet$ be the Hodge filtration of $\mathcal{F} \otimes_{K_0} K$. We say the refinement $\mathcal{F}_\bullet$ is *noncritical* if

$$(L \otimes_{K_0 \otimes_{\mathbb{Q}_p} L, \tau|_{K_0}} \mathcal{F}_i) \oplus (L \otimes_{K_0 \otimes_{\mathbb{Q}_p} L, \tau} \mathrm{Fil}^{-k_{\tau,i}}) = L \otimes_{K_0 \otimes_{\mathbb{Q}_p} L, \tau} \mathcal{F}_K$$

is satisfied for each $\tau \in \Sigma$ and $1 \leq i \leq m$. Similarly we may define the noncriticality of triangulation of D by the correspondence of triangulations and refinements.

If $\mathbf{k} = (k_\tau)_{\tau: K \hookrightarrow L} \in \mathbb{Z}^\Sigma$, let $z^{\mathbf{k}} \in \mathcal{T}(L)$ denote the character

$$z \mapsto \prod_{\tau \in \mathrm{Hom}(K, L)} \tau(z)^{k_\tau}$$

where $z \in K^*$. For

$$\mathbf{k} = (k_{\tau,i})_{1 \leq i \leq m, \tau: K \hookrightarrow L} \in (\mathbb{Z}^{\oplus m})^\Sigma,$$

let $z^{\mathbf{k}} \in \mathcal{T}^m(L)$ also denote the character

$$(z_1, \dots, z_m) \mapsto \prod_{\substack{1 \leq i \leq m \\ \tau \in \mathrm{Hom}(K, L)}} \tau(z_i)^{k_{\tau,i}}$$

where $(z_1, \dots, z_m) \in (K^*)^m$. We write $\mathrm{unr}(a)$ for the character $K^* \rightarrow A^*$ which sends $\mathcal{O}_K^*$ to 1 and ϖ_K to a for an L -algebra A and $a \in A^*$. By the argument of [6, Proposition 2.4.1] and [17, Lemma 2.1], we have the following lemma.

Lemma 3.7.3. *For a crystalline (φ, Γ) -module D over $\mathcal{R}_L$ of rank m and a refinement of $D_{\mathrm{crys}}(D)$, the corresponding triangulation of D has a parameter $(\delta_1, \dots, \delta_m)$ such that $\delta_i = z^{\mathbf{k}_i} \mathrm{unr}(\varphi_i)$ where $(\varphi_i)_i$ and $\mathbf{k}_i$ are constructed above.*

We introduce a notion of G -valued strictly dominance. We identify $\mathcal{W}_T$ (resp. $\mathcal{T}_T$) with $\mathcal{W}^d$ (resp. $\mathcal{T}^d$) by the fixed isomorphism $G \supset T \cong \mathbb{G}_m^d$ of algebraic tori. Let D be a crystalline frameable (φ, Γ) -module with G -structure and $\tilde{D} := r_*(D)$ be the pushout of D by the faithful representation $r: G \rightarrow \mathrm{GL}_m$ of regular type. We assume that the underlying G -torsor of $D_{\mathrm{crys}}(\tilde{D})$ is trivial and $D_{\mathrm{crys}}(\tilde{D})$ has a regular Hodge-Tate type. Note that the first assumption is satisfied if D is trianguline. Recall that there is the natural map $\mathcal{W}_T \rightarrow \mathcal{W}_{T_m}$.

- Definition 3.7.4.** (1) A parameter $x := (z^{-k_{\tau,1}}, \dots, z^{-k_{\tau,d}})_{\tau \in \Sigma} \in (\mathcal{W}_T(L))^\Sigma$ (resp. $(k_{\tau,1}, \dots, k_{\tau,d})_{\tau \in \Sigma} \in (\mathbb{Z}^d)^\Sigma$) is said to be *strictly dominant* if the image of x by the natural map $(\mathcal{W}_T(L))^\Sigma \rightarrow (\mathcal{W}_{T_m}(L))^\Sigma$ is strictly dominant.
- (2) For a (φ, Γ) -module with G -structure D , a refinement of $D_{\mathrm{crys}}(D)$ is said to be *noncritical* if the pushout of the refinement by $r: G \rightarrow \mathrm{GL}_m$ is a noncritical refinement of $r_*(D_{\mathrm{crys}}(D)) = D_{\mathrm{crys}}(D)$.
- (3) For a continuous representation $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GL}_m(k)$ or $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(k)$, we say a point $x = (\rho, \underline{\delta}) \in X_{\mathrm{tri}}(\bar{\rho})$ is *strictly dominant* if $\omega(x)$ is strictly dominant.

Remark 3.7.5. The notions of strictly dominance and noncriticality in Definition 3.7.4 may be defined not to depend on faithful representations $r: G \rightarrow \mathrm{GL}_m$. However, we adopt the definition depending r and considering (φ, Γ) -module with G -structures as classical (φ, Γ) -modules. We remark the definition is different from that not depending on r .

Example 3.7.6. In the case of $G = \mathrm{GSp}_{2n}$ and T is the maximal split torus of GSp_{2n} , the inclusion of tori $T \rightarrow T_{2n}$ induced by the standard representation $\mathrm{GSp}_{2n} \rightarrow \mathrm{GL}_{2n}$ determines that strictly dominant parameters are the form $(z^{-k_{\tau,1}}, \dots, z^{-k_{\tau,n+1}})_{\tau \in \Sigma} \in \mathcal{W}_T^\Sigma$ such that $k_{\tau,1} > \dots > k_{\tau,n}$ and $2k_{\tau,n} > k_{\tau,n+1}$ for all $\tau \in \Sigma$.

Let $\bar{\rho}: \mathcal{G}_K \rightarrow \mathrm{GSp}_{2n}(k_L)$ be a continuous representation. By the isomorphism $\mathcal{T}_T \cong \mathcal{T}^{n+1}$ by the isomorphism $T \cong \mathbb{G}_m^{m+1}$ in Section 3.2, we identify the two spaces with each other. We call a point $x = (\rho, (\delta_1, \dots, \delta_{n+1})) \in X_{\mathrm{tri}}(\bar{\rho})$ *saturated* if there exists a triangulation of the (φ, Γ) -module with GSp_{2n} -structure $D_{\mathrm{rig}}(\rho)$ with parameter $(\delta_1, \dots, \delta_{n+1})$.

Lemma 3.7.7. *Let $\bar{\rho}: \mathcal{G}_K \rightarrow \mathrm{GSp}_{2n}(k_L)$ be a $\mathrm{GSp}_{2n}(k_L)$ -valued representation and $x = (r, (\delta_1, \dots, \delta_{n+1})) \in X_{\mathrm{tri}}(\bar{\rho})$ be a strictly dominant point with $\omega(x) = \delta_{\mathbf{k}}$ for some $\mathbf{k} = (k_{\tau,i})_{1 \leq i \leq n+1, \tau \in \Sigma} \in (\mathbb{Z}^{n+1})^\Sigma$. For each $\tau \in \Sigma$, assume that*

$$k_{\tau,i} - k_{\tau,i+1} > [K: K_0] \mathrm{val}(\delta_1(\varpi_K) \cdots \delta_i(\varpi_K)) \text{ for } 1 \leq i \leq n-1, \text{ and} \\ 2k_{\tau,n} - k_{\tau,n+1} > [K: K_0] \mathrm{val}(\delta_1(\varpi_K) \cdots \delta_i(\varpi_K)),$$

where $\mathrm{val}(-)$ is the function of p -adic values. Then x is saturated and semistable. Moreover, if $(\delta_1, \dots, \delta_{n+1}) \in \mathcal{T}_{T, \mathrm{reg}}$, then r is crystalline strictly dominant noncritical.

PROOF. Let $\tilde{\delta} := (\delta_1, \dots, \delta_n, \delta_n^{-1} \delta_{n+1}, \dots, \delta_1^1 \delta_{n+1})$ be the image of δ by the natural map $\mathcal{T}_T \rightarrow \mathcal{T}^{2n}$ induced by the standard representation $\mathrm{GSp}_{2n} \rightarrow \mathrm{GL}_{2n}$.

The composite $r: \mathcal{G}_K \rightarrow \mathrm{GL}_{2n}(k_x)$ of ρ and the map $\mathrm{GSp}_{2n}(k_x) \rightarrow \mathrm{GL}_{2n}(k_x)$ is saturated and semistable. Moreover, if $\tilde{\delta} \in \mathcal{T}_{\mathrm{reg}}^{2n}$, then r is crystalline strictly dominant noncritical by [17, Lemma 2.10].

By Proposition 3.7.1 and symplectic diagonalization (Remark 3.7.2), the lemma follows in the case of ρ as well as r . $\square$

3.8. Explicit calculation of the tangent spaces of deformation functors

In this section, we calculate the dimensions of deformation functors of some crystalline (φ, Γ) -modules with GSp_{2n} -structures.

At first, we prove the additive version of the reconstruction theorem ([65, Theorem 11.2]). Let G be an affine algebraic monoid over a field k and R be a k -algebra. For a free module V_R over R , each element $Y \in \mathrm{Lie}_R(\mathrm{GL}(V_R))$ induces an endomorphism of the R -module V_R by the natural isomorphism $\mathrm{Lie}_R(\mathrm{GL}(V_R)) \xrightarrow{\sim} \mathrm{End}_R(V_R)$. We also write $Y: V_R \rightarrow V_R$ for the induced endomorphism by $Y \in \mathrm{Lie}_R(\mathrm{GL}(V_R))$. Let $X \in (\mathrm{Lie}_R(G))(R)$. For every finite-dimensional algebraic k -representation $r_V: G \rightarrow \mathrm{GL}(V)$ of G , we have an R -linear map $\lambda_V := (\mathrm{Lie}_R(r_{V,R}))(X): V_R \rightarrow V_R$. These maps satisfy the following conditions:

(C1) For all finite-dimensional representations V, W ,

$$\lambda_{V \otimes W} = \lambda_V \otimes 1 + 1 \otimes \lambda_W.$$

(C2) If we write $\mathbf{1}$ for the 1-dimensional trivial representation of G , induced $\lambda_{\mathbf{1}}: k \rightarrow k$ is the zero map.

(C3) For all G -equivariant maps $u: V \rightarrow W$,

$$\lambda_W \circ u_R = u_R \circ \lambda_V.$$

We write $G_R := G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ and $V_R := V \otimes_k R$ from here.

Lemma 3.8.1. *Let G be an affine algebraic monoid over a field k and R be a k -algebra. Suppose that we are given an R -linear map $\lambda_V: V_R \rightarrow V_R$ for every finite-dimensional algebraic k -representation $r_V: G \rightarrow \mathrm{GL}(V)$ of G . If the family $(\lambda_V)_V$ satisfies the conditions (C1), (C2), (C3) above, there exists a unique $X \in \mathrm{Lie}_R(G_R)$ such that $\lambda_V = ((\mathrm{Lie}_R(r_{V,R}))(X)): V_R \rightarrow V_R$ for all V .*

PROOF. The conditions (C1), (C2), (C3) are satisfied for the uniquely induced $\lambda_V: V_R \rightarrow V_R$ from the enlarged family of filtered unions V of the finite-dimensional representations. We set $A := \Gamma(G, \mathcal{O}_G)$. Considering A as a k -vector space, the regular representation $r_A: G \rightarrow \mathrm{GL}(A)$ is a filtered union of its finite-dimensional subrepresentation by [65, Proposition 4.6]. We write $\lambda_A: A \rightarrow A$ for the k -linear map induced from r_A .

By the condition (C1) and (C3), the diagram

$$\begin{array}{ccc} A \otimes_k A & \xrightarrow{m} & A \\ \lambda_A \otimes 1 + 1 \otimes \lambda_A \downarrow & & \downarrow \lambda_A \\ A \otimes_k A & \xrightarrow{m} & A, \end{array}$$

where m is the multiplication structure of the monoid G , is commutative, so that r_A is an element of the module $\mathrm{Der}_k(A, A)$ of k -derivations. We also have the natural isomorphism

$$\iota: \mathrm{Lie}_k(G) \xrightarrow{\sim} \mathrm{Der}_k(A, A)$$

of k -vector spaces by [65, Proposition 12.23, Proposition 12.24].

For an element $X \in \mathrm{Lie}_R(G_R)$ and a finite-dimensional algebraic representation $\lambda_V: G \rightarrow \mathrm{GL}(V)$, here X gives an element $X_\epsilon \in \mathrm{GL}(V_R[\epsilon])$. Writing ∂X for the composition

$$V_R \hookrightarrow V_R \oplus \epsilon V_R \xrightarrow{X_\epsilon} V_R \oplus \epsilon V_R \twoheadrightarrow \epsilon V_R \xrightarrow{\sim} V_R,$$

The element X is sent to $\partial X \in \mathrm{End}_R(V_R)$ by $\mathrm{Lie}_R(\mathrm{GL}(V_R)) \xrightarrow{\sim} \mathrm{End}_R(V_R)$. We may see that the composition

$$\mathrm{Lie}_R(G_R) \rightarrow \mathrm{Lie}_R(\mathrm{GL}(A_R)) \xrightarrow{\sim} \mathrm{End}_R(V_R)$$

is the same as the composition of ι_R and the inclusion $\mathrm{Der}_R(A_R, A_R) \subset \mathrm{End}_R(V_R)$. Then there exists $X \in \mathrm{Lie}_R(G_R)$ such that $\lambda_A = (\mathrm{Lie}_R(r_{A,R}))(X)$.

For each finite-dimensional algebraic k -representation $r_V: G \rightarrow \mathrm{GL}(V)$, taking V_0 for the underlying vector space of V that has the trivial G -action, there exists an injection $i: V \hookrightarrow V_0 \otimes A$ of k -vector spaces with G -actions by [65, Proposition 4.9]. In the diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & V_0 \otimes A \\ \lambda_V \downarrow & & \downarrow \lambda_{V_0 \otimes A} \\ V & \xrightarrow{i} & V_0 \otimes A, \end{array}$$

λ_V is also the endomorphism corresponding to the element $(\mathrm{Lie}_R(r_{V,R}))(X) \in \mathrm{Lie}_R(\mathrm{GL}(V_R))$ by the conditions (C1), (C2), (C3). Then the existence of $X \in \mathrm{Lie}_R(G_R)$ follows. The uniqueness of X follows from considering $\lambda_{W_0 \otimes A}$ for a finite-dimensional faithful representation $r_W: G \rightarrow \mathrm{GL}(W)$ and its underlying k -vector space W_0 . $\square$

Let G be an affine algebraic group over a field k and $X = \mathrm{Spec}(R)$ be a k -scheme. We recall the notation of exact tensor filtrations of the exact tensor functors $\omega: \mathrm{Rep}_K(G) \rightarrow \mathrm{Bun}(X)$ (cf. [78], [12, §2.7]). An *exact tensor filtration* of ω is the data of decreasing filtrations $(\mathrm{Fil}^\bullet(\omega(V)))_V$ for each $V \in \mathrm{Rep}_K(G)$ such that the following condition are satisfied:

- (1) the filtrations are functorial in V .
- (2) the filtrations are tensor compatible, in the sense that

$$\mathrm{Fil}^n \omega(V \otimes_k V') = \sum_{p+q=n} (\mathrm{Fil}^p \omega(V) \otimes_{\mathcal{O}_X} \mathrm{Fil}^q \omega(V')) \subset \omega(V) \otimes_{\mathcal{O}_X} \omega(V').$$

- (3) For the 1-dimensional trivial representation $\mathbf{1} \in \mathrm{Rep}_K(G)$ one has $\mathrm{Fil}^n(\omega(\mathbf{1})) = \omega(\mathbf{1})$ if $n \leq 0$ and $\mathrm{Fil}^n(\omega(\mathbf{1})) = 0$ if $n \geq 1$.
- (4) The associated functor from $\mathrm{Rep}_K(G)$ to the category of $\mathbb{Z}$ -graded objects of $\mathrm{Bun}(X)$ is exact.

For an exact tensor functor $\omega: \mathrm{Rep}_K(G) \rightarrow \mathrm{Bun}(X)$ and a morphism $Y \rightarrow X$ of affine schemes, let $\omega_Y: \mathrm{Rep}_K(G) \rightarrow \mathrm{Bun}(Y)$ be the composition of ω and the base change functor $\mathrm{Bun}(X) \rightarrow \mathrm{Bun}(Y)$. If $Y = \mathrm{Spec}(R)$ for a commutative ring R , we write $\omega_R = \omega_Y$.

Definition 3.8.2. Let $\omega, \eta: \mathrm{Rep}_K(G) \rightrightarrows \mathrm{Bun}(X)$ be exact tensor functors. Then $\underline{\mathrm{Hom}}^\otimes(\omega, \eta)$ is the presheaf on affine schemes $\mathrm{Spec}(R)$ over X given by

$$\underline{\mathrm{Hom}}^\otimes(\mathrm{Spec}(R)) := \mathrm{Hom}^\otimes(\omega_R, \eta_R),$$

where $\underline{\mathrm{Hom}}^\otimes$ is the set of natural transformations of functors which preserves tensor products. Let $\underline{\mathrm{Aut}}^\otimes(\omega)$ be the presheaf of groups of invertible sections of $\underline{\mathrm{Hom}}^\otimes(\omega, \omega)$.

We define two subfunctors of $\underline{\text{Aut}}^\otimes(\omega)$ dependent on a fixed exact tensor filtration $\text{Fil}^\bullet$ of ω . We remark that the notation $\text{Fil}^\bullet$ in this section is used in the different way from in the section 3.7 where it is used in the sense of decreasing filtrations on φ -modules.

Definition 3.8.3. Let $\text{Fil}^\bullet$ be a fixed exact tensor filtration of ω .

- (1) Let $P_{\text{Fil}^\bullet} = \underline{\text{Aut}}_{\text{Fil}^\bullet}^\otimes(\omega)$ be the functor on R -algebras such that $R' \mapsto \{\lambda \in \underline{\text{Aut}}^\otimes(\omega)(R') \mid \lambda(\text{Fil}^n \omega(V)) \subset \text{Fil}^n \omega(V) \text{ for all } V \in \text{Rep}_K(G) \text{ and } n \in \mathbb{Z}\}.$
- (2) Let $U_{\text{Fil}^\bullet} = \underline{\text{Aut}}_{\text{Fil}^\bullet}^{\otimes!}(\omega)$ be the functor on R -algebras such that $R' \mapsto \{\lambda \in \underline{\text{Aut}}^\otimes(\omega)(R') \mid (\lambda - \text{id})(\text{Fil}^n \omega(V)) \subset \text{Fil}^{n+1} \omega(V) \text{ for all } V \in \text{Rep}_K(G) \text{ and } n \in \mathbb{Z}\}.$

Let $\tilde{X} \rightarrow X$ be the G -torsor corresponding to ω .

Proposition 3.8.4. (1) The functor $\underline{\text{Aut}}^\otimes(\omega)$ is representable by the X -group scheme $\text{Aut}_G(\tilde{X})$, which is a form of $G \times_k X$.

(2) The functors $P_{\text{Fil}^\bullet}$ and $U_{\text{Fil}^\bullet}$ are representable by closed subschemes of $\text{Aut}_G(\tilde{X})$ and they are smooth if G is.

(3) If $X = \text{Spec}(R)$, we have

- (a) $\text{Lie}_R P_{\text{Fil}^\bullet} = \text{Fil}^0(\text{Lie}_R \underline{\text{Aut}}^\otimes(\omega))$ and
- (b) $\text{Lie}_R U_{\text{Fil}^\bullet} = \text{Fil}^1(\text{Lie}_R \underline{\text{Aut}}^\otimes(\omega)).$

PROOF. See [78, Chapter IV, 2.1.4.1], [12, Corollary 2.7.3] and its following argument. $\square$

Let $\text{ad}(\tilde{X}) := \text{Lie}_R \underline{\text{Aut}}^\otimes(\omega)$. For a parabolic subgroup P of $\text{Aut}_G(\tilde{X})$, let $\text{ad}_P(\tilde{X}) := \text{Fil}^0(\text{Lie}_R \underline{\text{Aut}}^\otimes(\omega))$ for an exact tensor filtration $\text{Fil}^\bullet$ such that $P_{\text{Fil}^\bullet} = P$. For a (φ, Γ) -module with G -structure D over $\mathcal{R}_L$ and its triangulation $\mathcal{P}$, we define $\text{ad}_{\mathcal{P}}(D) := \text{ad}_P(D)$ where P is the subgroup of $\text{Aut}_G(D)$ which preserves $\mathcal{P} \subset D$. If D is frameable, $\text{ad}_{\mathcal{P}}(D) \subset \text{Aut}_G(D) \cong G_{\mathcal{R}_L}$ is a Borel subgroup of $G_{\mathcal{R}_L}$ and is a conjugate of the standard Borel subgroup $B_{\mathcal{R}_L} \subset G_{\mathcal{R}_L}$.

We define the trianguline deformation functors of a frameable (φ, Γ) -module with G -structure D (associated to a triangulation $\mathcal{P}$ of D)

$$X_D, X_{D, \mathcal{P}}: \mathcal{C}_L \rightarrow \text{Set}$$

as follows. For an object $A \in \mathcal{C}_L$, let $X_{D, \mathcal{P}}(A)$ be the set of isomorphism classes of triples $(D_A, \mathcal{P}_A, \pi)$ where D_A is a frameable (φ, Γ) -module with G -structure over $\mathcal{R}_A$, $\mathcal{P}_A$ is a triangulation of D_A and $\pi: D_A \otimes_A L \xrightarrow{\sim} D$ is an isomorphism of (φ, Γ) -module with G -structure over $\mathcal{R}_L$ such that $\pi(\mathcal{P}_A) = \mathcal{P}$. Let $X_D(A)$ be the set of isomorphism classes of (D_A, π) .

Lemma 3.8.5. There is a natural isomorphism

- (1) $TX_D = X_D(L[\epsilon]) \xrightarrow{\sim} H^1(\text{ad}(D)).$
- (2) $TX_{D, \mathcal{P}} = X_{D, \mathcal{P}}(L[\epsilon]) \xrightarrow{\sim} H^1(\text{ad}_{\mathcal{P}}(D)).$

PROOF. We fix a framing $([\phi], [\gamma] \mid \gamma \in \Gamma)$ of D . For each algebraic representation $r: G \rightarrow \text{GL}(V)$, the pushout $r_*(D)$ is then the framed (φ, Γ) -module $([\phi_V], [\gamma_V] \mid \gamma \in \Gamma)$ of rank $\dim V$. For algebraic representations $r: G \rightarrow \text{GL}(V)$, $r': G \rightarrow \text{GL}(V')$, lifts of the (φ, Γ) -modules $r_*(D)$ and $r'_*(D)$ over $L[\epsilon]$ are the form

$$\begin{aligned} &([\phi_V + \epsilon \cdot \partial \phi_V], [\gamma_V + \epsilon \cdot \partial \gamma_V] \mid \gamma \in \Gamma), \\ &([\phi_{V'} + \epsilon \cdot \partial \phi_{V'}], [\gamma_{V'} + \epsilon \cdot \partial \gamma_{V'}] \mid \gamma \in \Gamma), \end{aligned}$$

where $\partial\phi_V, \partial\gamma_V \in \mathrm{End}(V)$ and $\partial\phi_{V'}, \partial\gamma_{V'} \in \mathrm{End}(V')$.

Then for the pushout $(r \otimes r')_*(D)$ of D by $r \otimes r': G \rightarrow \mathrm{GL}(V \otimes V')$, $\phi_{V \otimes V'}$ and $\gamma_{V \otimes V'}$ is written as follows:

$$\begin{aligned}\phi_{V \otimes V'} &= \phi_V \otimes \phi_{V'} + \epsilon \cdot (\partial\phi_V \otimes 1 + 1 \otimes \partial\phi_{V'}), \\ \gamma_{V \otimes V'} &= \gamma_V \otimes \gamma_{V'} + \epsilon \cdot (\partial\gamma_V \otimes 1 + 1 \otimes \partial\gamma_{V'}).\end{aligned}$$

By Lemma 3.8.1, functorial parameters $\partial\phi_V, \partial\gamma_V$ for all $V \in \mathrm{Rep}_K(G)$ is just given by an element of $\mathrm{Lie}_{\mathcal{R}_L}(D)$. The isomorphisms of the lemma follows by the proof of [26, Proposition 3.6, (ii)]. $\square$

From here we consider the case of $G = \mathrm{GL}_m$ over $k = K$. For $\mathbf{k} = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$, write $t^{\mathbf{k}} := \prod_{\tau \in \Sigma} t^{k_\tau} \in \mathcal{R}_L \left[\frac{1}{t} \right]$. We calculate bases that give each triangulation associated with the elements of the Weyl group and transformation matrices between them.

Let D be a (φ, Γ) -module over $\mathcal{R}_L$ of rank m . We assume that D is crystalline and $D_{\mathrm{crys}}(D)$ has m distinct Frobenius eigenvalues. Then fixing a refinement $\mathcal{F}_\bullet = \mathcal{F}_{\mathrm{id}, \bullet}$ of $D_{\mathrm{crys}}(D)$, we associate the elements $w \in W(\mathrm{GL}_m, T_m) \cong \mathfrak{S}_m$ of the Weyl group with a refinement $\mathcal{F}_{w, \bullet}$ of $D_{\mathrm{crys}}(D)$ as follows: Take a sequence $\varphi_1, \dots, \varphi_m$ of Frobenius eigenvalues of $D_{\mathrm{crys}}(D)$ and its corresponding eigenvector $x_1, \dots, x_m$. Then a refinement $\mathcal{F}_{w, \bullet}$ of $D_{\mathrm{crys}}(D)$ is defined as

$$0 \subset (x_{w(1)}) \subset (x_{w(1)}, x_{w(2)}) \subset \dots \subset (x_{w(1)}, \dots, x_{w(m)}) = D_{\mathrm{crys}}(D),$$

where $w: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ is the corresponding function to $w \in W(\mathrm{GL}_m, T_m)$ by $W(\mathrm{GL}_m, T_m) \xrightarrow{\sim} \mathfrak{S}_m \xrightarrow{\sim} \mathrm{Aut}(\{1, \dots, m\})$. Write $\mathcal{P}_w$ for the corresponding triangulation of D to $\mathcal{F}_{w, \bullet}$ by Proposition 3.7.1 and $\mathcal{P} := \mathcal{P}_{\mathrm{id}}$. Write $x_{w,i} := x_{w(i)}$ and $\mathcal{F}_{w,i} := (x_{w,1}, \dots, x_{w,i}) \subset D_{\mathrm{crys}}(D)$ for $i \in [0, m]$ and $w \in W(\mathrm{GL}_m, T_m)$.

We also write $D_{w,i} := \left(\mathcal{F}_{w,i} \otimes_{K \otimes_{\mathbb{Q}_p} L} \mathcal{R}_L \left[\frac{1}{t} \right] \right) \cap D$. For elements $y_1, \dots, y_m \in D$ and a triangulation $\mathcal{P}$ of D , we call $(y_1, \dots, y_m)$ a basis of $\mathcal{P}$ if the filtration

$$0 \subset (y_1) \subset (y_1, y_2) \subset \dots \subset (y_1, \dots, y_m) = D$$

is a triangulation of D and the same as $\mathcal{P}$. By the classical correspondence between triangulations and refinements, for $i \in [1, m]$ and $w \in W(\mathrm{GL}_m, T_m)$, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (D_{w,i-1} \left[\frac{1}{t} \right])^\Gamma & \longrightarrow & (D_{w,i} \left[\frac{1}{t} \right])^\Gamma & \longrightarrow & ((D_{w,i}/D_{w,i-1}) \left[\frac{1}{t} \right])^\Gamma \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & D_{\mathrm{crys}}(D_{w,i-1}) & \longrightarrow & D_{\mathrm{crys}}(D_{w,i}) & \longrightarrow & D_{\mathrm{crys}}(D_{w,i}/D_{w,i-1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D_{w,i-1} \left[\frac{1}{t} \right] & \longrightarrow & D_{w,i} \left[\frac{1}{t} \right] & \longrightarrow & (D_{w,i}/D_{w,i-1}) \left[\frac{1}{t} \right] \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & D_{w,i-1} & \longrightarrow & D_{w,i} & \longrightarrow & D_{w,i}/D_{w,i-1} \longrightarrow 0, \end{array}$$

where the horizontal maps form exact sequences, the vertical maps between the second row and the third row are the inclusions $- \mapsto \left(- \otimes_{K_0 \otimes_{\mathbb{Q}_p} L} \mathcal{R}_L \left[\frac{1}{t} \right] \right)$ and the bottom vertical maps are the inclusions $- \mapsto \left(- \otimes_{\mathcal{R}_L} \mathcal{R}_L \left[\frac{1}{t} \right] \right)$.

Moreover, the bottom right module $D_{w,i}/D_{w,i-1}$ is the rank 1 (φ, Γ) -module that is generated by $(t^{\mathbf{k}_{w,i}} x_{w,i} \bmod D_{w,i-1})$ over $\mathcal{R}_L$ for some $(\mathbf{k}_{w,1}, \dots, \mathbf{k}_{w,m}) \in (\mathbb{Z}^\Sigma)^m$. Let $U_m^- \subset \mathrm{GL}_m$ be the subgroup of lower triangular unipotent matrices. Then there exists a matrix $\mathbf{u}_w = (u_{w,i,j})_{i,j} \in U_m^-(\mathcal{R}_L \left[\frac{1}{t}\right])$ such that $\{x'_{w,1}, \dots, x'_{w,m}\}$ where

$${}^t(x'_{w,1}, \dots, x'_{w,m}) = \mathbf{u}_w \mathbf{t}_w {}^t(x_{w,1}, \dots, x_{w,m})$$

is a basis of $\mathcal{P}_w$ when we see $x'_{w,i}, x_{w,i}$ to be column vectors relative to a fixed basis of $D_w \left[\frac{1}{t}\right]$ and set $\mathbf{t}_w := \mathrm{diag}(t^{\mathbf{k}_{w,1}}, \dots, t^{\mathbf{k}_{w,m}})$ by Lemma 3.7.3.

For $w, w' \in W(\mathrm{GL}_m, T_m)$, write $\mathbf{a}_{w,w'} = (a_{w,w',i,j})_{i,j} \in \mathrm{GL}_m(\mathcal{R}_L \left[\frac{1}{t}\right])$ for a transformation matrix from the basis of $\mathcal{P}_w$ to that of $\mathcal{P}_{w'}$. Note that $\mathbf{u}_w$ and $\mathbf{a}_{w,w'}$ is not unique in $\mathrm{GL}_m(\mathcal{R}_L \left[\frac{1}{t}\right])$ but is unique in $U_m^-(\mathcal{R}_L) \backslash \mathrm{GL}_m(\mathcal{R}_L \left[\frac{1}{t}\right]) / U_m^-(\mathcal{R}_L)$. A matrix $\mathbf{a}_{w,w'}$ is written in the form $\mathbf{u}_{w'} \mathbf{t}_{w'} w \mathbf{t}_w^{-1} \mathbf{u}_w^{-1}$ where $w \in W(\mathrm{GL}_m, T_m) = N_{\mathrm{GL}_m}(T_m) / C_{\mathrm{GL}_m}(T_m)$ is the matrix associated with the element of the Weyl group. For $1 \leq i < j \leq m$, we call an element $w \in W(\mathrm{GL}_m, T_m) \cong \mathfrak{S}_m$ is of type (i, j) if $w(l) = l$ for $l < i$ or $j < l$, $w(i) = j$ and $w(j) = i$ as an element of $\mathfrak{S}_m = \mathrm{Aut}(\{1, \dots, m\})$. Let $M_{i \times j}(A)$ be the additive abelian group of $(i \times j)$ -matrices with coefficients in A .

Lemma 3.8.6. *For an element $\sigma \in W(\mathrm{GL}_m, T_m)$ of type (i, j) and $w, w' \in W(\mathrm{GL}_m, T_m)$ such that $w\sigma = w'$, we may take $\mathbf{a}_{w,w'}$ as the form*

$$\begin{pmatrix} I_{i-1} & 0_{(i-1) \times (j-i+1)} & 0_{(i-1) \times (m-j)} \\ \mathbf{n}_{w,w'} & \mathbf{a}'_{w,w'} & 0_{(j-i+1) \times (m-j)} \\ 0_{(m-j) \times (i-1)} & 0_{(m-j) \times (j-i+1)} & I_{m-j} \end{pmatrix}$$

where $\mathbf{a}'_{w,w'} \in \mathrm{GL}_{j-i+1}(\mathcal{R}_L)$ and $\mathbf{n}_{w,w'} \in M_{(j-i+1) \times (i-1)}(\mathcal{R}_L)$. Moreover, writing

$$(z^{\mathbf{k}_{w,1}} \mathrm{unr}(\varphi_{w,1}), \dots, z^{\mathbf{k}_{w,m}} \mathrm{unr}(\varphi_{w,m})), (z^{\mathbf{k}_{w',1}} \mathrm{unr}(\varphi_{w',1}), \dots, z^{\mathbf{k}_{w',m}} \mathrm{unr}(\varphi_{w',m})),$$

for the parameters associated to the triangulations $\mathcal{P}_w, \mathcal{P}_{w'}$, and when we write $\mathbf{a}'_{w,w'} = (a'_{w,w',\alpha,\beta})_{\alpha,\beta \in [1, j-i+1]}$, we have $a'_{w,w',1, j-i+1} = t^{\mathbf{k}_{w,i} - \mathbf{k}_{w',j}}$.

PROOF. By the classical correspondence between triangulations and refinements, $D_{w,l} = D_{w',l}$ for $l < i$ or $j < l$. So the first part is shown.

For the last part, since matrices $\mathbf{u}_w, \mathbf{t}_w, w$ are in $U_m^- \mathrm{diag}(\mathrm{GL}_{i-1}, \mathrm{GL}_{j-i+1}, \mathrm{GL}_{m-j})$, it suffices to calculate the middle $(j-i+1) \times (j-i+1)$ -matrix $\mathbf{a}'_{w,w'}$. Write $\mathbf{u}'_w, \mathbf{t}'_w, w'$ for the factors of the middle $(j-i+1) \times (j-i+1)$ -matrices of $\mathbf{u}_w, \mathbf{t}_w, w$. We have

$$\begin{aligned} \mathbf{a}'_{w,w'} &= \mathbf{u}'_{w'} \mathbf{t}'_{w'} w' \mathbf{t}'_w{}^{-1} \mathbf{u}'_w{}^{-1} \\ &= \mathbf{u}'_{w'} \begin{pmatrix} 0 & 0_{1 \times (m-2)} & t^{\mathbf{k}_{w,i} - \mathbf{k}_{w',j}} \\ 0_{(m-2) \times 1} & *_{(m-2) \times (m-2)} & 0_{(m-2) \times 1} \\ t^{\mathbf{k}_{w,j} - \mathbf{k}_{w',i}} & 0_{1 \times (m-2)} & 0 \end{pmatrix} \mathbf{u}'_w{}^{-1} \\ &= \begin{pmatrix} 0 & 0_{1 \times (m-2)} & t^{\mathbf{k}_{w,i} - \mathbf{k}_{w',j}} \\ 0_{(m-2) \times 1} & *_{(m-2) \times (m-2)} & *_{(m-2) \times 1} \\ t^{\mathbf{k}_{w,j} - \mathbf{k}_{w',i}} & *_{1 \times (m-2)} & * \end{pmatrix} \mathbf{u}'_w{}^{-1} \\ &= \begin{pmatrix} * & *_{1 \times (m-2)} & t^{\mathbf{k}_{w,i} - \mathbf{k}_{w',j}} \\ *_{(m-2) \times 1} & *_{(m-2) \times (m-2)} & *_{(m-2) \times 1} \\ * & *_{1 \times (m-2)} & * \end{pmatrix}. \end{aligned}$$

□

- Definition 3.8.7.** (1) We say that a crystalline (φ, Γ) -module D of rank m is φ -generic if the Frobenius $\Phi = \phi^{[K_0: \mathbb{Q}_p]}$ on $D_{\mathrm{crys}}(D)$ has m distinct eigenvalues $(\varphi_1, \dots, \varphi_m)$ in L and $\varphi_i \varphi_j^{-1} \neq p^{[K_0: \mathbb{Q}_p]}$ for $i \neq j$.
- (2) We say that a crystalline (φ, Γ) -module D with a regular Hodge-Tate type is *noncritical* if a trianguline parameter associated with each refinement of $D_{\mathrm{crys}}(D)$ is noncritical.
- (3) We say that a crystalline (φ, Γ) -module D is *benign* if D has a regular Hodge-Tate type, noncritical and φ -generic.

Remark 3.8.8. In [48, Definition 5.5], φ -genericity is defined as the definition above excluding only the last condition. However, we use the definition in [45, §3.3] for the simplification of the notation.

Lemma 3.8.9. *Let D be a rank m trianguline (φ, Γ) -module and $\mathcal{P}$ be a triangulation of D that has the parameter $(\delta_1, \dots, \delta_m)$ such that $\delta_i = z^{\mathbf{k}_i} \mathrm{unr}(\varphi_i)$ for $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_m) \in (\mathbb{Z}^\Sigma)^m$ and $i \in [1, m]$, D' be a rank m (φ, Γ) -submodule of D , $\mathcal{P}'$ be the triangulation of D' induced from $\mathcal{P}$ and $(\delta'_1, \dots, \delta'_m)$ be the parameter of $\mathcal{P}'$ such that $\delta'_i = z^{\mathbf{k}'_i} \mathrm{unr}(\varphi_i)$ for $\mathbf{k}' = (\mathbf{k}'_1, \dots, \mathbf{k}'_m) \in (\mathbb{Z}^\Sigma)^m$ and $i \in [1, m]$.*

Write $S(D', D, \mathcal{P}) \subset \{1, \dots, m\}$ for the subset of element $l \in \{1, \dots, m\}$ such that $\delta_l \neq \delta'_l$. For each $i \in S(D' \subset D, \mathcal{P})$, assume that $\varphi_i \notin \{1, p^{[K_0: \mathbb{Q}_p]}\}$ and $\max(k_{i, \tau}, 1 - k'_{i, \tau}) > 0$ for all $\tau \in \Sigma$. Then the induced map $H^1(D') \rightarrow H^1(D)$ of the first cohomology modules is surjective.

PROOF. Let D_i/D_{i-1} and $(D_i \cap D')/(D_{i-1} \cap D')$ for $i \in [1, m]$. Since D is trianguline with parameter $(\delta_1, \dots, \delta_m)$ and D' is a submodule of D , it suffices to show that $H^1(\mathcal{R}_L(\delta'_i)) \rightarrow H^1(\mathcal{R}_L(\delta_i))$ is surjective as δ, δ' are φ -generic and the snake lemma.

If $i \notin S(D' \subset D, \mathcal{P})$, we obtain $\dim_L H^1(\mathcal{R}_L(z^{\mathbf{k}_i} \delta)) = \dim_L H^1(\mathcal{R}_L(\delta))$ by Lemma 3.3.1. Then the surjectivity of the map $H^1(\mathcal{R}_L(\delta'_i)) \rightarrow H^1(\mathcal{R}_L(\delta_i))$ is reduced to show that $H^0(\mathcal{R}_L(\delta_i)/t^{\mathbf{k}''_i} \mathcal{R}_L(\delta_i)) = 0$ for $i \in S(D' \subset D, \mathcal{P})$ and $\mathbf{k}'' := \mathbf{k} - \mathbf{k}' \in \mathbb{Z}_{>0}^\Sigma$ by the long exact sequence. It is reduced to show $H^0(t^{\mathbf{k}'''} \mathcal{R}_L(\delta_i)/t_\tau t^{\mathbf{k}'''} \mathcal{R}_L(\delta_i)) = 0$ for all $\tau \in \Sigma$ and $\mathbf{k}''' \in \mathbb{Z}_{\geq 0}^\Sigma$ such that $\mathbf{k}_{i, \tau} + \mathbf{k}'''_{i, \tau} \neq 0$, as well by dévissage. This follows from that $k_{i, \tau} > 0$ for all $\tau \in \Sigma$ and [53, Lemma 6.2.12]. $\square$

From here, we make the following assumption:

Assumption 3.8.10. There is an element $w \in W(\mathrm{GL}_m, T_m)$ such that the triangulation $\mathcal{P}_w$ has a noncritical parameter.

Note that Assumption 3.8.10 is satisfied whenever $K = \mathbb{Q}_p$.

From now, we replace the triangulation $\mathcal{P}$ with the $\mathcal{P}_w$ in Assumption 3.8.10. We take a basis $(y_1, \dots, y_m)$, $y_i \in D$ for $i \in [1, m]$ of $\mathcal{P}_{\mathrm{id}}$. Let $(e_{\alpha, \beta})_{\alpha, \beta \in [1, m]}$ be the associated standard basis of $\mathrm{ad}(D)$ with the basis $(y_1, \dots, y_m)$ of D . Let $\mathbf{a}_w := \mathbf{a}_{\mathrm{id}, w}$. Then we may translate $\mathrm{ad}_{\mathcal{P}_w}(D)$ for $w \in W(\mathrm{GL}_m, T_m)$ in terms of the standard basis. We have

$$\begin{aligned} \mathrm{ad}_{\mathcal{P}}(D) &= \langle e_{\alpha, \beta} \mid 1 \leq \beta < \alpha \leq m \rangle =: \mathfrak{b}_{\mathcal{P}}^- \subset \mathrm{ad}(D), \\ \mathrm{ad}_{\mathcal{P}_w}(D) &= \mathbf{a}_w(\mathfrak{b}_{\mathcal{P}}^-). \end{aligned}$$

Let $\sigma_{i, j} \in W(\mathrm{GL}_m, T_m) \cong \mathfrak{S}_m$ be the (i, j) -transposition for $1 \leq i, j \leq m$, $i \neq j$ and $W' \subset W(\mathrm{GL}_m, T_m)$ be the subset of $\sigma_{i, j}$. For $\mathbf{a}, \mathbf{b} \in \mathrm{ad}(D)$, we write $\mathrm{ad}(\mathbf{a})(\mathbf{b}) := \mathbf{a} \mathbf{b} \mathbf{a}^{-1}$.

Lemma 3.8.11. *Let D' be the image of the map*

$$\bigoplus_{w \in W(\mathrm{GL}_m, T_m)} \mathrm{ad}_{\mathcal{P}_w}(D) \rightarrow \mathrm{ad}(D)$$

of (φ, Γ) -modules over $\mathcal{R}_L$. Then under Assumption 3.8.10, the saturation of $D' \subset \mathrm{ad}(D)$ is $\mathrm{ad}(D)$. Moreover, the inclusion of (φ, Γ) -modules $D' \subset \mathrm{ad}(D)$ satisfies the conditions in Lemma 3.8.9.

PROOF. We prove the assertion for the map

$$\bigoplus_{w \in W'} \mathrm{ad}_{\mathcal{P}_w}(D) \rightarrow \mathrm{ad}(D)$$

instead. As $\mathbf{a}'_{\sigma_{i,j}} \in \mathrm{GL}_{j-i+1}(\mathcal{R}_L)$, there exists $h = h(\tau, i, j) \in [0, j-i]$ such that $\mathbf{a}_{i+h,j} = \mathbf{a}'_{h+1,j-i+1} \in \mathcal{R}_L \setminus t_\tau \mathcal{R}_L$ for each $\tau \in \Sigma$ and fix such $h(\tau, i, j)$ for $\tau \in \Sigma$ and $1 \leq i < j \leq m$. The element $\mathrm{ad}(\mathbf{a}_{\sigma_{i,j}})(e_{j,i+h_\tau})$ is of the form

$$\theta e_{i,j} \bmod \langle e_{\alpha,\beta} \mid 1 \leq \alpha \leq i, j \leq \beta \leq i, (\alpha, \beta) \neq (i, j) \rangle$$

where $\theta \in (t_\tau^{\mathbf{k}_{\mathrm{id},i} - \mathbf{k}_{\sigma_{i,j},j}} \mathcal{R}) \setminus (t_\tau^{\mathbf{k}_{\mathrm{id},i} - \mathbf{k}_{\sigma_{i,j},j} + 1} \mathcal{R})$ by Lemma 3.8.6.

Then we can show that the saturation of the composition of the map in the assertion and the projection from D into

$$\mathfrak{b}_{\mathcal{P},l}^- := \langle e_{\alpha,\beta} \mid 1 \leq \alpha, \beta \leq m, \beta - \alpha \leq l \rangle$$

is $\mathfrak{b}_{\mathcal{P},l}^-$ for $1 \leq l \leq m-1$ by induction. When $l = m-1$, it is just as the assertion of the lemma. The last part of the Lemma follows by the fact that $(t_\tau) + (t_{\tau'}) = \mathcal{R}_L$ for $\tau \neq \tau'$ and Corollary 3.3.2. $\square$

We reprove [45, Theorem 3.11] under Assumption 3.8.10.

Proposition 3.8.12. *Let D be a φ -generic rank m crystalline (φ, Γ) -module of regular Hodge-Tate type. Then under Assumption 3.8.10, the map*

$$\bigoplus_{w \in W(\mathrm{GL}_m, T_m)} TX_{D, \mathcal{P}_w} \rightarrow TX_D$$

is surjective.

PROOF. Let D' be the image of the map

$$\bigoplus_{w \in W(\mathrm{GL}_m, T_m)} \mathrm{ad}_{\mathcal{P}_w}(D) \rightarrow \mathrm{ad}(D).$$

Then the inclusion $D' \subset \mathrm{ad}(D)$ satisfies the condition of Lemma 3.8.9 by Lemma 3.8.11 and we have $H^1(D') \cong H^1(\mathrm{ad}(D))$. We also obtain that $H^2(E) = 0$ for all (φ, Γ) -submodules E of the trianguline (φ, Γ) -module $\mathrm{ad}(D)$ because of the H^2 of the trianguline parameter of $\mathrm{ad}(D)$ is 0 by the φ -genericity of D and the right exactness of H^2 .

Let

$$D'' := \ker \left(\bigoplus_{w \in W(\mathrm{GL}_m, T_m)} \mathrm{ad}_{\mathcal{P}_w}(D) \rightarrow \mathrm{ad}(D) \right).$$

It follows that the map

$$H^1 \left(\bigoplus_{w \in W(\mathrm{GL}_m, T_m)} \mathrm{ad}_{\mathcal{P}_w}(D) \right) \rightarrow H^1(\mathrm{ad}(D))$$

is surjective by the long exact sequence associated with the exact sequence

$$0 \rightarrow D'' \rightarrow \bigoplus_{w \in W(\mathrm{GL}_m, T_m)} \mathrm{ad}_{\mathcal{P}_w}(D) \rightarrow D' \rightarrow 0$$

of (φ, Γ) -modules over $\mathcal{R}_L$ and $H^2(D'') = 0$. The result follows by Lemma 3.8.5, (1). $\square$

From here we consider the case of $G = \mathrm{GSp}_{2n}$ over $k = K$. We write $B \subset \mathrm{GSp}_{2n}$ for the standard Borel subgroup and $T \subset B$ for the maximal split torus. Let D be a frameable (φ, Γ) -module with GSp_{2n} -structure over $\mathcal{R}_L$ and $\mathcal{P}$ be a triangulation of D .

Let $r_{\mathrm{std}}: \mathrm{GSp}_{2n} \rightarrow \mathrm{GL}_{2n}$ denote the standard representation of GSp_{2n} . It is the embedding of regular type. Let $\mathbf{k} \in (\mathbb{Z}^\Sigma)^{2n}$ be a strictly dominant regular Hodge-Tate type such that there is a character $\lambda \in X^*(T)$ over L such that the element $r_{\mathrm{std},*}(\lambda) \in X^*(T_{2n}) \cong (\mathbb{Z}^\Sigma)^{2n}$ is $\mathbf{k}$.

We write $\tilde{D} := r_{\mathrm{std},*}(D)$. We assume that D is crystalline, $\tilde{D}$ is φ -generic with regular Hodge-Tate type $\mathbf{k}$ and there exists a refinement of $D_{\mathrm{crys}}(D)$. Moreover, we assume that there is a triangulation $\mathcal{P}$ which has a noncritical parameter and fix such $\mathcal{P}_{\mathrm{id}} = \mathcal{P}$.

For the corresponding refinement $\mathfrak{P} = \mathfrak{P}_{\mathrm{id}}$ of $D_{\mathrm{crys}}(D)$ to $\mathcal{P}$, we associate the elements $w \in W(\mathrm{GSp}_{2n}, T)$ of the Weyl group with a refinement $\mathfrak{P}_w$ of $D_{\mathrm{crys}}(D)$ and write $\mathcal{P}_w$ for the corresponding triangulation of D by Proposition 3.7.1. We also write $\mathcal{P}_w$ for the induced triangulation of $\tilde{D}$ from the triangulation $\mathcal{P}_w$ of D . It follows that $\mathrm{ad}_{\mathcal{P}_w}(D) = \mathrm{ad}(D) \cap \mathrm{ad}_{\mathcal{P}_w}(\tilde{D})$ by Proposition 3.8.4, (3) and $\mathrm{GSp}_{2n} \cap B_{2n} = B$.

Since we have the natural injective map $W(\mathrm{GSp}_{2n}, T) \hookrightarrow W(\mathrm{GL}_{2n}, T_{2n})$ of the Weyl groups, we have to show Lemma 3.8.11 restricting $W(\mathrm{GL}_{2n}, T_{2n})$ to $W(\mathrm{GSp}_{2n}, T)$ for obtaining a result of Proposition 3.8.12 in case of GSp_{2n} .

Lemma 3.8.13. *The saturation of the image of the map*

$$\bigoplus_{w \in W(\mathrm{GSp}_{2n}, T)} \mathrm{ad}_{\mathcal{P}_w}(D) \rightarrow \mathrm{ad}(D)$$

of (φ, Γ) -modules over $\mathcal{R}_L$ is $\mathrm{ad}(D)$. Moreover, this embedding of (φ, Γ) -modules satisfies the conditions in Lemma 3.8.9.

PROOF. By the above argument, the map in the assertion is the same as

$$\bigoplus_{w \in W(\mathrm{GSp}_{2n}, T) \subset W(\mathrm{GL}_{2n}, T_{2n})} (\mathrm{ad}(D) \cap \mathrm{ad}_{\mathcal{P}_w}(\tilde{D})) \rightarrow \mathrm{ad}(D).$$

Let I_{sat} be the image of the map. Calculating

$$\mathrm{ad}(\mathbf{a}_{\sigma_{i,j}\sigma_{2n+1-j,2n+1-i}})(e_{j,i} - e_{2n+1-i,2n+1-j})$$

for $1 \leq i < j \leq n$ instead of $\mathrm{ad}(\mathbf{a}_{\sigma_{i,j}})(e_{j,i})$ for $1 \leq i < j \leq n$ in the proof of Lemma 3.8.11, we may view I_{sat} contains the opposite Siegel parabolic submodule

$$\mathfrak{p}^- := \begin{pmatrix} *_{n \times n} & 0_{n \times n} \\ *_{n \times n} & *_{n \times n} \end{pmatrix} \cap \mathrm{ad}(D).$$

Then, in a similar way considering $\mathrm{ad}(\mathbf{a}_{\sigma_{n,n+1}})(\mathfrak{p}^-)$, we have

$$\langle e_{n,n+1}, e_{l,n+1} - e_{n,2n+1-l} \mid l \in [1, n-1] \rangle \subset I_{\mathrm{sat}}.$$

Finally considering

$$\mathrm{ad}(\mathbf{a}_{\sigma_{k,n}\sigma_{n+1,2n+1-k}})(\langle e_{n,n+1}, e_{l,n+1} - e_{n,2n+1-l} \mid l \in [1, n-1] \rangle),$$

we have

$$\begin{pmatrix} 0_{n \times n} & *_{n \times n} \\ 0_{n \times n} & 0_{n \times n} \end{pmatrix} \cap \text{ad}(D) \subset I_{\text{sat}}.$$

Consequently, we obtain $I_{\text{sat}} = \text{ad}(D)$. For the last part, additionally calculating the images of $\text{ad}_{\mathcal{P}_\sigma}(D) \rightarrow \text{ad}(D)$ for $\sigma = \sigma_{c, 2n+1-c}$, $c \in [1, n]$ as in Lemma 3.8.11, we can check that the embedding of (φ, Γ) -modules satisfies the conditions in Lemma 3.8.9 by the constructions and the calculations above. $\square$

Corollary 3.8.14. *Let D be a crystalline (φ, Γ) -module with GSp_{2n} -structure over $\mathcal{R}_L$ such that $\tilde{D}$ is φ -generic with regular Hodge-Tate type and assume that there exists a noncritical refinement of $D_{\text{crys}}(D)$. Then the map*

$$\bigoplus_{w \in W(\text{GSp}_{2n}, T)} TX_{D, \mathcal{P}_w} \rightarrow TX_D$$

is surjective.

PROOF. The map of (φ, Γ) -modules with GSp_{2n} -structures

$$\bigoplus_{w \in W(\text{GSp}_{2n}, T)} \text{ad}_{\mathcal{P}_w}(D) \rightarrow \text{ad}(D)$$

in the Lemma 3.8.13 satisfies the condition of Lemma 3.8.9 by its construction. Thus the result follows by Lemma 3.8.9. $\square$

3.9. Locally irreducibility and accumulation property

In this section, we prove locally irreducibility and accumulation property of the trianguline deformation spaces for GSp_{2n} .

3.9.1. Almost de Rham representations with G -structures. Let us recall the notations in [18, §3]. Let B_{dR} , B_{dR}^+ be the usual de Rham period rings of Fontaine with the Gal_K -action and the natural topology and all finite type modules over these rings are endowed with the natural topology (cf. [41, §3.2]). Let $\text{Rep}_{B_{\text{dR}}}(\text{Gal}_K)$ (resp. $\text{Rep}_{B_{\text{dR}}^+}(\text{Gal}_K)$) denote the category of B_{dR} -representations (resp. B_{dR}^+ -representations) of Gal_K (cf. [41, §3]). Let W be an object of $\text{Rep}_{B_{\text{dR}}}(\text{Gal}_K)$. It follows that W contains a B_{dR}^+ -lattice stable under Gal_K . We say that W is *almost de Rham* if it contains a Gal_K -stable B_{dR}^+ -lattice W^+ such that the Sen weights of the C -representation W^+/tW^+ are all in $\mathbb{Z}$ (cf. [41, §3.7]).

Let B_{pdR}^+ be the algebra $B_{\text{dR}}^+[\log(t)]$ and $B_{\text{pdR}} := B_{\text{dR}} \otimes_{B_{\text{dR}}^+} B_{\text{pdR}}^+$. The group Gal_K acts on these algebras as $g(\log(t)) = \log(t) + \log(\epsilon(g))$. Let $\nu_{B_{\text{pdR}}}$ be the unique B_{dR} -derivation of B_{pdR} such that $\nu_{B_{\text{pdR}}}(\log(t)) = -1$, which preserves the subalgebra $B_{\text{pdR}}^+ \subset B_{\text{pdR}}$. We set $D_{\text{pdR}}(W) := (B_{\text{pdR}} \otimes_{B_{\text{dR}}} W)^{\text{Gal}_K}$, which is a finite dimensional K -vector space of dimension $\leq \dim_{B_{\text{dR}}} W$. It follows that W is almost de Rham if and only if $\dim_K D_{\text{pdR}}(W) = \dim_{B_{\text{dR}}} W$ by [41, Théorème 4.1 (2)].

We say that W is *de Rham* if $\dim_K W^{\text{Gal}_K} = \dim_{B_{\text{dR}}} W$. If W is de Rham, W is also almost de Rham. We write $\text{Rep}_{\text{pdR}}(\text{Gal}_K)$ for the category of almost de Rham B_{dR} -representations, which is a Tannakian subcategory of $\text{Rep}_{B_{\text{dR}}}(\text{Gal}_K)$ and is stable under kernel, cokernel and extensions (cf. [41, §3.7]).

Let A be a finite dimensional $\mathbb{Q}_p$ -algebra. An $A \otimes_{\mathbb{Q}_p} B_{\text{dR}}$ -representations of Gal_K is a B_{dR} -representation W of Gal_K together with a morphism of $\mathbb{Q}_p$ -algebras

$$A \rightarrow \text{End}_{\text{Rep}_{B_{\text{dR}}}(\text{Gal}_K)}(W)$$

which makes W a finite free $A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ -module. Let $\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K)$ be the category of $A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ -representations of Gal_K . We similarly define $A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$ -representations of Gal_K and their category $\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+}(\mathrm{Gal}_K)$. We say that an $A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ -representation of Gal_K is *almost de Rham* if the underlying B_{dR} -representation is. Write $\mathrm{Rep}_{\mathrm{pdR},A}(\mathrm{Gal}_K)$ for the category of almost de Rham representations of Gal_K .

Let (G, B, T) be a split reductive group of regular type over K . For an abelian tensor category $\mathcal{A}$, let $G\text{-}\mathcal{A}$ denote the category of exact tensor functors $\omega: \mathrm{Rep}_K(G) \rightarrow \mathcal{A}$. For a exact tensor functor $F: \mathcal{A} \rightarrow \mathcal{A}'$ between abelian tensor categories, we also write $F: G\text{-}\mathcal{A} \rightarrow G\text{-}\mathcal{A}'$ the induced functor. An $A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ -representation with G -structure of Gal_K is an object of $G\text{-}\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K)$ and a $A \otimes_{\mathbb{Q}_p} K$ -representation with G -structure of $\mathbb{G}_a$ is an object of $G\text{-}\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} K}(\mathbb{G}_a)$. We say that $\omega \in \mathrm{ob}(G\text{-}\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K))$ is almost de Rham if $\omega(V)$ is almost de Rham for all $V \in \mathrm{Rep}_K(G)$. Write $G\text{-}\mathrm{Rep}_{\mathrm{pdR},A}(\mathrm{Gal}_K)$ for the category of almost de Rham $A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ -representations with G -structures of Gal_K .

Lemma 3.9.1. *The functor D_{pdR} induces an equivalence of categories between $G\text{-}\mathrm{Rep}_{\mathrm{pdR},A}(\mathrm{Gal}_K)$ and $G\text{-}\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} K}(\mathbb{G}_a)$.*

PROOF. This follows from [18, Lemma 3.1.4] and Tannakian formalism. $\square$

3.9.2. Trianguline deformation functors of almost de Rham representations with G -structures. Fix an almost $L \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ -representation with G -structure W . A *triangulation* $\mathcal{P}_W$ of W is a B -torsor $\mathrm{Rep}_K(B) \rightarrow \mathrm{Rep}_{L \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K)$ such that the composition with $\mathrm{Rep}_K(G) \rightarrow \mathrm{Rep}_K(B)$ is the same as the W . We assume that the underlying G -torsor over $L \otimes_{\mathbb{Q}_p} K$ of $D_{\mathrm{pdR}}(W)$ is trivial replacing K, L with sufficiently large finite extensions if necessary. Let $\omega_{\mathrm{st}}: \mathrm{Rep}_K(G) \rightarrow \mathrm{Bun}(L \otimes_{\mathbb{Q}_p} K)$ be the standard fiber functor. We also fix an isomorphism $\alpha: \omega_{\mathrm{st}} \xrightarrow{\sim} D_{\mathrm{pdR}}(W)$.

Definition 3.9.2. (1) Let X_W be the groupoid over $\mathcal{C}_L$ as follows:

- The objects of $X_W(A)$ are pairs (W_A, ι_A) where A is an object of $\mathcal{C}_L$, W_A is an object of $G\text{-}\mathrm{Rep}_{\mathrm{pdR},A}(\mathrm{Gal}_K)$ and $\iota_A: W_A \times_A L \xrightarrow{\sim} W$.
- A morphism $(W_A, \iota_A) \rightarrow (W_{A'}, \iota_{A'})$ is a map $A \rightarrow A'$ in $\mathcal{C}_L$ and an isomorphism $W_A \otimes_A A' \xrightarrow{\sim} W_{A'}$ compatible with the morphisms $\iota_A, \iota_{A'}$.

(2) Let $X_W^\square$ be the groupoid over $\mathcal{C}_L$ as follows:

- The objects of $X_W^\square(A)$ are (W_A, ι_A, α_A) with (W_A, ι_A) an object of $X_W(A)$ and $\alpha_A: \omega_{\mathrm{st}} \otimes_{\mathbb{Q}_p} A$ such that the following diagram commutes:

$$\begin{array}{ccc} \omega_{\mathrm{st}} \otimes_{\mathbb{Q}_p} A & \xrightarrow{1 \otimes \alpha_A} & D_{\mathrm{pdR}}(W_A) \otimes_A L \\ \parallel & & \downarrow \cong \\ \omega_{\mathrm{st}} \otimes_{\mathbb{Q}_p} A & \xrightarrow{\alpha} & D_{\mathrm{pdR}}(W). \end{array}$$

- A morphism $(W_A, \iota_A, \alpha_A) \rightarrow (W_{A'}, \iota_{A'}, \alpha_{A'})$ is a morphism $(W_A, \iota_A) \rightarrow (W_{A'}, \iota_{A'})$ in X_W such that the following diagram commutes:

$$\begin{array}{ccc} (\omega_{\mathrm{st}} \otimes_{\mathbb{Q}_p} A) \otimes_A A' & \xrightarrow{1 \otimes \alpha_A} & D_{\mathrm{pdR}}(W_A) \otimes_A A' \\ \parallel & & \downarrow \cong \\ \omega_{\mathrm{st}} \otimes_{\mathbb{Q}_p} A' & \xrightarrow{\alpha_{A'}} & D_{\mathrm{pdR}}(W_{A'}). \end{array}$$

- (3) For a fixed triangulation $\mathcal{P}_W$ of W , let $X_{W, \mathcal{P}_W}$ be the groupoid over $\mathcal{C}_L$ as follows:
- The objects of $X_{W, \mathcal{P}_W}(A)$ are $(W_A, \mathcal{P}_{W,A}, \iota_A)$ where (W_A, ι_A) is an object of $X_W(A)$ and $\mathcal{P}_{W,A}$ is a triangulation of W_A such that ι_A induces an isomorphism $\mathcal{P}_{W,A} \otimes_A L \xrightarrow{\sim} \mathcal{P}_W$.
 - The morphisms are the morphisms in X_W which induce an isomorphism $\mathcal{P}_{W,A} \otimes_A A' \xrightarrow{\sim} \mathcal{P}_{W,A'}$.
- (4) For a fixed triangulation $\mathcal{P}_W$ of W , let $X_{W, \mathcal{P}_W}^\square := X_{W, \mathcal{P}_W} \times_{X_W} X_W^\square$.

We set an algebraic group $\underline{G}$ over L as

$$\underline{G} := \text{Spec } L \times_{\text{Spec } \mathbb{Q}_p} \text{Res}_{K/\mathbb{Q}_p}(\text{GL}_{n,K}) \cong (\text{GL}_{n,L})^\Sigma.$$

Similarly we define $\underline{B}$, $\underline{T}$. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ be the Lie algebra over L of respectively $\underline{G}$, $\underline{B}$, $\underline{T}$. For a finite $\mathbb{Q}_p$ -algebra A , we write $\mathfrak{g}(A) := \mathfrak{g} \otimes_{\mathbb{Q}_p} A$ and do the same for $\mathfrak{b}$, $\mathfrak{t}$. Let $\widetilde{\mathfrak{g}}$ be the L -scheme defined by

$$\widetilde{\mathfrak{g}} := \{(g\underline{B}, \psi) \in (\underline{G}/\underline{B}) \times \mathfrak{g} \mid \text{ad}(g^{-1})\psi \in \mathfrak{b}\} \subset (\underline{G}/\underline{B}) \times \mathfrak{g}.$$

We write $q: \widetilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ for the Grothendieck's simultaneous resolution of singularities, which is defined by $(g\underline{B}, \psi) \mapsto \psi$. Let X be the L -scheme defined by

$$X := \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} = \{(g_1\underline{B}, g_2\underline{B}, \psi) \in (\underline{G}/\underline{B}) \times (\underline{G}/\underline{B}) \times \mathfrak{g} \mid \text{ad}(g_1^{-1})\psi \in \mathfrak{b}, \text{ad}(g_2^{-1})\psi \in \mathfrak{b}\},$$

where the fiber product is taken by the map q .

Lemma 3.9.3. *Let E be a field of characteristic 0 and A be a finite dimensional $\mathbb{Q}_p$ -algebra. There is a natural bijection between the set of homomorphisms of A -group schemes $\mathbb{G}_{a,A} \rightarrow G_A$ and the set of nilpotent elements of $\text{Lie}_A(G_A)$.*

Moreover, the full subcategory of the objects, whose underlying torsor are trivial, of $G\text{-Rep}_A(\mathbb{G}_a)$ is equivalent to the category of the pair (G_A, ν_A) where $\nu_A \in \text{Lie}_A(G_A)$ is a nilpotent element.

PROOF. This follows from the preceding argument of [18, Proposition 3.1.1], the construction of a quasi-inverse of the equivalence in Lemma 3.9.1 (cf. [18, Corollary 3.1.2]) and Lemma 3.8.1. $\square$

Let W be an almost de Rham $L \otimes_{\mathbb{Q}_p} B_{\text{dR}}$ -representation with G -structure and $\mathcal{P}_W$ be a triangulation of W . If (W_A, ι_A) is an object of $X_W(A)$ for $A \in \mathcal{C}_L$, let $\nu_{W_A} := \nu_{D_{\text{pdR}}(W_A)}$ denote the nilpotent endomorphism of $D_{\text{pdR}}(W_A)$ corresponding by Lemma 3.9.3 to the action of $\mathbb{G}_a$, which is given by Lemma 3.9.1. If (W_A, ι_A, α_A) is an object of $X_W^\square(A)$, we define $N_{W_A} \in \mathfrak{g}(A)$ as the matrix of $\alpha_A^{-1} \circ \nu_{W_A} \circ \alpha_A$ in the canonical basis of ω_{st} . In the case $A = L$, we simply write $N_W = N_{W_L}$.

Let $\widehat{\mathfrak{g}}$ (resp. $\widehat{\mathfrak{t}}$) denote the completion of $\mathfrak{g}$ (resp. $\mathfrak{t}$) at the point $N_W \in \mathfrak{g}(L)$ (resp. $0 \in \mathfrak{t}$). Note that the set of triangulations of $\omega_{\text{st}, A}: \text{Rep}_K(G) \rightarrow \text{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\text{dR}}}(\text{Gal}_K)$ is bijective to the subset of elements of $\{gBg^{-1} \mid g \in \omega_{\text{st}}^{\text{geom}}(A \otimes_{\mathbb{Q}_p} B_{\text{dR}})\}$, where $\omega_{\text{st}}^{\text{geom}}$ is the underlying scheme of ω_{st} naturally isomorphic to the group scheme $G_{A \otimes_{\mathbb{Q}_p} B_{\text{dR}}}$, which are invariant in the action of Gal_K .

It is also bijective to the subset of elements of $(G/B)(A \otimes_{\mathbb{Q}_p} B_{\text{dR}})$, which are invariant in the action of Gal_K , by $gBg^{-1} \mapsto gB$. let $\widehat{\widetilde{\mathfrak{g}}}$ denote the completion of $\widetilde{\mathfrak{g}}$ at the point $(\alpha^{-1}(\mathcal{P}_W), N_W) \in \widehat{\widetilde{\mathfrak{g}}}(L)$, where $\alpha^{-1}(\mathcal{P}_W) \in (\underline{G}/\underline{B})(L)$ is the object corresponding to the triangulation $\mathcal{P}_W$. If $A \in \mathcal{C}_L$, we similarly define $\alpha_A^{-1}(\mathcal{P}_{W,A})$ for triangulations $\mathcal{P}_W$ of $D_{\text{pdR}}(D_A)$. For a groupoid X over $\mathcal{C}_L$, let $|X|$ denote the associated functor with X .

Corollary 3.9.4. (1) The groupoid $X_W^\square$ over $\mathcal{C}_L$ is pro-representable and the functor

$$(W_A, \iota_A, \alpha_A) \mapsto N_{W_A}$$

induces an isomorphism between $X_W^\square$ and $\widehat{\mathfrak{g}}$.

(2) The groupoid $X_{W, \mathcal{P}_W}^\square$ over $\mathcal{C}_L$ is pro-representable and the functor

$$(\mathcal{P}_{W,A}, W_A, \iota_A, \alpha_A) \mapsto (\alpha_A^{-1}(\mathcal{P}_{W,A}), N_{W_A})$$

induces an isomorphism between $X_{W, \mathcal{P}_W}^\square$ and $\widehat{\mathfrak{g}}$.

PROOF. In the case of $G = \mathrm{GL}_m$, we have $X_W^\square = |X_W^\square|$ and $X_{W, \mathcal{P}_W}^\square = |X_{W, \mathcal{P}_W}^\square|$ by the preceding argument of [18, Corollary 3.6]. For a general G , considering the pushout $r_*(D)$ of D by a faithful representation $G \rightarrow \mathrm{GL}_m$, we also have the similar two equations as above. Then the two pro-representabilities follows from Lemma 3.8.1 and Lemma 3.9.1. $\square$

3.9.3. Formally smoothness of the functor W_{dR} . Let A be a finite dimensional $\mathbb{Q}_p$ -algebra. For an object $\mathcal{M}$ of $\Phi\Gamma(\mathcal{R}_A[\frac{1}{t}])$ and a $\mathcal{R}_A$ -lattice D of $\mathcal{M}$ stable under φ, Γ , there exists covariant functors

$$\begin{aligned} W_{\mathrm{dR}}^+ : \Phi\Gamma(\mathcal{R}_A) &\rightarrow \mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+}(\mathrm{Gal}_K) \quad D \mapsto W_{\mathrm{dR}}^+(D) \\ W_{\mathrm{dR}} : \Phi\Gamma(\mathcal{R}_A[\frac{1}{t}]) &\rightarrow \mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K) \quad \mathcal{M} \mapsto W_{\mathrm{dR}}(\mathcal{M}), \end{aligned}$$

which are induced from the functors in the case of $A = \mathbb{Q}_p$ in [8, Proposition 2.2.6 (2)], constructed by [18, Lemma 3.3.5].

Let $\mathcal{M}$ be an object of $G\text{-}\Phi\Gamma(\mathcal{R}_A[\frac{1}{t}])$ and D be an object of $G\text{-}\Phi\Gamma(\mathcal{R}_A)$ whose composition with the base change functor $\Phi\Gamma(\mathcal{R}_A) \rightarrow \Phi\Gamma(\mathcal{R}_A[\frac{1}{t}])$ of (φ, Γ) -modules is isomorphic to $\mathcal{M}$. It follows that the functors

$$\begin{aligned} W_{\mathrm{dR}}^+ : G\text{-}\Phi\Gamma(\mathcal{R}_A) &\rightarrow G\text{-}\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+}(\mathrm{Gal}_K) \\ W_{\mathrm{dR}} : G\text{-}\Phi\Gamma(\mathcal{R}_A[\frac{1}{t}]) &\rightarrow G\text{-}\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K), \end{aligned}$$

are induced by ones constructed above since W_{dR}^+ is an exact tensor functor, that is seen by the construction. Since B_{dR}^+ is a complete discrete valuation ring whose residue field is algebraically closed and $W_{\mathrm{dR}}(\mathcal{M})$ is defined over B_{dR}^+ as $W_{\mathrm{dR}} = W_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$, the underlying torsors of the objects $W_{\mathrm{dR}}^+(D)$ and $W_{\mathrm{dR}}(\mathcal{M})$ are trivial by [24, Proposition 6.1.1].

We define a (φ, Γ) -module with G -structure $\mathcal{M}$ over $\mathcal{R}_A[\frac{1}{t}]$, its *triangulation* $\mathcal{P}_{\mathcal{M}}$ and its *parameter*, and when $A = L$ the groupoid $X_{\mathcal{M}, \mathcal{P}_{\mathcal{M}}}$ of deformations over $\mathcal{C}_L$ in the same way as those over $\mathcal{R}_A$ for an affinoid algebra A over $\mathbb{Q}_p$, especially in the case of A is an object of $\mathcal{C}_L$. Note that a parameter of a fixed triangulation over $\mathcal{R}_A[\frac{1}{t}]$ is not unique.

Let $\mathcal{T}_0$ be the Zariski open subspace of the rigid space $\mathcal{T}$, which is the complement of the characters $z^{\mathbf{k}, \epsilon \cdot z^{\mathbf{k}}}$ where $\mathbf{k} \in \mathbb{Z}^\Sigma$, and $\mathcal{T}_0^n$ be the Zariski open subspace of the rigid space $\mathcal{T}^d$, which is the complement of the characters $(\delta_1, \dots, \delta_d)$ such that $\delta_i \delta_j^{-1} \in \mathcal{T}_0$ for $i \neq j$. For a fixed embedding $r : G \rightarrow \mathrm{GL}_m$ of regular type, we define $\mathcal{T}_{T,0}$ be the inverse image of $\mathcal{T}_0^m \subset \mathcal{T}^m$ by the map $\mathcal{T}_T \rightarrow \mathcal{T}_{T^m} = \mathcal{T}^m$.

Let $\mathcal{M}$ be a trianguline (φ, Γ) -module with G -structure over $\mathcal{R}_L[\frac{1}{t}]$, $\mathcal{P}_{\mathcal{M}}$ be a triangulation of $\mathcal{M}$ with a parameter $\underline{\delta} := (\delta_i)_{i \in [1, d]} \in (\delta_1, \dots, \delta_d) \in \mathcal{T}^d$, $W := W_{\mathrm{dR}}(\mathcal{M})$, $\mathcal{P}_W$ be the induced triangulation $W_{\mathrm{dR}}(\mathcal{M})$. Arguing as [18, Corollary 3.3.9], we obtain a morphism of groupoids over $\mathcal{C}_L$:

$$X_{\mathcal{M}, \mathcal{P}_{\mathcal{M}}} \rightarrow \widehat{\mathcal{T}}_{\underline{\delta}}^n \times_{\widehat{\mathfrak{t}}} X_{W, \mathcal{P}_W}.$$

Proposition 3.9.5. *Assume the parameter $\underline{\delta}$ is locally algebraic and in $\mathcal{T}_{T,0}$. Then the morphism*

$$X_{\mathcal{M}, \mathcal{P}_{\mathcal{M}}} \rightarrow \widehat{\mathcal{T}}_{\underline{\delta}}^n \times_{\mathfrak{t}} X_{W, \mathcal{P}_W}$$

of groupoids over $\mathcal{C}_L$ is formally smooth.

PROOF. Let $A \twoheadrightarrow A'$ be a surjective map in $\mathcal{C}_L$, $x_{A'} = (\mathcal{M}_{A'}, \mathcal{P}_{\mathcal{M}_{A'}}, \pi_{A'})$ be an object of $X_{\mathcal{M}, \mathcal{P}_{\mathcal{M}}}(A')$, $y_{A'} = (\underline{\delta}_{A'}, W_{A'}, \mathcal{P}_{W_{A'}}, \iota_{A'})$ be its image in $(\widehat{\mathcal{T}}_{\underline{\delta}}^n \times_{\mathfrak{t}} X_{W, \mathcal{P}_W})(A')$. Let $y_A = (\underline{\delta}_A, W_A, \mathcal{P}_{W_A}, \iota_A)$ be an object of $(\widehat{\mathcal{T}}_{\underline{\delta}}^n \times_{\mathfrak{t}} X_{W, \mathcal{P}_W})(A)$ such that $\underline{\delta}_A = \underline{\delta}_B$ modulo $\ker(A \rightarrow A')$ and $A' \otimes_A (W_A, \mathcal{P}_{W_A}, \iota_A) = (W_{A'}, \mathcal{P}_{W_{A'}}, \iota_{A'})$. We prove that there exists an object $x_A = (\mathcal{M}_A, \mathcal{P}_{\mathcal{M}_A}, \pi_A)$ in $X_{\mathcal{M}, \mathcal{P}_{\mathcal{M}}}(A)$ whose image in $X_{\mathcal{M}, \mathcal{P}_{\mathcal{M}}}(A')$ is $x_{A'}$ and whose image in $(\widehat{\mathcal{T}}_{\underline{\delta}}^n \times_{\mathfrak{t}} X_{W, \mathcal{P}_W})(A)$ is isomorphic to y_A .

Since the embedding $G \rightarrow \mathrm{GL}_m$ is of regular type, we use the notations and the constructions in Proposition 3.5.10. By induction on $i \in [0, s-1]$, we will construct (φ, Γ) -modules with $L_{i+1} \cap B$ -structures $\mathcal{M}_{A, i+1}$, which are the parabolic extensions of the (φ, Γ) -module with $L_i \cap B$ -structure $\mathcal{M}_{A, i}$, which is already known by the assumption and compatible with isomorphisms $A' \otimes_A \mathcal{F}_{A, i} \cong \mathcal{F}_{A', i}$ and $W_{\mathrm{dR}}(\mathcal{M}_{A, i}) \cong W_{A, i}$, where $W_{A, i}$ is the i successive parabolic extension of the T -torsor $\mathcal{P}_{W_{A'}} \times^B T$ given by the assumption.

By the assumption of the induction, i successive parabolic extensions of $\mathcal{P}_{W_A} \times^B T$, $\mathcal{P}_{W_{A'}} \times^B T$, $\mathcal{P}_{\mathcal{M}_A} \times^B T$ and $\mathcal{P}_{\mathcal{M}_{A'}} \times^B T$ are given by respectively $W_{A, i}$, $W_{A', i}$, $\mathcal{M}_{A, i}$ and $\mathcal{M}_{A', i}$. For each $i \in [1, s-1]$, there exist objects $\mathcal{M}_{\mathrm{ad}, A, i}$, $\mathcal{M}_{\mathrm{ad}, A', i}$, $W_{\mathrm{ad}, A, i}$ and $W_{\mathrm{ad}, A', i}$ in each category and the corresponding $(i+1)$ -th extension classes are respectively written as follows:

$$\begin{aligned} E_1 &:= \mathrm{Ext}_{\mathrm{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K)}^1(A \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}, W_{\mathrm{ad}, A, i}), \\ E &:= \mathrm{Ext}_{\mathrm{Rep}_{A' \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}}(\mathrm{Gal}_K)}^1(A' \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}, W_{\mathrm{ad}, A', i}), \\ E_2 &:= \mathrm{Ext}_{\Phi\Gamma(\mathcal{R}_{A'}[\frac{1}{t}])}^1(\mathcal{R}_{A'}[\frac{1}{t}], \mathcal{M}_{\mathrm{ad}, A', i}), \end{aligned}$$

where $W_{\mathrm{ad}, A, i} \otimes_A A' \cong W_{\mathrm{ad}, A', i}$, $\mathcal{M}_{\mathrm{ad}, A, i} \otimes_A A' \cong \mathcal{M}_{\mathrm{ad}, A', i}$, $W_{\mathrm{dR}}(\mathcal{M}_{\mathrm{ad}, A, i}) \cong W_{\mathrm{ad}, A, i}$ and $W_{\mathrm{dR}}(\mathcal{M}_{\mathrm{ad}, A', i}) \cong W_{\mathrm{ad}, A', i}$ and the parameter of $\mathcal{P}_{\mathcal{M}_{A', i}}$ is in $\mathcal{T}_{T,0}(A')$. There is the map

$$\mathrm{Ext}_{\Phi\Gamma(\mathcal{R}_A[\frac{1}{t}])}^1(\mathcal{R}_A[\frac{1}{t}], \mathcal{M}_{\mathrm{ad}, A, i}) \rightarrow E_1 \times_E E_2$$

induced by $W_{\mathrm{dR}} \times (B \otimes_A 1)$. Then it suffices to prove the map is surjective and this follows by the argument of [18, Theorem 3.4.4]. $\square$

3.9.4. Locally irreducibility and accumulation property. From this section, let B, T denote the standard Borel subgroup and the standard maximal split torus of GSp_{2n} again. Let $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(k_L)$ be a GSp_{2n} -valued representation. We assume that $\bar{\rho}$ has a crystalline lift in this subsection. We may similarly do the arguments of [18, §3] in the case of GSp_{2n} , using the results of [18, §2] and Tannakian formalism. Especially, we note the GSp_{2n} -variant of [18, Corollary 3.7.10]. We say a crystalline frameable (φ, Γ) -module with GSp_{2n} -structure D is *benign* if $r_{\mathrm{std},*}(D)$ is benign.

Proposition 3.9.6. *Let $x = (\rho, \underline{\delta})$ be a point of $X_{\mathrm{tri}}(\bar{\rho})$ such that $D_{\mathrm{rig}}(\rho)$ is benign. Then the rigid space $X_{\mathrm{tri}}(\bar{\rho})$ is normal, irreducible and Cohen-Macaulay at x .*

Proposition 3.9.7. *Let $x = (\rho, \underline{\delta}) \in X_{\mathrm{tri}}(\bar{\rho})$ be a crystalline point such that $D_{\mathrm{rig}}(r_{\mathrm{std},*}(\rho))$ is φ -generic with regular Hodge-Tate type and assume that there exists a refinement of $D_{\mathrm{crys}}(D)$, then the morphism $\omega': X_{\mathrm{tri}}(\bar{\rho}) \rightarrow \mathcal{T}_T$ is flat in a neighborhood of x .*

PROOF. This is shown by Theorem 3.6.2, Proposition 3.9.5, [18, Proposition 3.5.1, Corollary 3.5.9, Corollary 3.7.8] in the $G = \mathrm{GSp}_{2n}$ case, which are similarly shown with Tannakian formalism, and the proof of [18, Proposition 4.1.1]. $\square$

We introduce accumulation property for $X_{\mathrm{tri}}(\bar{\rho})$ (cf. [17, Definition 2.11] or [18, Definition 4.1.3]).

Definition 3.9.8. Let $x = (\rho, \underline{\delta}) \in X_{\mathrm{tri}}(\bar{\rho})$ such that $\omega(x)$ is algebraic. We say that $X_{\mathrm{tri}}(\bar{\rho})$ satisfies the *accumulation property* if the set of the points $(\rho', \underline{\delta}') \in X_{\mathrm{tri}}(\bar{\rho})$ such that ρ' is benign accumulate at x in $X_{\mathrm{tri}}(\bar{\rho})$ in the sense of [6, §3.3.1].

Proposition 3.9.9. Let $x = (\rho, \underline{\delta}) \in X_{\mathrm{tri}}(\bar{\rho})$ be a crystalline point such that $D_{\mathrm{rig}}(r_{\mathrm{std},*}(x))$ is φ -generic with regular Hodge-Tate type and assume that $\omega(x)$ is algebraic. Then $X_{\mathrm{tri}}(\bar{\rho})$ satisfies the accumulation property at x .

PROOF. We may show the assertion by the proof of [18, Proposition 4.1.4] from Lemma 3.7.7 and Proposition 3.9.7. $\square$

3.10. Zariski density theorem for irreducible components which has a regular crystalline point

Let $G = \mathrm{GSp}_{2n}$ and B, T be respectively the standard Borel subgroup and the standard maximal split torus of GSp_{2n} . Let $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(k_L)$ be a continuous GSp_{2n} -representation. Let $\mathbf{k} \in (\mathbb{Z}^\Sigma)^{2n}$ be a strictly dominant regular Hodge-Tate type such that there is a character $\lambda \in X^*(T)$ over L such that the element $r_{\mathrm{std},*}(\lambda) \in X^*(T_{2n}) \cong (\mathbb{Z}^\Sigma)^{2n}$ is $\mathbf{k}$.

There is a quotient $R_{\bar{\rho}, \mathrm{crys}}^{\mathbf{k}}$ of $R_{\bar{\rho}}$, corresponding Zariski closed subspace $\mathfrak{X}_{\bar{\rho}, \mathrm{crys}}^{\mathbf{k}} \subset \mathfrak{X}_{\bar{\rho}}$ consists of the crystalline GSp_{2n} -representations with Hodge-Tate type $\mathbf{k}$ by [4, Proposition 3.0.9].

Lemma 3.10.1. Let x be a point of $\mathfrak{X}_{\bar{\rho}, \mathrm{crys}}^{\mathbf{k}}$ and U be an admissible open neighborhood of x . Then there exists a point $z \in U$ whose corresponding representation is benign.

PROOF. We will prove étale locally. There exists an étale morphism $U' := \mathrm{Sp}(R) \rightarrow \mathfrak{X}_{\bar{\rho}, \mathrm{crys}}^{\mathbf{k}}$ whose image contains x such that

$$D_{\mathrm{crys}}(V_R) := ((R \hat{\otimes}_{\mathbb{Q}_p} B_{\mathrm{crys}}) \otimes_R V_R)^{\mathrm{Gal}_K}$$

is a finite free $K_0 \otimes_{\mathbb{Q}_p} R$ -module and an isomorphism $D_{\mathrm{dR}}(V_R) \xrightarrow{\sim} K \otimes_{K_0} D_{\mathrm{crys}}(V_R)$ which are compatible with base changes, where V_R is the restriction to U' of the universal framed deformation of $\bar{\rho}$ by [10, Corollary 6.3.3], [25, Corollary 3.19] and Tannakian formalism. So it suffices to construct a deformation of the $\mathrm{GSp}_{2n}(k_z)$ -representation ρ_z on L such that $D_{\mathrm{rig}}(\rho_z)$ satisfies φ -genericity and noncriticality for any $z \in U \times_{\mathfrak{X}_{\bar{\rho}, \mathrm{crys}}^{\mathbf{k}}} U'$ because $R_{\bar{\rho}}^{\mathbf{k}}[1/p]$ is an integral domain, which follows by [4, Proposition 4.1.5]. That is done by constructing an appropriate lift of K -filtered φ -module $D_{\mathrm{crys}}(\rho_z)$ over L on $L[\epsilon] \in \mathcal{C}_L$ by Tannakian formalism and the construction after Lemma 3.7.3 since weak admissibility of K -filtered φ -modules over L is preserved by deformations on $A \in \mathcal{C}_L$, and that is done by the proof of [68, Lemma 4.2]. $\square$

Proposition 3.10.2. The Zariski closure $\overline{\mathfrak{X}}_{\mathrm{reg-crys}}$ of the following subset of $\mathfrak{X}_{\bar{\rho}}$:

$$\mathfrak{X}_{\mathrm{reg-crys}} := \{x \in \mathfrak{X}_{\bar{\rho}} \mid \rho_x \text{ is crystalline with regular Hodge-Tate type}\}$$

is a union of irreducible components of $\mathfrak{X}_{\bar{\rho}}$.

PROOF. We may follow the proof of [48, Proposition 5.10]. Let $\mathfrak{Z}$ be an irreducible component of the Zariski closure $\overline{\mathfrak{X}}_{\text{reg-crys}}$ of $\mathfrak{X}_{\text{reg-crys}}$. Endowing the closed subsets of $\mathfrak{X}_{\bar{\rho}}$ with the reduced closed structures ([14, 9.5.3, Proposition 4]), the complement $\mathfrak{U} := \mathfrak{Z} \setminus \mathfrak{Z}_{\text{sing}}$ of the singular locus $\mathfrak{Z}_{\text{sing}} \subset \mathfrak{Z}$ is an admissible open in $\mathfrak{Z}$.

We may pick a smooth point $x \in \mathfrak{U}$ with the corresponding representation $\rho_x: \text{Gal}_K \rightarrow \text{GSp}_{2n}(k(x))$, and we may also assume that ρ_x is benign by Lemma 3.10.1. To see that x is also smooth, we have to show that $H^2(\text{Gal}_K, \text{ad}(D_{\text{rig}}(\rho_x))) = 0$. Then it suffices to show $H^2(\text{ad}(D_{\text{rig}}(r_{\text{std},*}(\rho_x)))) = 0$ since $\text{ad}(D_{\text{rig}}(\rho_x))$ is a saturated (φ, Γ) -submodule of $\text{ad}(D_{\text{rig}}(r_{\text{std},*}(\rho_x)))$, and this follows from the argument in the first half part of the proof of [48, Proposition 5.10] by the φ -genericity of $D_{\text{rig}}(r_{\text{std},*}(\rho_x))$.

The irreducible set $\mathfrak{Z}$ is contained in a unique irreducible component of $\mathfrak{X}_{\bar{\rho}}$. So the assertion is reduced to the surjectivity of the injection

$$T_x \mathfrak{Z} \hookrightarrow T_x \mathfrak{X}_{\bar{\rho}}$$

of the tangent spaces.

Recall that the triangulations of $D_{\text{rig}}(\rho_x)$ are exactly parametrized by $w \in W(\text{GSp}_{2n}, T)$ by Proposition 3.7.1, and we write them as $\mathcal{P}_w$. Each pair $(x, \mathcal{P}_w)$ defines a point $y_w \in \mathfrak{X}_{\text{tri}}(\bar{\rho})$. Replace L with the field of definition $k(x)$ of x , which is a finite extension of L . Let $\mathcal{P}_w \left[\frac{1}{t} \right]$ denote the induced triangulation of the (φ, Γ) -module with GSp_{2n} -structure $D_{\text{rig}}(\rho_x) \otimes_{\mathcal{R}_L} \mathcal{R}_L \left[\frac{1}{t} \right]$ over $\mathcal{R}_L \left[\frac{1}{t} \right]$ from the base change by $\mathcal{R}_L \rightarrow \mathcal{R}_L \left[\frac{1}{t} \right]$.

Write $\mathcal{X}_{\rho}$ for the usual functor of framed deformations over $\mathcal{C}_L$ of continuous $\rho: \text{Gal}_K \rightarrow \text{GSp}_{2n}(L)$, which is pro-represented by R_{ρ} . We define

$$\begin{aligned} X_{D, \mathcal{M}} &:= X_D \times_{X_D \left[\frac{1}{t} \right]} X_{D \left[\frac{1}{t} \right], \mathcal{M}}, \\ \mathcal{X}_{D, \mathcal{M}} &:= X_{D, \mathcal{M}} \times_{X_D} \mathcal{X}_D \end{aligned}$$

as an GSp_{2n} -analogue of the groupoids over $\mathcal{C}_L$ constructed in [18, §3.5, §3.6] for a triangulation $\mathcal{M}$ of $D_{\text{rig}}(\rho) \left[\frac{1}{t} \right]$. Note that there is the natural monomorphism of groupoids $X_{D, \mathcal{P}_w} \rightarrow X_{D, \mathcal{P}_w \left[\frac{1}{t} \right]}$.

Since $D_{\text{rig}}(r_{\text{std},*}(\rho_x))$ is benign, it follows that $\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}), y_w} \cong R_{\mathcal{P}_w \left[\frac{1}{t} \right]}$ is an integral domain by arguing [18, Theorem 3.6.2 (i), Corollary 3.7.8] (see also [48, Proposition 5.7]) with Tannakian formalism and Proposition 3.9.6.

It follows that there is a unique irreducible component $\mathfrak{Y}_w$ of $\mathfrak{X}_{\text{tri}}(\bar{\rho})$, which contains y_w . Let $\mathfrak{X}_{\text{cr}}$ be the subset of points of $(\rho', \underline{\delta}') \in X_{\text{tri}}(\bar{\rho})$ such that ρ' is benign. By Proposition 3.9.9, there exists an affinoid neighborhood $\mathfrak{U}_w$ of y_w such that $\mathfrak{U}_w \cap \mathfrak{X}_{\text{cr}}$ is dense in $\mathfrak{U}_w$. Then $\mathfrak{U}_w \cap \mathfrak{X}_{\text{cr}} \cap \mathfrak{Y}_w$ is dense in $\mathfrak{U}_w \cap \mathfrak{Y}_w$, which is Zariski open and Zariski dense in $\mathfrak{Y}_w$, so $\mathfrak{X}_{\text{cr}} \cap \mathfrak{Y}_w$ is dense in $\mathfrak{Y}_w$. Then the image of $\mathfrak{Y}_w$ by the natural projection

$$X_{\text{tri}}(\bar{\rho}) \rightarrow \mathfrak{X}_{\bar{\rho}} \times_L \mathcal{T}_T \rightarrow \mathfrak{X}_{\bar{\rho}}$$

is an irreducible component $\mathfrak{V}$ of $\mathfrak{X}_{\bar{\rho}}$. So we get the maps

$$\bigoplus_{w \in W(\text{GSp}_{2n}, T)} T \left(\mathcal{X}_{\rho_x, \mathcal{P}_w \left[\frac{1}{t} \right]} \right) = \bigoplus_{w \in W(\text{GSp}_{2n}, T)} T_{y_w} \mathfrak{Y}_w \rightarrow T_x \mathfrak{Z} \hookrightarrow T_x \mathfrak{V} \cong T(\mathcal{X}_{\rho_x})$$

The map is compatible with the forgetful maps $X_{\rho_x, \mathcal{P}_w \left[\frac{1}{t} \right]} \rightarrow X_{\rho_x}$. By the definition of $\mathcal{X}_{\rho_x, \mathcal{P}_w \left[\frac{1}{t} \right]}$ and that $\mathcal{X}_{\rho_x} \rightarrow X_{\rho_x}$ is formally smooth, there exists a finite dimensional vector

space V over L and the above sequence of the maps factors as

$$\begin{aligned} \bigoplus_{w \in W(\mathrm{GSp}_{2n}, T)} T\left(\mathcal{X}_{\rho_x, \mathcal{P}_w\left[\frac{1}{t}\right]}\right) &\cong \bigoplus_{w \in W(\mathrm{GSp}_{2n}, T)} \left(T\left(X_{\rho_x, \mathcal{P}_w\left[\frac{1}{t}\right]}\right) \oplus V\right) \\ &\rightarrow T(X_{\rho_x}) \oplus V \cong T(\mathcal{X}_{\rho_x}) \end{aligned}$$

where the middle map is defined by $\sum_w T(\pi_w) \oplus \mathrm{id}_V$ for each projection π_w to the w -component. By Corollary 3.8.14, $\sum_w T(\pi_w)$ is surjective, so $T_x \mathfrak{Z} \rightarrow T_x \mathfrak{V}$ is an isomorphism. $\square$

3.11. Partial lifts and their partial extensions

In this section, we extend the argument of partial lifts and partial extension to the case that p is any prime, including the case of $p = 2$.

3.11.1. Quadratic functions and parabolic subgroups. We first recall the definition of quadratic functions, which is used in this section, especially over a field of characteristic 2.

Definition 3.11.1. For a ring R and R -modules M and N , a *quadratic function* (over R) is a function $q: M \rightarrow N$ such that the following two conditions hold:

- (1) $q(ax) = a^2 q(x)$ for all $a \in R$.
- (2) The function $M \times M \rightarrow N$, $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is bilinear.

A *quadratic form* (over R) is a quadratic function (over R) whose target is free of rank one.

Note that if M is free and $N = R$ is free of rank one, a function $q: M \rightarrow R$ is a quadratic form if and only if there is a (not necessarily symmetry) bilinear function $b: M \times M \rightarrow R$.

Let a, b, c, m be positive integers such that $a + b + c = m$. We assume that $a = c$. Let

$$P := \begin{pmatrix} \mathrm{GL}_a & * & * \\ 0 & \mathrm{GL}_b & * \\ 0 & 0 & \mathrm{GL}_c \end{pmatrix},$$

which is a parabolic subgroup of GL_m . Write $P = U \rtimes L$ where U is the unipotent radical and L is a Levi subgroup of P and $\mathrm{ad}: L \rightarrow \mathrm{Aut}(U)$ for the conjugate action. We have the upper central series $1 \subset U^{\mathrm{der}} \subset U$ where

$$U^{\mathrm{der}} := [U, U] = \begin{pmatrix} I_a & 0 & * \\ 0 & I_b & 0 \\ 0 & 0 & I_c \end{pmatrix}.$$

Let $U^{\mathrm{ab}} := U/U^{\mathrm{der}}$. There is the natural isomorphism of schemes

$$U \cong \mathrm{Mat}_{a \times b} \oplus \mathrm{Mat}_{b \times c} \oplus \mathrm{Mat}_{a \times c}$$

which induces the isomorphisms of abelian group schemes $U^{\mathrm{ab}} \cong \mathrm{Mat}_{a \times b} \oplus \mathrm{Mat}_{b \times c}$ and $U^{\mathrm{der}} \cong \mathrm{Mat}_{a \times c}$. Write $\mathrm{ad}^1: L \rightarrow \mathrm{Aut}(U^{\mathrm{ab}})$ and $\mathrm{ad}^0: L \rightarrow \mathrm{Aut}(U^{\mathrm{der}})$ for the conjugate actions induced from ad .

3.11.2. Structures of the cup products for parabolic extensions. From here on, for the sake of simplicity, frameable (φ, Γ) -modules $([\phi], [\gamma] \mid \gamma \in \Gamma)$ will be denoted as $([\phi], [\gamma])$. Fix a Galois representation $\bar{\tau}: \text{Gal}_K \rightarrow M(\bar{\mathbb{F}}_p)$ and its lift $\tau: \text{Gal}_K \rightarrow M(\bar{\mathbb{Z}}_p)$. and fix a frameable (φ, Γ) -module with L -structure $([\phi_L], [\gamma_L])$, which represents the associated (φ, Γ) -module to $\bar{\tau}$. Let $([\phi_1], [\gamma_1])$ be a frameable (φ, Γ) -module with P/U^{der} -structure which lifts $([\phi_L], [\gamma_L])$ and write $[\phi_1] = [\phi_L][\phi_U^1]$ and $[\gamma_1] = [\gamma_L][\gamma_U^1]$ where $[\phi_U^1], [\gamma_U^1] \in U^{\text{ab}}$. Fix isomorphisms $b_1: U^{\text{ab}} \xrightarrow{\sim} \mathbb{G}_a^{\oplus s_1}$ and $U^{\text{der}} \xrightarrow{\sim} \mathbb{G}_a^{\oplus s_0}$ of algebraic groups. Then we write $\text{ad}_{[-]}^i$ for the matrix $b_i \circ \text{ad}_{[-]}^i \circ b_i^{-1} \in \text{GL}(\mathbb{G}_a^{\oplus s_i})$ for $i \in \{0, 1\}$. Let V be a vector space over K and $s, t \in \text{GL}(V \otimes_K \mathbb{A}_A)$. We define a *framed Herr complex* $C_{\text{Herr}}^\bullet(s, t)$ as

$$C_{\text{Herr}}^\bullet(s, t) := [V \otimes_K \mathbb{A}_A \xrightarrow{(s \circ \varphi - 1, t \circ \gamma - 1)} (V \otimes_K \mathbb{A}_A) \oplus (V \otimes_K \mathbb{A}_A) \xrightarrow{(t \circ \gamma - 1) \oplus (1 - s \circ \varphi)} V \otimes_K \mathbb{A}_A]$$

and write

$$H_{\text{Herr}}^i(s, t) := Z_{\text{Herr}}^i(s, t) / B_{\text{Herr}}^i(s, t)$$

for the i -th cohomology of $C_{\text{Herr}}^\bullet(s, t)$ for $i \in [0, 2]$. We have natural isomorphisms

$$\begin{aligned} H^i(\text{Gal}_K, U^{\text{ab}}) &\cong H_{\text{Herr}}^i(\text{ad}_{[\phi_L]}^1, \text{ad}_{[\gamma_L]}^1), \\ H^i(\text{Gal}_K, U^{\text{der}}) &\cong H_{\text{Herr}}^i(\text{ad}_{[\phi_L]}^0, \text{ad}_{[\gamma_L]}^0) \end{aligned}$$

for $i \in [0, 2]$ by [39, Theorem 5.1.29]. We set $\lambda(s, t) := \gamma(s)^{-1} t^{-1} s \varphi(t)$ when (s, t) be a (ϕ, Γ) -module with P -structure. For an element $\alpha \in C_{\text{Herr}}^i(s, t)$ for an integer i , write $[\alpha]$ for the image of α in $H_{\text{Herr}}^i(s, t)$.

Lemma 3.11.2. *For an element $\beta \in H_{\text{Herr}}^1(\text{ad}_{[\phi_L]}^1, \text{ad}_{[\gamma_L]}^1)$, write for $([\phi_1], [\gamma_1])$ the naturally associated frameable (φ, Γ) -module with P/U^{der} -structure to β , which lifts $([\phi_L], [\gamma_L])$. Let $[\phi_0]$ and $[\gamma_0]$ be elements of $P(\mathbb{A}_A)$ which are lifts of respectively $[\phi_1]$ and $[\gamma_1]$. Then the map*

$$Q: H_{\text{Herr}}^1(\text{ad}_{[\phi_L]}^1, \text{ad}_{[\gamma_L]}^1) \rightarrow H_{\text{Herr}}^2(\text{ad}_{[\phi_L]}^0, \text{ad}_{[\gamma_L]}^0), \quad \beta \mapsto \lambda([\phi_0], [\gamma_0])$$

is well-defined.

PROOF. To see the map is well-defined, it suffices to show that the images of $\lambda([\phi_0], [\gamma_0]) \in C_{\text{Herr}}^2(\text{ad}_{[\phi_L]}^0, \text{ad}_{[\gamma_L]}^0)$ in $H_{\text{Herr}}^2(\text{ad}_{[\phi_L]}^0, \text{ad}_{[\gamma_L]}^0)$ does not depend on a choice of lifts $([\phi_0], [\gamma_0])$ of $([\phi_1], [\gamma_1])$. This follows by the equation in the definition of λ and that $\lambda([\phi_0], [\gamma_0]) = 0 \bmod U^{\text{der}}$ obtained from the commutativity of $[\phi_1]$ and $[\gamma_1]$, and that U^{der} is abelian and a center of U (also cf. [59, Lemma 9.4.1]). $\square$

Lemma 3.11.3. *The map*

$$Q: H_{\text{Herr}}^1(\text{ad}_{[\phi_L]}^1, \text{ad}_{[\gamma_L]}^1) \rightarrow H_{\text{Herr}}^2(\text{ad}_{[\phi_L]}^0, \text{ad}_{[\gamma_L]}^0)$$

is a quadratic function over A .

PROOF. We take a frameable (φ, Γ) -module with P/U^{der} -structure $([\phi_1], [\gamma_1])$, which lifts $([\phi_L], [\gamma_L])$. Let

$$([\phi_U^1], [\gamma_U^1]) \in Z_{\text{Herr}}^1(\text{ad}_{[\phi_L]}^1, \text{ad}_{[\gamma_L]}^1)$$

be a cocycle which represents $([\phi_1], [\gamma_1])$. Note that $[\phi_1] = [\phi_L][\phi_U^1]$ and $[\gamma_1] = [\gamma_L][\gamma_U^1]$. Let $([\phi_{U,1}^1], [\phi_{U,2}^1])$ (resp. $([\gamma_{U,1}^1], [\gamma_{U,2}^1])$) be the corresponding element in $\text{Mat}_{a \times b}(\mathbb{A}_A) \oplus \text{Mat}_{b \times c}(\mathbb{A}_A)$ to $[\phi_U^1]$ (resp. $[\gamma_U^1]$) via the isomorphism

$$U^{\text{ab}} \cong \text{Mat}_{a \times b} \oplus \text{Mat}_{b \times c}.$$

We take bases $\{z_1, \dots, z_{ab}\}, \{z'_1, \dots, z'_{bc}\}$ of the free $\mathbb{A}_A$ -modules $\mathrm{Mat}_{a \times b}(\mathbb{A}_A)$, $\mathrm{Mat}_{a \times b}(\mathbb{A}_A)$ and write $[\phi_1^1], \dots, [\phi_{ab}^1], [\phi_1^1]', \dots, [\phi_{bc}^1]'$ and $[\gamma_1^1], \dots, [\gamma_{ab}^1], [\gamma_1^1]', \dots, [\gamma_{bc}^1]'$ for the corresponding factors of $[\phi_U^1], [\gamma_U^1]$. Since there is the natural inclusion of schemes $i: P/U^{\mathrm{der}} \hookrightarrow P$, which is a section of the quotient $P \rightarrow P/U^{\mathrm{der}}$, we take the section $[\phi_0] := i([\phi_1]), [\gamma_0] := i([\gamma_1])$ of $[\phi_1], [\gamma_1]$. Since $\lambda(\phi_L, \gamma_L) = 1$, we have

$$\lambda([\phi_0], [\gamma_0]) = \gamma([\phi_U^1]^{-1}) \mathrm{ad}_{\gamma([\phi_L]^{-1})}([\gamma_U^1]^{-1}) \mathrm{ad}_{\varphi([\gamma_L]^{-1})}([\phi_U^1]) \varphi([\gamma_U^1]).$$

Then the factors in the upper right $(a \times c)$ -submatrices of the products are the form

$$\sum_{i=1}^{ab} (f_{\phi,i}([\phi_1^i]) + f_{\gamma,i}([\gamma_1^i])) + \sum_{j=1}^{bc} (f'_{\phi,j}([\phi_1^j]') + f'_{\gamma,j}([\gamma_1^j]')),$$

where $f_{\phi,i}, f_{\gamma,i}, f'_{\phi,j}$ and $f'_{\gamma,j}$ are A -linear functions $\mathbb{A}_A \rightarrow \mathbb{A}_A$. So the lemma follows by Lemma 3.11.2. $\square$

We fix a Galois representation

$$\bar{\tau} = \mathrm{diag}(\bar{\tau}_a, \bar{\tau}_b, \bar{\tau}_c): \mathrm{Gal}_K \rightarrow L(\bar{\mathbb{F}}_p)$$

as well as a lift

$$\tau = \mathrm{diag}(\tau_a, \tau_b, \tau_c): \mathrm{Gal}_K \rightarrow L(\bar{\mathbb{Z}}_p)$$

of $\bar{\tau}$ which are associated to the (φ, Γ) -modules with L -structures $([\bar{\phi}_L], [\bar{\gamma}_L])$ and $([\phi_L], [\gamma_L])$. Let

$$\cup: H_{\mathrm{Herr}}^1(\mathrm{ad}_{[\phi_L]}^1, \mathrm{ad}_{[\gamma_L]}^1) \times H_{\mathrm{Herr}}^1(\mathrm{ad}_{[\phi_L]}^1, \mathrm{ad}_{[\gamma_L]}^1) \rightarrow H_{\mathrm{Herr}}^2(\mathrm{ad}_{[\phi_L]}^0, \mathrm{ad}_{[\gamma_L]}^0)$$

be the cup product over $\bar{\mathbb{Z}}_p$, defined as

$$(x, y) \mapsto Q(x + y) - Q(x) - Q(y).$$

We assume that $\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, \mathrm{Mat}_{a \times c}(\bar{\mathbb{F}}_p)) = 1$.

Definition 3.11.4. A *partial extension* of τ is a continuous group homomorphism $\rho': \mathrm{Gal}_K \rightarrow (P/U^{\mathrm{der}})(\bar{\mathbb{Z}}_p)$ extending τ . The set of equivalence classes of partial extensions of τ are in natural bijection with $H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(\bar{\mathbb{Z}}_p))$.

Let $\bar{\rho}'_P: \mathrm{Gal}_K \rightarrow P(\bar{\mathbb{F}}_p)$ be a continuous group homomorphism. A *partial lift* of $\bar{\rho}'_P$ is a continuous group homomorphism $\rho': \mathrm{Gal}_K \rightarrow (P/U^{\mathrm{der}})(\bar{\mathbb{Z}}_p)$ which lifts $\bar{\rho}'_P \bmod U^{\mathrm{der}}$.

Lemma 3.11.5. Let A be $\bar{\mathbb{Z}}_p$ or $\bar{\mathbb{F}}_p$. A partial extension $\rho': \mathrm{Gal}_K \rightarrow (P/U^{\mathrm{der}})(A)$ of a continuous homomorphism $\tau_A: \mathrm{Gal}_K \rightarrow L(A)$ extends to a full extension $\rho: \mathrm{Gal}_K \rightarrow P(A)$ if and only if $Q(\beta) = 0$ where $\beta \in H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))$ is the cohomology class corresponding to ρ' .

PROOF. Let $([\phi_L], [\gamma_L])$ be a framed (φ, Γ) -module with L -structure over $\mathbb{A}_A$ which represents the associated (φ, Γ) -module to τ_A . Let $H_{\mathrm{Herr}}^1([\phi_L], [\gamma_L])$ be the equivalence classes of all extensions of $([\phi_L], [\gamma_L])$ to a framed (φ, Γ) -module with P -structure.

We define $Z_{\mathrm{Herr}}^1([\phi_L], [\gamma_L])$ as the set of elements

$$((x_0, y_0), (x_1, y_1)) \in C_{\mathrm{Herr}}^1(\mathrm{ad}_{[\phi_L]}^0, \mathrm{ad}_{[\gamma_L]}^0) \oplus C_{\mathrm{Herr}}^1(\mathrm{ad}_{[\phi_L]}^1, \mathrm{ad}_{[\gamma_L]}^1)$$

such that $d(x_1, y_1) = 0$ and $d(x_0, y_0) + Q((x_1, y_1)) = 0$. Then there exists a surjective map $Z_{\mathrm{Herr}}^1([\phi_L], [\gamma_L]) \rightarrow H_{\mathrm{Herr}}^1([\phi_L], [\gamma_L])$ by the argument of Lemma 3.11.2. So the lemma follows. $\square$

3.11.3. Classical actions on the Galois cohomologies. Let $\Delta = \{1, j\}$ be the finite group with two elements. Toward an application to the GSp_{2n} -valued representations and etc., we introduce the classical actions of Δ in [61, Definition 6.2].

Definition 3.11.6. An action of Δ on each $H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))$ and $H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(A))$ is said to be *classical* if the following conditions hold:

- (1) $jH^1(\mathrm{Gal}_K, \mathrm{Mat}_{a \times b}) \subset H^1(\mathrm{Gal}_K, \mathrm{Mat}_{b \times c})$,
- (2) $jH^1(\mathrm{Gal}_K, \mathrm{Mat}_{b \times c}) \subset H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a \times b})$,
- (3) The action of Δ is compatible with the cup products.

Example 3.11.7. Let $G = \mathrm{GSp}_{2n}$. The maximal proper Levi subgroups of G are of the form $\mathrm{GL}_k \times \mathrm{GSp}_{2(n-k)}$ or GL_n and set

$$M_k := \begin{cases} \mathrm{GL}_k \times \mathrm{GSp}_{2(n-k)} & (1 \leq k < n) \\ \mathrm{GL}_n & (k = n) \end{cases}$$

for $1 \leq k \leq n$. Let

$$P_k := \mathrm{GSp}_{2n} \cap \begin{pmatrix} \mathrm{GL}_k & \mathrm{Mat}_{k \times (2n-2k)} & \mathrm{Mat}_{k \times k} \\ & \mathrm{GL}_{2n-2k} & \mathrm{Mat}_{(2n-2k) \times k} \\ & & \mathrm{GL}_k \end{pmatrix}$$

be the corresponding parabolic subgroup of G to M_k . Let U_k be the unipotent radical of P_k .

Set $\Omega_k := \begin{pmatrix} & I_{n-k} \\ -I_{n-k} & \end{pmatrix}$. Define an action of $\Delta = \{1, j\}$ on $\mathrm{Mat}_{k \times (2n-2k)} \times \mathrm{Mat}_{(2n-2k) \times k}$ by

$$\begin{cases} j(x, y) := ({}^t y \Omega_k, \Omega_k {}^t x) & (x, y) \in \mathrm{Mat}_{k \times 2(n-k)} \times \mathrm{Mat}_{2(n-k) \times k} \\ jz := {}^t z & z \in \mathrm{Mat}_{k \times k} \end{cases}$$

Then the induced actions of Δ on $H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))$ and $H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(A))$ is classical.

For a classical action Δ in the above definition, write

$$Q^\Delta: H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^\Delta \rightarrow H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(A))$$

for the restriction of the quadratic function $Q: H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A)) \rightarrow H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(A))$ defined in Lemma 3.11.2.

Lemma 3.11.8. *If A is a $\mathbb{F}_2$ -algebra, the map*

$$Q^\Delta: H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(A))^\Delta \rightarrow H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(A))$$

is Frobenius semilinear.

PROOF. Since $Q(x, 0) = Q(0, y) = 0$ for $x \in H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a \times b})$ and $y \in H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a \times b})$ by Lemma 3.11.5, $(x, y) \mapsto Q((x, y))$ is a bilinear. Then for $x, x' \in H^1(\mathrm{Gal}_K, \mathrm{Mat}_{a \times b})$,

$$\begin{aligned} & Q((x + x', \Delta(x) + \Delta(x'))) \\ &= Q((x, \Delta(x)) + Q((x, \Delta(x'))) + Q((x', \Delta(x)) + Q((x', \Delta(x'))) \\ &= Q((x, \Delta(x)) + Q((x', \Delta(x'))) + 2Q((x, \Delta(x'))) \\ &= Q((x, \Delta(x)) + Q((x', \Delta(x'))) \end{aligned}$$

by the classicality of the action Δ . We also have $\alpha^2 Q((x, \Delta(x))) = Q((\alpha x, \alpha \Delta(x)))$ for $\alpha \in A$ since Q is a quadratic function. $\square$

3.12. Existence of crystalline lifts from partial lifts

In this section, we prove the existence of crystalline lifts from partial lifts under Property SSD (see Condition 3.12.1). We define the following map

$$Q_{\mathrm{red}}^{\Delta}: H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(\overline{\mathbb{Z}}_p))^{\Delta} \rightarrow H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(\overline{\mathbb{F}}_p))$$

induced by Q^{Δ} and the surjection $\overline{\mathbb{Z}}_p \twoheadrightarrow \overline{\mathbb{F}}_p$.

Note that if the Levi-factor $\bar{\rho}_M$ of the representation $\bar{\rho}$ is written by $\mathrm{diag}(\bar{\rho}_1, \dots, \bar{\rho}_k)$ when G is embedded in a GL_m such that each $\bar{\rho}_i$ is irreducible and $\bar{\rho}_i \neq \bar{\rho}_j$ for $i \neq j$, the map $Q_{\mathrm{red}}^{\Delta}$ is surjective because $H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(\overline{\mathbb{Z}}_p)) = H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(\overline{\mathbb{F}}_p)) = 0$.

Let G be a connected split reductive group over K , $P \subset G$ be a parabolic subgroup and $P = U \rtimes L$ be the Levi decomposition. Write $\mathcal{X}_{K,L,\mathrm{red}}$ for the reduced Emerton–Gee stacks for L defined in [59]. We introduce the property SSD (Sufficiently Small Dimension) in [61, Introduction].

Condition 3.12.1 (Property SSD for G). Write $X_s \subset \mathcal{X}_{K,L,\mathrm{red}}$ for the scheme-theoretical closure of the locus

$$\{x \mid \dim_{\overline{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, U^{\mathrm{ab}}(\overline{\mathbb{F}}_p)) \geq s\}.$$

For any L obtained as a Levi subgroup of G as above, $\dim X_s + s \leq [K: \mathbb{Q}_p] \dim L / B_L$ where B_L is a Borel subgroup of L .

Definition 3.12.2. Let $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_p)$ be a continuous representation. We say that $\bar{\rho}$ is $Q_{\mathrm{red}}^{\Delta}$ -surjective if it satisfies one of the following conditions:

- (1) $\bar{\rho}$ is elliptic.
- (2) $\bar{\rho}$ factors through a maximal parabolic subgroup $P \subset \mathrm{GSp}_{2n}$ whose Levi subgroup is isomorphic to $\mathrm{GL}_k \times \mathrm{GSp}_{2(n-k)}$ (and fix an isomorphism), there is a crystalline lift $\mathrm{diag}(\rho_a, \rho_b, \rho_c)$ of the Levi component $\mathrm{diag}(\bar{\rho}_a, \bar{\rho}_b, \bar{\rho}_c)$ of $\bar{\rho}$, where $\bar{\rho}_a, \bar{\rho}_c$ are irreducible, and there is a sufficiently large integer n such that $(\lambda \rho_a, \rho_b, \lambda^{-1} \rho_c)$, where $\lambda = \overline{\mathbb{Z}}_p(n)$ with trivial reduction, is regular crystalline and the corresponding map

$$Q_{\mathrm{red}}^{\Delta}: H^1(\mathrm{Gal}_K, U^{\mathrm{ab}}(\overline{\mathbb{Z}}_p))^{\Delta} \rightarrow H^2(\mathrm{Gal}_K, U^{\mathrm{der}}(\overline{\mathbb{F}}_p)),$$

constructed after Lemma 3.11.8 is surjective.

Remark 3.12.3. If $p \neq 2$, any continuous representation $\mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_p)$ is $Q_{\mathrm{red}}^{\Delta}$ -surjective by the argument of [61, §6, §8].

Theorem 3.12.4. For each integer $n \geq 1$, assume that Property SSD for $G = \mathrm{GSp}_{2n}$ holds. Let $\bar{\rho}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_p)$ be a continuous representation whose similitude character is $\bar{\psi}: \mathrm{Gal}_K \rightarrow \overline{\mathbb{F}}_p^*$ and $\psi: \mathrm{Gal}_K \rightarrow \overline{\mathbb{Z}}_p^*$ be a crystalline lift of $\bar{\psi}$. Assume that $\bar{\rho}$ is $Q_{\mathrm{red}}^{\Delta}$ -surjective. Then $\bar{\rho}$ admits a crystalline lift ρ which has the abelianization ψ and regular Hodge–Tate weights.

PROOF. Note that the abelianization of a GSp_{2n} -valued representation is its similitude character. By Theorem 2.3.6, we may assume that $\bar{\rho}$ is not elliptic. There exists the maximal proper Levi subgroup of the form $M_k := \mathrm{GL}_k \times \mathrm{GSp}_{2(n-k)} \subset \mathrm{GSp}_{2n}$ and the corresponding parabolic subgroup

$$P_k := \mathrm{GSp}_{2n} \cap \begin{pmatrix} \mathrm{GL}_k & \mathrm{Mat}_{k \times (2n-2k)} & \mathrm{Mat}_{k \times k} \\ & \mathrm{GL}_{2n-2k} & \mathrm{Mat}_{(2n-2k) \times k} \\ & & \mathrm{GL}_k \end{pmatrix}$$

which $\bar{\rho}$ factors through and that $\bar{\rho}$ is a Heisenberg-type extension, i.e. a representation such that $\dim_{\bar{\mathbb{F}}_p} H^2(\mathrm{Gal}_K, \mathrm{gr}^0 \mathrm{Lie} U(\bar{\mathbb{F}}_p)) \leq 1$ as in [61, Definition 5.1], of a continuous representation $\bar{\rho}_M: \mathrm{Gal}_K \rightarrow M_k(\bar{\mathbb{F}}_p)$ for some $k \in [1, n]$. Write

$$\bar{\rho}_M = \begin{pmatrix} \bar{\rho}_a & & \\ & \bar{\rho}_b & \\ & & \bar{\rho}_c \end{pmatrix}$$

such that $\bar{\rho}_b: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2(n-k)}(\bar{\mathbb{F}}_p)$ has the similitude character $\bar{\psi}$ and

$$\bar{\rho}_a: \mathrm{Gal}_K \rightarrow \mathrm{GL}_k(\bar{\mathbb{F}}_p), \quad \bar{\rho}_c = \bar{\psi}^t \bar{\rho}_a^{-1}: \mathrm{Gal}_K \rightarrow \mathrm{GL}_k(\bar{\mathbb{F}}_p)$$

are irreducible.

By the assumption and Theorem 2.3.6, $\bar{\rho}_b$ admits a crystalline lift

$$\rho_b: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2(n-k)}(\bar{\mathbb{Z}}_p)$$

with similitude character ψ . Let $\rho_a: \mathrm{Gal}_K \rightarrow \mathrm{GL}_k(\bar{\mathbb{Z}}_p)$ be a crystalline lift and set $\rho_c := \psi^t \rho_a^{-1}$.

By replacing (ρ_a, ρ_b, ρ_c) with $(\lambda \rho_a, \rho_b, \lambda^{-1} \rho_c)$ for a sufficiently large n and $\lambda = \bar{\mathbb{Z}}_p(n)$ with trivial reduction, we may assume

$$H^1(\mathrm{Gal}_K, \mathrm{Lie} U(\bar{\mathbb{Q}}_p)) = H_{\mathrm{crys}}^1(\mathrm{Gal}_K, \mathrm{Lie} U(\bar{\mathbb{Q}}_p)).$$

Then the theorem follows by the argument of the proof of [39, Theorem 6.3.2], Lemma 3.11.5 and the last argument of the proof of [61, Theorem 8.5]. $\square$

Remark 3.12.5. The Property SSD for $G = \mathrm{GSp}_{2n}$ will be proved in the section 3.13.

3.13. Dimension calculations of the Emerton–Gee stacks for GSp_{2n}

In this section, we calculate the dimensions of the Emerton–Gee stacks $\mathcal{X}_{K, \mathrm{GSp}_{2n}, \mathrm{red}}$ defined in [59] along with the method of [61, §10]. Let $\mathbf{P}$ be a condition of $\bar{\mathbb{F}}_p$ -points in a moduli stack. We mean a “locus” of $\mathbf{P}$ in the moduli stack the scheme-theoretic closure of the scheme-theoretic image of all families whose $\bar{\mathbb{F}}_p$ -points satisfy $\mathbf{P}$. See also [62] and the proof of [61, Theorem 2] for the contents of this subsection.

We prove the following theorem, that is stronger than Property SSD (Condition 3.12.1) for GSp_{2n} .

Let $i^{\mathrm{std}}: \mathcal{X}_{K, \mathrm{GSp}_{2n}, \mathrm{red}} \rightarrow \mathcal{X}_{K, \mathrm{GL}_{2n}, \mathrm{red}}$ be the map of stacks induced by the standard representation $\mathrm{GSp}_{2n} \rightarrow \mathrm{GL}_{2n}$. In this section, we often think of GSp_{2n} -valued representations embedded in GL_{2n} by the standard representation.

Theorem 3.13.1. *Let $\bar{\alpha}: \mathrm{Gal}_K \rightarrow \mathrm{GL}_a(\bar{\mathbb{F}}_p)$ be an irreducible Galois representation. The locus of $\bar{x} \in \mathcal{X}_{K, \mathrm{GSp}_{2n}, \mathrm{red}}$ such that $\dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\mathrm{Gal}_K}(\bar{\alpha}, i^{\mathrm{std}}(\bar{x})) \geq r$ is of dimension at most*

$$d_{2n, r} := [K: \mathbb{Q}_p]n^2 - \left(\frac{r^2}{2} - \frac{r}{2} + 1 \right).$$

We write $\mathcal{X}_{2n}^{\bar{\alpha}^{\oplus r}}$ for the locus considered in Theorem 3.13.1.

3.13.1. The semisimple loci. It suffices to check the case of $2n = ra$ for considering the elliptic locus of $\mathcal{X}_{2n}^{\bar{\alpha}^{\oplus r}}$. We have the following lemma.

Lemma 3.13.2. *If the image of $\mathrm{diag}(\bar{\alpha}(-1), \dots, \bar{\alpha}(-1))$ is in $\mathrm{GSp}_{2n}(\bar{\mathbb{F}}_p)$, the semisimple locus of $\mathcal{X}_{ra}^{\bar{\alpha}^{\oplus r}}$ has dimension at most $-\left(\frac{r^2}{2} - \frac{r}{2} + 1\right)$. If otherwise, the elliptic locus is empty.*

PROOF. This follows because the dimension of the group of automorphisms of the Galois representation $\mathrm{diag}(\bar{\alpha}(-1), \dots, \bar{\alpha}(-1))$, which is seen to be valued in GSp_{2n} , is 1 if $r = 1$ otherwise at least $\dim \mathrm{GSp}_{2 \lfloor \frac{r}{2} \rfloor} \geq \frac{r^2}{2} - \frac{r}{2} + 1$. $\square$

3.13.2. Reduction to the maximally non-split parabolic loci. Let $\psi: \mathrm{Gal}_K \rightarrow \bar{\mathbb{F}}_p^*$ be a character.

Lemma 3.13.3. *The dimension of $X_{2n}^{\bar{\alpha}^{\oplus r}}$ is at most $d_{2n,r}$, or equal to the dimension of the locus of*

$$\begin{pmatrix} \bar{\alpha}(-1)^{\oplus r} & * & * \\ & \bar{\tau} & * \\ & & \psi(\bar{\alpha}^\vee(1))^{\oplus r} \end{pmatrix}$$

where $\bar{\tau}: \mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2(n-ar)}(\bar{\mathbb{F}}_p)$ is a continuous representation.

PROOF. It suffices to show the following inequalities

$$\begin{aligned} -k_1^2 + d_{2n-2k_1a, r-2k_1} &\leq d_{2n,r}, \\ d_{2n-2k_1a-k_3a, k_2} + d_{k_3a, k_3} &\leq d_{2n-2k_1a, r-2k_1} \end{aligned}$$

under the conditions of $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$, $2k_1 + k_2 + k_3 = r < n/a$ and $2|ak_3$ and they follow by the proof of [61, Lemma 10.6]. $\square$

3.13.3. The maximally non-split parabolic loci. Let M_k be the Levi subgroup of GSp_{2n} , P_k be the corresponding parabolic subgroup of GSp_{2n} and U_k be the unipotent radical of P_k for $1 \leq k \leq n$ as in Example 3.11.7.

We introduce some notations. Let A be a reduced finite type $\bar{\mathbb{F}}_p$ -algebra and $\mathrm{Spec} A \rightarrow \mathcal{X}_{K, M_{ra}, \mathrm{red}}$ be a morphism. By replacing A by its scheme-theoretically surjective cover, we may assume $\mathrm{Spec} A \rightarrow \mathcal{X}_{K, M_{ra}, \mathrm{red}}$ is basic in the sense of [59, Definition 10.1.2]. Set

$$\begin{aligned} X_A &:= \mathrm{Spec} A, \\ Y_A &:= \mathrm{Spec} A \times_{\mathcal{X}_{K, M_{ra}, \mathrm{red}}} \mathcal{X}_{K, P_{ra}/U_{ra}^{\mathrm{der}}}, \\ Z_A &:= \mathrm{Spec} A \times_{\mathcal{X}_{K, M_{ra}, \mathrm{red}}} \mathcal{X}_{K, P_{ra}}. \end{aligned}$$

These spaces are algebraic stacks by [59, Proposition 10.1.8].

Write Y_A^C , Z_A^C for the coarse moduli sheaves of respectively Y_A , Z_A and W_A for the scheme-theoretic image of Z_A^C in Y_A^C . Let c be the codimension of W_A in Y_A^C .

Lemma 3.13.4. *In the above situation, we assume $\mathrm{Spec} A$ is irreducible and each $x \in (\mathrm{Spec} A)(\bar{\mathbb{F}}_p)$ corresponds to a representation*

$$\begin{pmatrix} \bar{\alpha}(-1)^{\oplus r} & * & * \\ & \bar{\tau} & * \\ & & (\psi \bar{\alpha}^\vee(1))^{\oplus r} \end{pmatrix}$$

such that $\bar{\tau} \in \mathcal{X}_{2n-2r}^{\bar{\alpha}(-1)^{\oplus s}} \setminus \mathcal{X}_{2n-2r}^{\bar{\alpha}(-1)^{\oplus s+1}}$ for an $s \geq 0$.

Write d_X for the dimension of the scheme-theoretic image of $X_A \rightarrow \mathcal{X}_{K, M_{ra}, \text{red}}$ and d_Z for the dimension of the scheme-theoretic image of $Z_A \rightarrow \mathcal{X}_{K, \text{GSp}_{2n}, \text{red}}$. Then we have

$$d_Z \leq d_X + rs - c + [K : \mathbb{Q}_p] \left(ra(n - 2ra) + \frac{ra(ra + 1)}{2} \right) + \begin{cases} \frac{r(r+1)}{2} & (\psi \bar{\alpha}^\vee(1) = \bar{\alpha}(-2)) \\ 0 & (\psi \bar{\alpha}^\vee(1) \neq \bar{\alpha}(-2)) \end{cases},$$

$$d_X \leq d_{2n-2ra, s} - r^2.$$

PROOF. We have

$$\dim Z_A^C \leq \dim X_A - c + \text{rank } H^1(\text{Gal}_K, U_{ra}^{\text{ab}}) + \text{rank } H^1(\text{Gal}_K, U_{ra}^{\text{der}})$$

by the definitions. Then the lemma follows by the argument of the proof of [61, Lemma 10.8]. Note that $\dim U_{ra} = ra(n - 2ra) + \frac{ra(ra + 1)}{2}$. $\square$

PROOF OF THEOREM 3.13.1. We need to show $d_Z \leq d_{2n, r}$ for all $0 \leq s \leq \frac{n - ar}{a}$. It suffices to prove

$$(3.13.1) \quad rs - \frac{r(r-1)}{2} + \frac{r^2 - s^2}{2} - \frac{r-s}{2} \leq [K : \mathbb{Q}_p] \left(\frac{1}{2} a^2 r^2 + \left(n - \frac{1}{2} \right) ar \right) + c$$

by Lemma 3.13.4. The LHS of (3.13.1) is a quadratic polynomial in s and achieves maximum at $s = r + 1/2$. Since s only takes integral values, we may assume $s = r$. So the equation (3.13.1) is reduced to

$$(3.13.2) \quad \frac{1}{2} r^2 + \frac{1}{2} r \leq [K : \mathbb{Q}_p] \left(\frac{1}{2} a^2 r^2 + \left(n - \frac{1}{2} \right) ar \right) + c.$$

The RHS of (3.13.2) is at least $\frac{1}{2} r^2 + \frac{1}{2} r$ since $[K : \mathbb{Q}_p] \geq 1$, $a \geq 1$ and $n \geq 1$. $\square$

3.14. Zariski density of crystalline points

In this section, we prove the main theorem of this chapter.

We fix a continuous representation $\bar{\rho} : \text{Gal}_K \rightarrow \text{GSp}_{2n}(\overline{\mathbb{F}}_p)$. Let $R_{\bar{\rho}}$ be the framed universal deformation ring over the ring of integer $\mathcal{O} := \mathcal{O}_L$ of a sufficiently large finite extension L/K and write $\mathfrak{X}_{\bar{\rho}} := (\text{Spf } R_{\bar{\rho}})^{\text{rig}}$ for the rigid generic fiber of $\text{Spf } R_{\bar{\rho}}$. Let $\mu := \mu_{p^\infty}(K)$ denote the group of p -power roots of unity in K .

Theorem 3.14.1. *Let $\mathfrak{X}_{\text{reg-crys}}$ be the subset of regular crystalline points in $\mathfrak{X}_{\bar{\rho}}$. If*

- (1) $p \geq 3$, or
- (2) $p = 2$ and $\bar{\rho}$ is Q_{red}^Δ -surjective,

then $\mathfrak{X}_{\text{reg-crys}}$ is Zariski dense in $\mathfrak{X}_{\bar{\rho}}$.

PROOF. Let $\rho^{\text{univ}} : \text{Gal}_K \rightarrow \text{GSp}_{2n}(R_{\bar{\rho}})$ be the universal representation. Let $\bar{\psi}$ be the composite

$$\text{Gal}_K \xrightarrow{\rho^{\text{univ}}} \text{GSp}_{2n}(R_{\bar{\rho}}) \rightarrow (R_{\bar{\rho}})^*,$$

where the last map is the similitude character. Replacing L with its finite extension if necessary, we may take and fix a homomorphism $\mathcal{O}[\mu] \rightarrow R_{\bar{\psi}}$ as constructed in [70, §15.2].

Since $\mathcal{O}[\mu] \left[\frac{1}{p} \right] \cong \prod_{\chi: \mu \rightarrow \mathcal{O}^*} L$, we get decompositions

$$R_{\overline{\psi}} \left[\frac{1}{p} \right] = \prod_{\chi: \mu \rightarrow \mathcal{O}^*} R_{\overline{\psi}}^{\chi} \left[\frac{1}{p} \right], \quad R_{\overline{\rho}} \left[\frac{1}{p} \right] = \prod_{\chi: \mu \rightarrow \mathcal{O}^*} R_{\overline{\rho}}^{\chi} \left[\frac{1}{p} \right]$$

where $R_{\overline{\psi}}^{\chi} := R_{\overline{\psi}} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$ and $R_{\overline{\rho}}^{\chi} := R_{\overline{\rho}} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$ for each homomorphism of groups $\chi: \mu \rightarrow \mathcal{O}^*$.

By [70, Theorem 1.2], there is a natural bijection between the set of irreducible components of $\mathrm{Spec} R_{\overline{\rho}}$ and those of $\mathrm{Spec} R_{\overline{\psi}}$. Then it follows that the irreducible components of $\mathfrak{X}_{\overline{\rho}}$ are given by the rigid generic fiber $\mathfrak{X}_{\overline{\rho}}^{\chi}$ of $\mathrm{Spf} R_{\overline{\rho}}^{\chi}$ for $\chi: \mu \rightarrow \mathcal{O}^*$ by [28, Theorem 2.3.1].

By Proposition 3.10.2, the Zariski closure of $\mathcal{S}_{\mathrm{cr}}$ is a union of irreducible components of $\mathfrak{X}_{\overline{\rho}}$. Note that if $p \geq 3$, any continuous representation $\mathrm{Gal}_K \rightarrow \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_p)$ is $Q_{\mathrm{red}}^{\Delta}$ -surjective (cf. Remark 3.12.3). When the assumption of the theorem is satisfied, each irreducible component of $\mathfrak{X}_{\overline{\rho}}$ has a point, which is crystalline with regular Hodge-Tate weight, by Theorem 3.12.4 and Theorem 3.13.1. Then the assertion follows. $\square$

CHAPTER 4

Locally analytic completed cohomology of Shimura varieties of Hodge type

4.1. Introduction

In this chapter, we study locally analytic subrepresentations of completed cohomology of Shimura varieties of Hodge type.

In Section 4.2, we recall the notation of Shimura varieties of Hodge type and their special case, unitary Shimura varieties. In Section 4.3, we recall the construction of the Hodge-Tate period map associated with general Shimura varieties. In Section 4.4, we slightly generalize the result of Camargo [77, Theorem 1.1.8] in the case of J -locally analytic cohomology with coefficient of local systems attached to p -adic algebraic representations of reductive groups. In Section 4.5, we will compare the local sections on perfectoid Siegel modular varieties and on perfectoid Shimura varieties of Hodge type. We also prove the locally analyticity of perfectoid covers of general Shimura varieties of Hodge type generalizing the method of Pan [69, §3.5, §3.6, §4.2]. From them, we will deduce the main theorem.

4.2. Shimura varieties of Hodge type and unitary Shimura curves

4.2.1. Generalities of Shimura varieties. Let G be a reductive group over $\mathbb{Q}$ and (G, X) a Shimura datum (e.g. see [64, Definition 5.5]). Let $\mathcal{F}/\mathbb{Q}$ be the reflex field of (G, X) .

A natural projective system $\mathrm{Sh}_K(\mathbb{C})$ of complex manifolds over $\mathbb{C}$ indexed by open compact subgroups $K \subset G(\mathbb{A}^\infty)$ is associated with the datum (G, X) , which is defined by

$$\mathrm{Sh}_K(\mathbb{C}) := G(\mathbb{Q}) \backslash (X \times (G(\mathbb{A}^\infty)/K)).$$

Let Sh_K denote the canonical model of $\mathrm{Sh}_K(\mathbb{C})$ over $\mathrm{Spec} \mathcal{F}$.

Let Z be the center of G and $Z_c \subset Z$ its maximal $\mathbb{Q}$ -anisotropic torus which is $\mathbb{R}$ -split. Let $G^c := G/Z_c$. For any subgroup H of $G(\mathbb{A})$, let H^c denote the image of H in G^c .

Given neat open compact subgroups $K_1 \subset K_2 \subset G(\mathbb{A}^\infty)$ where K_1 is a normal subgroup of K_2 , the induced map $\mathrm{Sh}_{K_1} \rightarrow \mathrm{Sh}_{K_2}$ of Shimura varieties is Galois finite étale with Galois group K_2^c/K_1^c .

For a Shimura variety Sh_K and an auxiliary K -admissible cone decomposition Σ , let $\mathrm{Sh}_{K,\Sigma}^{\mathrm{tor}}$ denote the toroidal compactification as in [72] (see also [40] for an algebraic construction in the Siegel modular varieties). When the choice of the cone decomposition will not be important, we write $\mathrm{Sh}_K^{\mathrm{tor}} = \mathrm{Sh}_{K,\Sigma}^{\mathrm{tor}}$. A cone decomposition Σ can be taken as follows and assume the conditions are always satisfied:

- (1) Σ is a smooth cone decomposition relative to K so that the toroidal compactification $\mathrm{Sh}_K^{\mathrm{tor}}$ is smooth, projective with boundary divisor given by normal crossings ([72, Theorem 9.21]).

- (2) The divisor of boundary $D_K := \text{Sh}_K^{\text{tor}} \setminus \text{Sh}_K$ is a union $D_K = \cup_{a \in I} D_{K,a}$ of finite irreducible smooth divisors and $\cup_{a \in I'} D_{K,a}$ is also a smooth divisor for any subset $I' \subset I$ ([72, Proposition 9.20]).

4.2.2. Shimura varieties of Hodge type. We introduce the notation of the *Siegel modular varieties* and *Shimura varieties of Hodge type* (cf. [83], etc.). Fix an integer $g \geq 1$. Let (V, ψ) be a symplectic space of dimension $2g$ over $\mathbb{Q}$, that is, V is a vector space of dimension $2g$ over $\mathbb{Q}$ and $\psi: V \times V \rightarrow \mathbb{Q}$ is a non-degenerate skew-symmetric bilinear form. Note that (V, ψ) is (non-canonically) isomorphic to the standard symplectic space (V_{2g}, ψ_{2g}) , which is, $V_{2g} := \mathbb{Q}^{2g}$ and ψ is the standard symplectic pairing defined by

$$\psi((a_1, \dots, a_g, b_1, \dots, b_g), (a'_1, \dots, a'_g, b'_1, \dots, b'_g)) = \sum_{i=1}^g (a_i b'_i - a'_i b_i).$$

We fix the self-dual lattice $V_{2g, \mathbb{Z}} := \mathbb{Z}^{2g}$. Let $\text{GSp}_{2g}/\mathbb{Z}$ be the group of symplectic similitudes of $V_{2g, \mathbb{Z}}$. Note that the definition of “ GSp_{2g} ” is slightly different from that of “ GSp_{2n} ” in the last two chapters. For an integer $m \geq 1$, we define the following congruence subgroups of $\text{GSp}_{2g}(\mathbb{Z}_p)$

$$\begin{aligned} \Gamma_0(p^m) &:= \left\{ \gamma \in \text{GSp}_{2g}(\mathbb{Z}_p) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p^m}, \det \gamma \equiv 1 \pmod{p^m} \right\}, \\ \Gamma_1(p^m) &:= \left\{ \gamma \in \text{GSp}_{2g}(\mathbb{Z}_p) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^m} \right\}, \\ \Gamma(p^m) &:= \left\{ \gamma \in \text{GSp}_{2g}(\mathbb{Z}_p) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p^m} \right\}, \end{aligned}$$

where the all blocks has size $g \times g$.

A Shimura datum (G, D) is said to be *of Hodge type* if it admits a closed embedding $(G, D) \hookrightarrow (\text{GSp}_{2g}, D_{\text{GSp}_{2g}})$ into the group of symplectic similitude group GSp_{2g} , with $D_{\text{GSp}_{2g}}$ given by the Siegel upper half space. The associated variety Sh_K with (G, D) of Hodge type and a compact open subgroup $K \subset G(\mathbb{A}^\infty)$ is called a *Shimura variety of Hodge type*.

4.2.3. Unitary Shimura curves. We introduce the notation of the *unitary Shimura curves* handled within the paper (cf. [33], [23], [35]). Let F be a totally real number field of finite degree $d_F := [F: \mathbb{Q}] > 1$. Let B be a quaternion algebra over F . We assume that there is a unique real place τ_∞ where B splits.

Let $\mathcal{E}$ be a quadratic imaginary extension of $\mathbb{Q}$ and $\mathcal{F} := F \cdot \mathcal{E}$. Let $c: \mathcal{F} \rightarrow \mathcal{F}$ be the conjugation induced by that of $\mathcal{E}$. Fix an embedding $u_\infty: \mathcal{E} \hookrightarrow \mathbb{C}$ and write $u_\mathbb{C}: \mathcal{F} \rightarrow \mathbb{C}$ for the unique embedding such that $u_\mathbb{C}|_F = \tau_\infty$ and $u_\mathbb{C}|_\mathcal{E} = u_\infty$. Let $D := B \otimes_F \mathcal{F}$. Let $l \mapsto \bar{l}$ denote the tensor product of the canonical involution of B and the conjugation $c: \mathcal{F} \rightarrow \mathcal{F}$. Let $\delta \in D$ be a invertible and symmetric (i.e. $\bar{\delta} = \delta$) element and set $l^* := \delta^{-1} \bar{l} \delta$. The map $l \mapsto l^*$ is an involution of second kind on D .

Write V for the underlying $\mathbb{Q}$ -vector space of D , which is equipped with the action of $D \otimes_{\mathcal{F}} D^{\text{op}}$ defined by $(d_1 \otimes d_2) \cdot v = d_1 v d_2$ for any $d_1, d_2 \in D$ and $v \in V$. Choose an element $\alpha \in \mathcal{F}$ that is non-zero and imaginary (i.e. $\bar{\alpha} = -\alpha$). We define a pairing $\psi: V \times V \rightarrow \mathcal{F}$ as

$$\psi(v, w) := \text{tr}_{\mathcal{F}/\mathbb{Q}}(\alpha \text{tr}_{D/\mathcal{F}}(v \delta w^*))$$

for $v, w \in V$. It is an alternating and non-degenerate form and the following equation is verified:

$$\psi(lv, w) = \psi(v, l^*w).$$

For any $\mathbb{Q}$ -algebra R , regard elements $g \in D^{\text{op}} \otimes_{\mathbb{Q}} R$ as D -linear R -endomorphisms $v \mapsto vg$ of $V \otimes_{\mathbb{Q}} R$. Let G be the algebraic group such that $G(R)$ is the subgroup of the multiplicative group $(D^{\text{op}} \otimes_{\mathbb{Q}} R)^{\times}$ which satisfies $g \in G(R)$ if and only if there exists an element $r \in R^{\times}$ such that

$$\psi(vg, wg) = r\psi(v, w)$$

for any $v, w \in V \otimes_{\mathbb{Q}} R$.

Let $\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. Let $\mu: \mathbb{S} \rightarrow G_{\mathbb{R}}$ be the homomorphism defined in [23, §2.1.3]. The map μ defines the $G(\mathbb{R})$ -conjugation class X of homomorphisms $\mathbb{S} \rightarrow G_{\mathbb{R}}$, which is identified with the Poincaré upper half plane. Then the composition of h and the embedding $G_{\mathbb{R}} \hookrightarrow \text{GL}(V_{\mathbb{R}})$ defines a Hodge structure of type $\{(-1, 0), (0, -1)\}$ (cf. [23, §2.2.4]). In other words, the map induced a closed immersion $(G, X) \hookrightarrow (\text{GSp}(V, \psi), D)$ where D is given by the Siegel upper half space and so (G, X) is a Shimura data of Hodge type.

A natural projective system $\text{Sh}_K(\mathbb{C})$ of curves over $\mathbb{C}$ indexed by open compact subgroups $K \subset G(\mathbb{A}^{\infty})$ is associated with the data (G, X) as in Section 4.2.1. The curve $M_K(\mathbb{C})$ admits a proper smooth canonical model over the reflex field $\mathcal{F}$ via the embedding $u_{\mathbb{C}}: \mathcal{F} \hookrightarrow \mathbb{C}$.

Fix a place $\wp$ of F over p . Write $\wp_1 = \wp, \wp_2, \dots, \wp_s$ for the places of F over p . Assume that p decomposes in $\mathcal{E}$ and B splits at $\wp_i$ for all place $\wp_i$ of F over p . Fix isomorphisms $D \otimes_{\mathbb{Q}} F_{\wp_i} \xrightarrow{\sim} M_2(F_{\wp_i})$ for each place $\wp_i$ of F over p and an embedding $\lambda: \mathcal{E} \hookrightarrow \mathbb{Q}_p$, which corresponds to a place of $\mathcal{E}$ over p . Let $\mathcal{O}_D$ be an order of D which is stable by the involution $l \mapsto l^*$. We have the following decomposition

$$\begin{aligned} D \otimes_{\mathbb{Q}} \mathbb{Q}_p &\cong (D \otimes_{\mathcal{E}, \lambda} \mathbb{Q}_p) \oplus (D \otimes_{\mathcal{E}, \lambda \circ c} \mathbb{Q}_p) \\ &\cong \left(\prod_{\wp_i | p} M_2(F_{\wp_i}) \right) \oplus \left(\prod_{\wp_i | p} M_2(F_{\wp_i}) \right) =: \left(\prod_{\wp_i | p} D_{\wp_i}^+ \right) \oplus \left(\prod_{\wp_i | p} D_{\wp_i}^- \right) \end{aligned}$$

and similarly we have

$$\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{\wp_i | p} (\mathcal{O}_{D_{\wp_i}^+} \oplus \mathcal{O}_{D_{\wp_i}^-}).$$

We assume the following condition on λ :

- The ring $\mathcal{O}_{D_{\wp_i}^+}$ (resp. $\mathcal{O}_{D_{\wp_i}^-}$) is a maximal order in $D_{\wp_i}^+$ (resp. $D_{\wp_i}^-$) and the ring $\mathcal{O}_{D_{\wp_i}^-}$ is identified with $M_2(\mathcal{O}_{F_{\wp_i}})$ via the composite $\mathcal{O}_{D_{\wp_i}^-} \hookrightarrow D_{\wp_i}^- \cong M_2(F_{\wp_i})$ for all places $\wp_i | p$.

Let $e_{\wp_i}^{-,1} \in \mathcal{O}_{D_{\wp_i}^-}$ (resp. $e_{\wp_i}^{-,2} \in \mathcal{O}_{D_{\wp_i}^-}$) be the element which is identified with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$) via the isomorphism $\mathcal{O}_{D_{\wp_i}^-} \xrightarrow{\sim} M_2(\mathcal{O}_{F_{\wp_i}})$. Set $e_{\wp_i}^{+,k} := (e_{\wp_i}^{-,k})^*$ for $k = 1, 2$. For any $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ -module Λ , we have the following decomposition

$$\begin{aligned} \Lambda &\cong \prod_{\wp_i | p} (\Lambda_{\wp_i}^+ \oplus \Lambda_{\wp_i}^-) \cong \prod_{\wp_i | p} (\Lambda_{\wp_i}^{+,1} \oplus \Lambda_{\wp_i}^{+,2} \oplus \Lambda_{\wp_i}^{-,1} \oplus \Lambda_{\wp_i}^{-,2}) \\ &\cong \prod_{\substack{\wp_i | p \\ \sigma \in \Sigma_{\wp_i}}} (\Lambda_{\wp_i, \sigma}^{+,1} \oplus \Lambda_{\wp_i, \sigma}^{+,2} \oplus \Lambda_{\wp_i, \sigma}^{-,1} \oplus \Lambda_{\wp_i, \sigma}^{-,2}) \end{aligned}$$

where $\Lambda_{\wp_i}^{\pm,k} := e_{\wp_i}^{\pm,k} \Lambda$ is an $\mathcal{O}_{F_{\wp_i}}$ -module.

For any $\mathbb{Q}$ -algebra R , we have the isomorphisms

$$\begin{aligned} G(R) &\xrightarrow{\sim} \{(r, g) \in R^\times \times (D^{\text{op}} \otimes_{\mathbb{Q}} R)^\times \mid gg^* = r\} \\ &\xrightarrow{\sim} \{(r, g) \in R^\times \times (D \otimes_{\mathbb{Q}} R)^\times \mid gg^* = r\} \end{aligned}$$

functorial in R , where the first map is the natural isomorphism by the definition of G and the second map is defined by

$$\begin{aligned} R^\times \times (D^{\text{op}} \otimes_{\mathbb{Q}} R)^\times &\xrightarrow{\sim} R^\times \times (D \otimes_{\mathbb{Q}} R)^\times, \\ (r, g) &\mapsto (r^{-1}, g^{-1}). \end{aligned}$$

Thus, we have an isomorphism

$$(4.2.1) \quad G_{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{G}_{m, \mathbb{Q}_p} \times \prod_{\wp_i | p} \text{Res}_{F_{\wp_i}/\mathbb{Q}_p} \text{GL}_{2, F_{\wp_i}}$$

of algebraic groups over $\mathbb{Q}_p$ by the above condition on λ . We also set

$$\begin{aligned} G(\mathbb{Q}_p^\wp) &:= \mathbb{Q}_p^\times \times \prod_{\wp_i \neq \wp} \text{GL}_2(F_{\wp_i}) \subset G(\mathbb{Q}_p), \\ G(\mathbb{A}^{\infty, \wp}) &:= G(\mathbb{Q}_p^\wp) \times G(\mathbb{A}^{\infty, p}) \subset G(\mathbb{A}^\infty). \end{aligned}$$

We fix a finite extension $E/\mathbb{Q}_p$ such that $|\text{Hom}_{\mathbb{Q}_p}(F_{\wp_i}, E)| = [F_{\wp_i} : \mathbb{Q}_p]$ for all $i \in [1, s]$ and write $\Sigma_{\wp_i} := \text{Hom}_{\mathbb{Q}_p}(F_{\wp_i}, E)$.

We fix an embedding $\tau : F_\wp \hookrightarrow E$ in $\Sigma_\wp$ and a lattice $V_{\mathbb{Z}} \subset V$ where ψ induces a perfect duality. Let $V_{\widehat{\mathbb{Z}}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Theorem 4.2.1. *Suppose that $K \subset G(\mathbb{A}^\infty)$ is a neat subgroup. Then the curve $\text{Sh}_{K, \wp} := \text{Sh}_K \times_{\mathcal{F}} F_\wp$ is represented by a functor $(\text{Sch} / \text{Spec } F_\wp) \rightarrow \text{Set}$, which associates each scheme S over $\text{Spec } F_\wp$ with the set of isomorphism classes of the data $(A, \iota, \theta, \bar{\alpha})$ where*

- (1) *A is an abelian scheme of relative dimension $4d_F$ over S , with an action of $\mathcal{O}_D$ via $\iota : \mathcal{O}_D \hookrightarrow \text{End}_S(A)$ (Note that ι induces a natural $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Q}_p$ -module structure on $\text{Lie}(A)$) which satisfies the following two conditions:*
 - (a) *Here $\text{Lie}(A)_{\wp}^{-1}$ is a locally free $\mathcal{O}_S$ -module of rank 1, where $F_\wp$ acts via the structural morphism.*
 - (b) *$\theta : A \rightarrow A^\vee$ is a polarization of A where the associated Rosati involution sends $\iota(\gamma)$ to $\iota(\gamma^*)$.*
- (2) *Let $\widehat{T}(A) := (\varprojlim_n A[n]) \otimes_{\mathbb{Z}} \mathbb{Q}$. Here $\bar{\alpha}$ is a modulo K class of*

$$\alpha : V_{\widehat{\mathbb{Z}}} \xrightarrow{\sim} \widehat{T}(A)$$

which are $\mathcal{O}_D$ -linear symplectic isomorphisms local on the étale topology on S .

PROOF. See [33, Théorème 4.19], [23, §2.3], and also [35, §3.1.1]. $\square$

4.3. The Hodge-Tate period map

We fix an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. In this section, we recall the construction of the Hodge-Tate period maps of general Shimura varieties using (logarithmic) p -adic Riemann-Hilbert correspondence.

4.3.1. Log adic spaces. We recall the notation of Kummer-étale sites on log adic spaces ([36], [37]).

Let X be a log smooth adic space ([37, Definition 3.1.1]) over $\mathrm{Spa}(k, k^+)$, where (k, k^+) is a non-archimedean extension of $(\mathbb{Q}_p, \mathbb{Z}_p)$. We endow X with the log structure induced by a normal crossing divisor $D \subset X$ ([37, Example 2.3.17]). Let X_{an} , $X_{\mathrm{két}}$, $X_{\mathrm{prokét}}$ denote the analytic, Kummer-étale and pro-Kummer étale sites of X respectively ([37, Definition 4.1.16, Definition 5.1.2]). If D is empty, $X_{\mathrm{két}}$ and $X_{\mathrm{prokét}}$ are the same as the (pro-)étale sites $X_{\mathrm{ét}}$ and $X_{\mathrm{proét}}$ of X as in [82] respectively. Log perfectoid affinoids ([37, Definition 5.3.1]) forms a basis of $X_{\mathrm{prokét}}$ by [37, Proposition 5.3.12]. We also write $\mathcal{O}_X$ for the inverse images to $X_{\mathrm{prokét}}$ of the structure sheaf.

For a topological abelian group V , let $\underline{V}$ denote the pro-Kummer étale sheaf defined as

$$\underline{V}((\mathrm{Spa}(R, R^+), \mathcal{M})) := \mathrm{Hom}_{\mathrm{cont}}(|\mathrm{Spa}(R, R^+)|, V)$$

for each log affinoid perfectoid object $(\mathrm{Spa}(R, R^+), \mathcal{M})$ in $X_{\mathrm{prokét}}$ ([37, Definition 5.3.1]), where $|\mathrm{Spa}(R, R^+)|$ denotes the underlying topological space of $\mathrm{Spa}(R, R^+)$. When $V = \mathbb{Z}_p$ or $\mathbb{Q}_p$, write $\widehat{\mathbb{Z}}_p := \mathbb{Z}_p$ and $\widehat{\mathbb{Q}}_p := \mathbb{Q}_p$.

Let $\widehat{\mathcal{O}}_X^+$, $\widehat{\mathcal{O}}_X$, $\mathbb{B}_{\mathrm{dR}, X}^+$, $\mathbb{B}_{\mathrm{dR}, X}$, $\mathcal{O}_{\mathbb{B}_{\mathrm{dR}, \log, X}}^+$, $\mathcal{O}_{\mathbb{B}_{\mathrm{dR}, \log, X}}$, etc. be the period sheaves on $X_{\mathrm{proét}}$ or $X_{\mathrm{prokét}}$ as in [36, §2.2].

4.3.2. The Hodge-Tate period map for Shimura varieties of Hodge type. In this subsection, we use the notation in Section 4.2.1. Fix a Shimura datum (G, X) of Hodge type. Let μ be the $G(\mathbb{C})$ -conjugacy class of Hodge cocharacters (defined over $\mathcal{F}$) attached to X , which gives rise to associated subgroups P_μ , N_μ , M_μ of $G_{\mathcal{F}}$ and a flag variety $\mathrm{Fl}_\mu = G/P_\mu$ over $\mathrm{Spec} \mathcal{F}$ (cf. [22, §2.1], [77, §3.1]). Let $\mathcal{F}\ell_\mu$ denote the adic space attached to $\mathrm{Fl}_\mu \times_{\mathrm{Spec} \mathcal{F}} \mathrm{Spec} \mathbb{C}_p$.

We recall the notation of G -equivariant Lie algebroids (cf. [5, §3], [69, §4.2.6], [77, §3.2]). Let $\mathfrak{g}$, $\mathfrak{g}^c$, $\mathfrak{p}_\mu$, $\mathfrak{n}_\mu$ and $\mathfrak{m}_\mu$ be the Lie algebras of G , G^c , P_μ , N_μ and M_μ respectively. Let $\mathfrak{g}^0 := \mathcal{O}_{\mathcal{F}\ell_\mu} \otimes \mathfrak{g}$. We also have a natural filtration $\mathfrak{n}_\mu^0 \subset \mathfrak{p}_\mu^0 \subset \mathfrak{m}_\mu^0$ depending on μ and set $\mathfrak{m}_\mu^0 := \mathfrak{p}_\mu^0 / \mathfrak{n}_\mu^0$.

Let $\mathcal{S}h_K = \mathcal{S}h_{K, \mathbb{C}_p}$ denote the adic space over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ attached to $\mathrm{Sh}_{K, \mathbb{C}_p} := \mathrm{Sh}_K \times_{\mathrm{Spec} \mathcal{F}} \mathrm{Spec} \mathbb{C}_p$, (cf. [47]). We also let $\mathcal{S}h_K^{\mathrm{tor}} = \mathcal{S}h_{K, \mathbb{C}_p}^{\mathrm{tor}}$ denote the adic space attached to the toroidal compactification. It is seen as a log adic space with log structure defined by the normal crossing divisor of the boundary.

Suppose that $K = K_p K^p$ for some open compact subgroups $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}^{\infty, p})$. Let $\varprojlim_{K_p} \mathcal{S}h_{K_p K^p}$, where $K_p \subset G(\mathbb{Q}_p)$ lies on open compact subgroups, be the infinite level Shimura variety considered as an object in $\mathcal{S}h_{K_p K^p, \mathrm{proét}}$. Assume K is neat. Then we can take the same smooth cone decomposition Σ relative to $K = K'_p K^p$ for all sufficiently small $K'_p \subset K_p$ ([72, Corollary 9.33]) which is also compatible with an closed embedding $\iota: (G, X) \hookrightarrow (\mathrm{GSp}_{2g}, D_{\mathrm{GSp}_{2g}})$ of the Shimura data ([54, Proposition 4.10]). Assume that such Σ is taken. Then consider also $\varprojlim_{K_p} \mathcal{S}h_{K_p K^p}^{\mathrm{tor}}$ to be an object in $\mathcal{S}h_{K_p K^p, \mathrm{prokét}}^{\mathrm{tor}}$.

Theorem 4.3.1. *For any neat compact open subgroup $K^p \subset G(\mathbb{A}^{\infty, p}) \subset G(\mathbb{A}^\infty)$ contained in the level- N -subgroup*

$$\{\gamma \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}^p) \mid \gamma \equiv 1 \pmod{N}\}$$

of $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}^p)$ for some $N \geq 3$ prime to p , there exist perfectoid spaces $\mathcal{S}h_{K^p}$ and $\mathcal{S}h_{K^p}^{\mathrm{tor}}$ such that

$$\mathcal{S}h_{K^p} \sim \varprojlim_{K_p} \mathcal{S}h_{K_p K^p} \quad \text{and} \quad \mathcal{S}h_{K^p}^{\mathrm{tor}} \sim \varprojlim_{K_p} \mathcal{S}h_{K_p K^p}^{\mathrm{tor}},$$

where “ $\sim$ ” is referred to [84, Definition 2.4.1]. Moreover, the closed embedding of Shimura data ι induces natural closed immersions of perfectoid spaces

$$\mathcal{S}h_{K^p} \hookrightarrow \mathcal{Y}_{K'^p}, \quad \text{and} \quad \mathcal{S}h_{K^p}^{\mathrm{tor}} \hookrightarrow \mathcal{Y}_{K'^p}^{\mathrm{tor}}$$

where $K'^p \subset \mathrm{GSp}_{2g}(\mathbb{A}^\infty)$ is a neat compact open subgroup such that $K'^p \cap G(\mathbb{A}^\infty) = K^p$ and $\mathcal{Y}_{K'^p}$, $\mathcal{Y}_{K'^p}^{\mathrm{tor}}$ be the perfectoid Siegel modular variety of level K'^p and its toroidal compactification.

PROOF. The proof is similar to that of [83, Theorem IV.1.1] and [54, Proposition 6.1]. $\square$

Write $\mathcal{S}h := \mathcal{S}h_{K^p K^p}^{\mathrm{tor}}$. Let V be an object in $\mathrm{Rep}_{\mathbb{Q}_p}(G^c)$. The object V naturally induces an étale local system $V_{\mathrm{\acute{e}t}}$ (resp. Kummer étale local system $V_{\mathrm{k\acute{e}t}}$ on $\mathcal{S}h$ (resp. $\mathcal{S}h^{\mathrm{tor}}$) (see [63, §4.2] and [36, §5.2]). Let $V_{\mathrm{prok\acute{e}t}}$ denote the associated pro-Kummer étale local system by [37, Lemma 6.3.3]. It is de Rham in the sense of [77, Definition 2.2.1] by [36, Theorem 5.3.1]. We note that the pullback of $V_{\mathrm{prok\acute{e}t}}$ by the natural map of sites $\mathcal{S}h_{K^p, \mathrm{prok\acute{e}t}} \rightarrow \mathcal{S}h_{K_p K^p, \mathrm{prok\acute{e}t}}$ is the constant pro-Kummer étale sheaf $\underline{V}$ defined in Section 4.3.1.

Proposition 4.3.2. (1) For $K^p \subset G(\mathbb{A}^{\infty, p})$ which satisfies the condition in Theorem 4.3.1, there exists a natural $G(\mathbb{Q}_p)$ -equivariant map of adic spaces

$$\pi_{\mathrm{HT}}^{\mathrm{tor}}: \mathcal{S}h_{K^p}^{\mathrm{tor}} \rightarrow \mathcal{F}\ell_\mu,$$

which is functorial in Shimura data.

(2) There exists a basis $\mathfrak{B}$ of open affinoid subsets of $\mathcal{F}\ell$ stable under finite intersections and each $V \in \mathfrak{B}$ has the following properties:

- (a) Its preimage $U_\infty := \pi_{\mathrm{HT}}^{\mathrm{tor}, -1}(V)$ is affinoid perfectoid.
- (b) U_∞ is the preimage of an affinoid subset $U_{K'_p} \subset \mathcal{S}h_{K'_p K^p}^{\mathrm{tor}}$ for sufficiently small compact open subgroup $K'_p \subset G(\mathbb{A}^\infty)$.
- (c) The map $\varinjlim_{K'_p} H^0(U_{K'_p}, \mathcal{O}_{\mathcal{S}h_{K'_p K^p}^{\mathrm{tor}}}) \rightarrow H^0(U_\infty, \mathcal{O}_{\mathcal{S}h_{K^p}^{\mathrm{tor}}})$ has dense image.

PROOF. The proof of (1) is similar to [15, Theorem 4.40] and [77, Theorem 4.2.1]. We explain the detailed argument. For each object $U = \mathrm{Spa}(R, R^+)$ in $\mathcal{U}_\infty$ and an object V in $\mathrm{Rep}_{\mathbb{Q}_p}(G^c)$, we have a natural identification

$$H^0(U, V_{\mathrm{prok\acute{e}t}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{S}h}) = V \otimes_{\mathbb{Q}_p} R,$$

where the tensor product of the left hand side is taken in the category of pro-Kummer étale presheaves of rings over the presheaf associated with $\mathbb{Q}_p$.

On the other hand, the pro-Kummer étale local system $V_{\mathrm{prok\acute{e}t}}$ is de Rham, i.e., there is a filtered log-connection

$$(V_{\mathrm{dR}, \log}, \nabla_{\log}, \mathrm{Fil}^\bullet)$$

and an isomorphism of pro-Kummer étale sheaves

$$V_{\mathrm{prok\acute{e}t}} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathbb{B}_{\mathrm{dR}, \log}, \mathcal{S}h} \cong V_{\mathrm{dR}, \log} \otimes_{\mathcal{O}_{\mathcal{S}h}} \mathcal{O}_{\mathbb{B}_{\mathrm{dR}, \log}, \mathcal{S}h}$$

on $\mathcal{S}h$ by [36, Theorem 5.3.1] compatible with connections and filtrations.

Then the Hodge-Tate filtration defined by

$$\mathrm{Fil}_{-j}(V_{\mathrm{prok\acute{e}t}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{Sh}) := (\mathbb{M} \cap \mathrm{Fil}^j \mathbb{M}^0) / (\mathrm{Fil}^1 \mathbb{M} \cap \mathrm{Fil}^j \mathbb{M}^0)$$

of $V_{\mathrm{prok\acute{e}t}} \otimes_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{Sh}$, where

$$\mathbb{M} := V_{\mathrm{prok\acute{e}t}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}, Sh}^+ \quad \text{and} \quad \mathbb{M}^0 := (V_{\mathrm{dR}, \log} \otimes_{\mathcal{O}_{Sh}} \mathcal{O}_{\mathbb{B}_{\mathrm{dR}, \log, Sh}}^+)^{\nabla_{\log}=0}$$

is compatible with the Hodge filtration $\mathrm{Fil}^\bullet$ on $V_{\mathrm{dR}, \log}$ by [77, Lemma 2.2.3] (see also [15, §4.4.38]). So we obtain an increasing μ -filtration on $V \otimes_{\mathbb{Q}_p} R$ and this induces the map $\pi_{\mathrm{HT}}^{\mathrm{tor}}$ in this proposition.

The all construction in the proof is $G(\mathbb{Q}_p)$ -equivariant by the moduli interpretation of Siegel modular varieties (cf. [40, §1], [83, §III.3]). The functoriality of $\pi_{\mathrm{HT}}^{\mathrm{tor}}$ in Shimura data follows by the construction and Tannakian formalism. Then we have proved (1).

For (2), see [83, Theorem IV.1.1]. $\square$

Remark 4.3.3. By considering pro-Kummer étale sites on infinite level Shimura varieties instead of perfectoid Shimura varieties of Hodge type in Theorem 4.3.1, it is possible to construct Hodge-Tate period maps for general Shimura varieties ([77, Theorem 4.2.1]).

We call the map $\pi_{\mathrm{HT}}^{\mathrm{tor}}$ the *Hodge-Tate period map*. We write $\pi_{\mathrm{HT}}: Sh_{K^p} \rightarrow \mathcal{F}\ell_\mu$ for the restriction of $\pi_{\mathrm{HT}}^{\mathrm{tor}}$ to Sh_{K^p} . Note that the construction produces the same Hodge-Tate period map as the previously constructed one for Shimura varieties of Hodge type using minimal compactifications instead of toroidal compactifications ([83, Theorem IV.1.1], [22, Theorem 2.1.3]).

4.3.3. Flag varieties associated with unitary Shimura curves. At the end of this section, we make an observation about flag varieties associated with unitary Shimura curves. Let (G, X) be as in Section 4.2.3 and Sh_K is the associated unitary Shimura curve of level $K = K_p K^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}^{\infty, p})$. Fix an embedding $\tau: F_\varphi \hookrightarrow E$. We let

$$\varepsilon: \mathcal{A} \rightarrow (\mathrm{Sh}_K)_{\tau, E} := \mathrm{Sh}_K \times_{\mathrm{Spec} \mathcal{F}} \mathrm{Spec} F_\varphi \times_{\mathrm{Spec} F_\varphi, \tau} \mathrm{Spec} E$$

be the universal abelian variety for the moduli interpretation of the unitary Shimura curve in Theorem 4.2.1. Then $R^1 \varepsilon_* E$ defines a rank two E -local system on $\mathrm{Sh}_{K, \acute{e}t}$, which is isomorphic to the E -local system $V_{\mathrm{std}, \acute{e}t}$ corresponds to the standard representation $\mathrm{GL}_2(F_\varphi) \rightarrow \mathrm{GL}(V_{\mathrm{std}})$ over E via the embedding τ . The local system $R^1 \varepsilon_* E$ is de Rham and the associated filtered module with integrable connection is given by $R^1 \varepsilon_*(\Omega_{\mathcal{A}/\mathrm{Sh}_K}^\bullet \otimes_{\mathbb{Q}_p} E)$ with the Hodge filtration and the Gauss-Manin connection by [82, Theorem 1.10].

Consider the exact sequence

$$(4.3.1) \quad 0 \rightarrow \varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^1 \rightarrow R^1 \varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^\bullet \rightarrow R^1 \varepsilon_* \mathcal{O}_{\mathcal{A}} \rightarrow 0,$$

which defines the Hodge-Tate filtration of the rank two vector bundle $R^1 \varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^\bullet$ over $(\mathrm{Sh}_K)_{\tau, E}$.

Proposition 4.3.4. *Let $\varphi_i | p$ and $\sigma \in \Sigma_{\varphi_i}$.*

- (1) *Unless $\varphi_i = \varphi$ and $\sigma = \tau$, the $\mathcal{O}_{(\mathrm{Sh}_K)_{\tau, E}}$ -module $(R^1 \varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^\bullet)_{\varphi_i, \sigma}^{-, 1}$ is locally free of rank two and naturally isomorphic to the $\mathcal{O}_{(\mathrm{Sh}_K)_{\tau, E}}$ -module $(R^1 \varepsilon_* \mathcal{O}_{\mathcal{A}})_{\varphi_i, \sigma}^{-, 1}$.*
- (2) *If $\varphi_i = \varphi$ and $\sigma = \tau$, the induced maps*

$$0 \rightarrow (\varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^1)_{\varphi, \tau}^{-, 1} \rightarrow (R^1 \varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^\bullet)_{\varphi, \tau}^{-, 1} \rightarrow (R^1 \varepsilon_* \mathcal{O}_{\mathcal{A}})_{\varphi, \tau}^{-, 1} \rightarrow 0$$

is exact sequence of locally free $\mathcal{O}_{(\mathrm{Sh}_K)_{\tau, E}}$ -modules. The rank of second and fourth term is one and the third term is rank two.

PROOF. See [35, Proposition 3.3.2]. $\square$

Recall that a closed immersion $(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), D)$ to a Siegel data is given by the element $h: \mathbb{S} \rightarrow G_{\mathbb{R}}$ in X . Then it follows the exact sequence (4.3.1) naturally comes from the exact sequences from de Rham cohomology of the universal abelian variety of the Siegel modular variety attached with $(\mathrm{GSp}(V, \psi), D)$.

Proposition 4.3.5. *The flag variety $\mathrm{Fl}_{G,E}$ over E associated with the unitary Shimura curve is isomorphic to $\mathrm{GL}_{2,E}/B_E \cong \mathbb{P}_E^1$, where B_E is the standard Borel subgroup of $\mathrm{GL}_{2,E}$.*

PROOF. Since the completed colimit along $K_p \rightarrow 1$ gives the Hodge-Tate filtration defining the Hodge-Tate period map of the infinite level perfectoid Siegel modular variety, it follows that the subgroup

$$\mathbb{G}_{m,E} \times_{\mathrm{Spec} E} \left(\prod_{\wp_i \neq \wp} \mathrm{Res}_{F_{\wp_i}/\mathbb{Q}_p} \mathrm{GL}_2(F_{\wp_i}) \right)_E \times_{\mathrm{Spec} E} \prod_{\sigma \neq \tau} \mathrm{GL}_{2,E} \subset G_E$$

stabilizes the Hodge-Tate filtration by Proposition 4.3.4 (1). On the other hand, the E -vector space $V_{\wp, \tau}^{-,1}$ is identified with the contragredient of the standard representation of $\mathrm{GL}_2(E)$. Let $B \subset \mathrm{GL}_2(E)$ be the Borel subgroup which stabilizes the standard representation. So a subgroup of B stabilizes the Hodge-Tate filtration by Proposition 4.3.4 (2). The closed immersion of the Shimura data induces a closed immersion $G_E/P \hookrightarrow \mathrm{GSp}(V, \psi)_E/\tilde{P}$, where $P \subset G_E$ is a parabolic subgroup and $\tilde{P} \subset \mathrm{GSp}(V, \psi)_{\mathbb{Q}_p}$ is the Siegel parabolic subgroup. In particular, G_E/P is proper over E , so just the Borel subgroup $B \subset \mathrm{GL}_2(E)$ is the stabilizer of the whole Hodge-Tate filtration. $\square$

The exact sequence obtained by tensoring (4.3.1) with $\hat{\mathcal{O}}_{(\mathrm{Sh}_K)_{\tau, E}}$ produces a Hodge-Tate period map $\pi_{\mathrm{HT}, G}: \mathcal{S}h_{K^p} \rightarrow \mathcal{F}\ell_{\mu}$, which is compatible with the Hodge-Tate period map $\pi_{\mathrm{HT}, \mathrm{GSp}(V, \psi)}^{\mathrm{tor}}$ of the Siegel modular variety via the closed immersion $(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), D)$ of the Shimura data by Proposition 4.3.2.

4.4. J -locally analytic completed cohomology of unitary Shimura curves

We recall the notion of J -locally analytic actions of p -adic Lie groups and slightly generalize the results in [69] and [77] to the case of J -locally analytic vectors in the completed cohomology of unitary Shimura curves.

4.4.1. J -Locally analytic vectors. Let $\Sigma_{\wp} := \mathrm{Hom}_{\mathbb{Q}_p}(F_{\wp}, E)$ and $J \subset \Sigma_{\wp}$. Let $\mathcal{G}$ be a p -adic Lie group of dimension d . Then there exists a compact open subgroup $\mathcal{G}_0 \subset \mathcal{G}$ which is a uniform pro- p group (cf. e.g., [74, §4.1.1]). Consequently, there exist $g_1, \dots, g_d \in \mathcal{G}_0$ such that the map

$$c: \mathbb{Z}_p^d \rightarrow \mathcal{G}_0, \quad (x_1, \dots, x_d) \mapsto g_1^{x_1} \cdots g_d^{x_d}$$

is a homeomorphism. Fix such c . Let $\mathcal{G}_n := \mathcal{G}_0^{p^n} = \{g^{p^n} \mid g \in \mathcal{G}_0\}$ for $n \geq 1$. It follows that $\{\mathcal{G}_n\}_n$ forms a fundamental system of open compact subgroups of the identity element in $\mathcal{G}$.

Let $F_{\wp}$ be the p -adic field as in Section 4.2.3 and $\tilde{\mathcal{G}}$ an algebraic group over $\mathcal{O}_{\wp} = \mathcal{O}_{F_{\wp}}$. From here, let $\mathcal{G} := \mathrm{Res}_{\mathcal{O}_{\wp}/\mathbb{Z}_p} \tilde{\mathcal{G}}$ and we also see $\mathcal{G}$ as a p -adic Lie group. Let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\tilde{\mathcal{G}}$ over $F_{\wp}$, which can be also naturally seen to be the Lie algebra of $\mathcal{G}_0$ over $\mathbb{Q}_p$.

We have a decomposition

$$\mathcal{G} \times_{\mathrm{Spec} \mathbb{Z}_p} \mathrm{Spec} \mathcal{O}_E \xrightarrow{\sim} \bigsqcup_{\sigma \in \Sigma_\varphi} \mathcal{G}_\sigma, \quad \text{and} \quad \mathfrak{g} \otimes E \xrightarrow{\sim} \prod_{\sigma \in \Sigma_\varphi} \mathfrak{g}_\sigma,$$

where $\mathcal{G}_\sigma := \widetilde{\mathcal{G}} \times_{\mathrm{Spec} \mathcal{O}_\varphi} \mathrm{Spec} \mathcal{O}_E$ and $\mathfrak{g}_\sigma := \widetilde{\mathfrak{g}} \otimes_{F_\varphi} E$. Write $\mathcal{G}_J := \prod_{\sigma \in J} \mathcal{G}_\sigma$, $\mathfrak{g}_J := \prod_{\sigma \in J} \mathfrak{g}_\sigma$, and $\mathfrak{g}^J := \prod_{\sigma \notin J} \mathfrak{g}_\sigma$.

Let B be a Banach space over E . Let $C^{\mathrm{an}}(\mathcal{G}_n, B)$ denote the E -vector space of $(\mathbb{Q}_p$ -)analytic functions from $\mathcal{G}_n$ to B (cf. [80, §2], in which B -valued locally $(\mathbb{Q}_p$ -)analytic functions are defined). We equip the space $C^{\mathrm{an}}(\mathcal{G}_n, B)$ with a norm $\| - \|_{\mathcal{G}_n}$ defined by $\|f\|_{\mathcal{G}_n} := \sup_{\mathbf{k} \in \mathbb{N}^d} \|p^{n|\mathbf{k}|} b_{\mathbf{k}}\|$, where $b_{\mathbf{k}}$ are coefficient when the element $f \in C^{\mathrm{an}}(\mathcal{G}_n, B)$ is expanded by the coordinates $x_1, \dots, x_d$ as

$$(x_1, \dots, x_d) \mapsto \sum_{\mathbf{k}=(k_1, \dots, k_d) \in \mathbb{N}^d} b_{\mathbf{k}} x_1^{k_1} \cdots x_d^{k_d}$$

for some $b_{\mathbf{k}} \in B$ such that $p^{n|\mathbf{k}|} b_{\mathbf{k}} \rightarrow 0$ as $|\mathbf{k}| \rightarrow 0$.

Let I, J be a subset of Σ_φ such that $I \cap J = \emptyset$. A function $f \in C^{\mathrm{an}}(\mathcal{G}_n, B)$ is said to be $\mathcal{G}_{n,I}$ -smooth $\mathcal{G}_{n,J}$ -analytic if it is invariant under the action of $\mathcal{G}_{n,I}$ and the action of $\mathfrak{g} \otimes E$ factors through $\mathfrak{g}_J$ (cf. [86, Définition, 2.1]). If $I = \emptyset$, we say $\mathcal{G}_{n,J}$ -analytic instead of $\mathcal{G}_{n,\emptyset}$ -smooth $\mathcal{G}_{n,J}$ -analytic.

Let $C^{\mathcal{G}_{n,J}-\mathrm{an}}(\mathcal{G}_n, B)$ (resp. $C^{\mathcal{G}_{n,I}-\mathrm{sm}, \mathcal{G}_{n,J}-\mathrm{an}}(\mathcal{G}_n, B)$) denote the E -vector subspace of $C^{\mathrm{an}}(\mathcal{G}_n, B)$ generated by $\mathcal{G}_{n,J}$ -analytic functions (resp. $\mathcal{G}_{n,I}$ -smooth $\mathcal{G}_{n,J}$ -analytic functions). The subspace are closed (cf. [86, Lemme 2.5]), so it has the natural structure of Banach subspace induced from $C^{\mathrm{an}}(\mathcal{G}_n, B)$. There is a natural isomorphism of Banach spaces

$$C^{\mathcal{G}_{n,I}-\mathrm{sm}, \mathcal{G}_{n,J}-\mathrm{an}}(\mathcal{G}_n, E) \widehat{\otimes}_E B \cong C^{\mathcal{G}_{n,I}-\mathrm{sm}, \mathcal{G}_{n,J}-\mathrm{an}}(\mathcal{G}_n, B),$$

which preserves the norms.

We extend the definition of the spaces of $(\mathbb{Q}_p$ -)locally analytic vectors ([80, §2]) and its $\mathfrak{L}\mathfrak{A}$ -acyclicity (cf. [69, Definition 2.2.1] and also [74, Definition 4.40]) to J -locally analytic case.

Definition 4.4.1. For a positive integer $n \geq 1$, a Banach space B equipped with a continuous action of $\mathcal{G}$, and a subset $J \subset \Sigma_\varphi$,

- (1) An element $b \in B$ is said to be $\mathcal{G}_{n,I}$ -smooth $\mathcal{G}_{n,J}$ -analytic if b is in the image of the evaluation map at the identity element $C^{\mathcal{G}_{n,I}-\mathrm{sm}, \mathcal{G}_{n,J}-\mathrm{an}}(\mathcal{G}_n, B)^{\mathcal{G}_n} \rightarrow B$. Let $B^{\mathcal{G}_{n,I}-\mathrm{sm}, \mathcal{G}_{n,J}-\mathrm{an}} \subset B$ denote the subspace of $\mathcal{G}_{n,I}$ -smooth $\mathcal{G}_{n,J}$ -analytic vectors.
- (2) The space of J -locally analytic vectors in B is the subspace $B^{J-\mathrm{la}} := \bigcup_{N \geq n} B^{\mathcal{G}_N, J-\mathrm{an}}$. It is stable under the $\mathcal{G}$ -action and independent of the choice of $\mathcal{G}_0$ and n .
- (3) For $i \geq 0$, the i -th J -locally analytic cohomology of B is defined by

$$R^i \mathfrak{L}\mathfrak{A}_J(B) := \varinjlim_N H_{\mathrm{cont}}^i(\mathcal{G}_N, B \widehat{\otimes}_E C^{\mathcal{G}_N, J-\mathrm{an}}(\mathcal{G}_N, E)).$$

We say B is $\mathfrak{L}\mathfrak{A}_J$ -acyclic if $R^i \mathfrak{L}\mathfrak{A}_J(B) = 0$ for $i \geq 1$. We often say $J = \mathbb{Q}_p$ instead of $J = \Sigma_\varphi$. If $J = \mathbb{Q}_p$, we say $\mathfrak{L}\mathfrak{A}$ -acyclic instead of $\mathfrak{L}\mathfrak{A}_{\Sigma_\varphi}$ -acyclic.

Note that $\mathfrak{L}\mathfrak{A}$ -acyclicity can be also defined for Banach spaces with continuous $\mathcal{G}_n$ -actions when $n \geq 1$.

In the case of $J = \mathbb{Q}_p$ (in other words, if “ J -an” = “an”), the following theorem is well-known.

Theorem 4.4.2. *Any admissible $\mathbb{Q}_p$ -Banach representation B of a p -adic Lie group is $\mathfrak{LA}$ -acyclic (in the sense of [69, Definition 2.2.1]).*

PROOF. See [81, Theorem 7.1] (and also [69, Theorem 2.2.3], [77, Proposition 2.3.1]). $\square$

For a later purpose, we need to introduce the derived version of $\mathfrak{LA}$, using the condensed settings as in [74]. We use the notation of [74, §4]. For $h \geq 0$, we can attach a rigid analytic group $\mathbb{G}^{(h)}$ such that $\mathcal{G}_0 \subset \mathbb{G}^{(h)}(\mathbb{Q}_p)$ as in [74, Definition 4.7] or [75, Definition 2.1.2]. Note that $\mathbb{G}_I^{(h)}$ -smooth $\mathbb{G}_J^{(h)}$ -analytic functions on $\mathbb{G}^{(h)}$ can be similarly defined in the case of p -adic Lie groups and let

$$C^{\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h)}-\text{an}}(\mathbb{G}^{(h)}, E) \subset C^{\mathbb{G}_J^{(h)}-\text{an}}(\mathbb{G}^{(h)}, E)$$

denote the subspace of $\mathbb{G}_I^{(h)}$ -smooth $\mathbb{G}_J^{(h)}$ -analytic functions. We can consider the subalgebra of $\mathbb{G}_I^{(h)}$ -smooth $\mathbb{G}_J^{(h)}$ -analytic functions (cf. [86, Example 2.3])

$$C^{\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h)}-\text{an}}(\mathbb{G}^{(h)}, E) \subset C^{\mathbb{Q}_p-\text{an}}(\mathbb{G}^{(h)}, E) := C^{\text{an}}(\mathbb{G}^{(h)}, E).$$

The subalgebra $C^{\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h)}-\text{an}}(\mathbb{G}^{(h)}, E)$ is also an affinoid ring, and so it provides a natural analytic ring that we write $C^{\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h)}-\text{an}}(\mathbb{G}^{(h)}, E)_{\blacksquare}$. For an object C in the category $D(E_{\blacksquare}[\mathcal{G}_0])$ of solid $\mathcal{G}_0$ -modules, we define the *derived $\mathbb{G}_I^{(h)}$ -smooth $\mathbb{G}_J^{(h)}$ -analytic vectors* of C

$$C^{R\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h)}-\text{an}} := R\text{Hom}_{E_{\blacksquare}[G]}(E, (C^{R\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h)}-\text{an}}(\mathbb{G}^{(h)}, E)_{\blacksquare} \otimes_{E_{\blacksquare}} C)_{\star_{1,3}})$$

as an object in $D(E_{\blacksquare}[\mathcal{G}])$, where the action of G is induced by the $\star_2$ -action. See [74, Remark 4.23, Definition 4.24] for the notation of $(-)_{\star_I}$ for $I \subset \{1, 2, 3\}$. We set

$$C^{R\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h^+)}-\text{an}} := R\varprojlim_{h' < h} C^{R\mathbb{G}_I^{(h)}-\text{sm}, \mathbb{G}_J^{(h')}-\text{an}}.$$

We also define the *derived J -locally analytic vectors* of C

$$C^{RJ-\text{la}} := \text{hocolim}_{h \rightarrow \infty} C^{R\mathbb{G}_J^{(h)}-\text{an}} = \text{hocolim}_{h \rightarrow \infty} C^{R\mathbb{G}_J^{(h^+)}-\text{an}}$$

as a homotopy colimit in $D(E_{\blacksquare}[\mathcal{G}_0])$. We also set

$$(4.4.1) \quad C^{R\mathcal{G}_0, I-\text{sm}, J-\text{la}} := \text{hocolim}_{h \rightarrow \infty} C^{R\mathbb{G}_I^{(h)}-\text{sm}, R\mathbb{G}_J^{(h)}-\text{an}} = \text{hocolim}_{h \rightarrow \infty} C^{R\mathbb{G}_I^{(h^+)}-\text{sm}, R\mathbb{G}_J^{(h^+)}-\text{an}}.$$

We note that if the underlying topological module of $B = B[0]$ in $D(E_{\blacksquare}[\mathcal{G}_0])$ is a Banach space, we have

$$H^0(B^{RJ-\text{la}}) = R^0\mathfrak{LA}_J(B) = B^{J-\text{la}}.$$

In the remaining of this subsection, suppose that $\mathcal{G}$ be a p -adic Lie group such that $\mathcal{G} = \mathcal{G}' \times \mathcal{G}''$ for p -adic Lie groups $\mathcal{G}', \mathcal{G}''$. For a Banach E -representation B of $\mathcal{G}$, Let us define a subspace of $\mathcal{G}'$ -locally analytic vectors (resp. $\mathcal{G}''$ -locally analytic vectors) $B^{\mathcal{G}'-\text{la}}$ (resp. $B^{\mathcal{G}''-\text{la}}$) as the subspace of elements in B^{la} killed by the Lie algebra $\text{Lie}(\mathcal{G}'')$ (resp. $\text{Lie}(\mathcal{G}')$).

For $h > 0$, let $\mathbb{G}'^{(h)}$ and $\mathbb{G}''^{(h^+)}$ be the rigid analytic groups associated with $\mathcal{G}'_0$, and similarly define $\mathbb{G}''^{(h)}$ and $\mathbb{G}'^{(h^+)}$ as before. Let $\mathcal{G}'_0, \mathcal{G}''_0$ be uniform pro- p subgroups of $\mathcal{G}', \mathcal{G}''$ respectively, with $\mathcal{G}_0 := \mathcal{G}'_0 \times \mathcal{G}''_0$. For $(\mathcal{H}, \mathcal{H}_0) = (\mathcal{G}', \mathcal{G}'_0)$ or $(\mathcal{G}'', \mathcal{G}''_0)$, let $\mathcal{D}_{(h)}(\mathcal{H}_0, \mathcal{O}_E)$ and $\mathcal{D}_{(h)}(\mathcal{H}_0, E)$ denote the (condensed) distribution algebras as defined in [74, Definition 4.12]. We also set $\mathcal{D}^{(h)}(\mathcal{H}_0, E) := \varinjlim_{h' > h} \mathcal{D}^{(h')}(\mathcal{H}_0, E)$. We may also define the spaces of

derived $\mathcal{G}'_0$ -(locally) analytic vectors $C^{R\text{an}'}, C^{R\text{la}'}$ (resp. derived $\mathcal{G}''_0$ -(locally) analytic vectors $C^{R\text{an}''}, C^{R\text{la}''}$) of an object C in $D(E_{\blacksquare}[\mathcal{G}_0])$.

The following lemma is obtained by the same argument of the proof of [74, Theorem 4.36 (1)].

Lemma 4.4.3. *Let V be an object in $D(E_{\blacksquare})$ and C be an object in $D(E_{\blacksquare}[\mathcal{G}_0])$. We naturally extend the action of $\mathcal{G}'_0$ on $\mathcal{D}^{(h^+)}(\mathcal{G}'_0, E)$ to $\mathcal{G}_0 = \mathcal{G}'_0 \times \mathcal{G}''_0$ so that $\mathcal{G}''_0$ acts trivially. There is a natural quasi-isomorphism in $D(E_{\blacksquare}[\mathcal{G}_0])$*

$$R\text{Hom}_{E_{\blacksquare}[\mathcal{G}_0]}(\mathcal{D}^{(h^+)}(\mathcal{G}'_0, E) \otimes_{K_{\blacksquare}}^L V, C) \cong R\text{Hom}_E(V, C^{RG''^{(h^+)}-\text{sm}, \mathcal{G}'^{(h^+)}-\text{an}}).$$

We will use the following proposition in the later.

Proposition 4.4.4. *For an object C in $D(E_{\blacksquare}[\mathcal{G}_0])$, there exists a natural quasi-isomorphism*

$$R\text{Hom}_{E_{\blacksquare}[\mathcal{G}_0]}(E, C^{R\text{la}}) \xrightarrow{\sim} C^{RG''_0-\text{sm}, \text{la}'},$$

where $RG''_0 - \text{sm}, \text{la}'$ is defined as the equation (4.4.1).

PROOF. The proof is similar to that of [74, Theorem 5.3]. By [74, Remark 4.13], we have a natural isomorphism

$$\mathcal{D}_{(h)}(\mathcal{G}'_0, E) \otimes_{E_{\blacksquare}} \mathcal{D}_{(h)}(\mathcal{G}''_0, E) \cong \mathcal{D}_{(h)}(\mathcal{G}_0, E).$$

By the isomorphism and taking $-\otimes_{E_{\blacksquare}} \mathcal{D}_{(h)}(\mathcal{G}'_0, E)$ to the quasi-isomorphism in Kohlhaase's theorem [74, Theorem 5.8], we have a quasi-isomorphism

$$\mathcal{O}_E \otimes_{\mathcal{O}_{E_{\blacksquare}[\mathcal{G}_0]}}^L \mathcal{D}_{(h)}(\mathcal{G}_0, E) \cong \mathcal{D}_{(h)}(\mathcal{G}'_0, E).$$

By taking colimits, we also have

$$(4.4.2) \quad \mathcal{O}_E \otimes_{\mathcal{O}_{E_{\blacksquare}[\mathcal{G}_0]}}^L \mathcal{D}^{(h^+)}(\mathcal{G}_0, E) \cong \mathcal{D}^{(h^+)}(\mathcal{G}'_0, E).$$

We now compute

$$\begin{aligned} & R\text{Hom}_{E_{\blacksquare}[\mathcal{G}_0]}(E, C^{RG^{(h^+)}-\text{an}}) \\ & \cong R\text{Hom}_{E_{\blacksquare}[\mathcal{G}_0]}(E, R\text{Hom}_{E_{\blacksquare}[\mathcal{G}]}(\mathcal{D}^{(h^+)}(\mathcal{G}_0, E), C)) && ([74, \text{Theorem 4.36}]) \\ & \cong R\text{Hom}_{E_{\blacksquare}[\mathcal{G}_0]}(E \otimes_{E_{\blacksquare}[\mathcal{G}_0]}^L \mathcal{D}^{(h^+)}(\mathcal{G}_0, E), C) && (\text{tensor-Hom adjunction}) \\ & \cong R\text{Hom}_{E_{\blacksquare}[\mathcal{G}_0]}(\mathcal{D}^{(h^+)}(\mathcal{G}'_0, E), C) && (4.4.2) \\ & \cong C^{RG''^{(h^+)}-\text{sm}, \mathcal{G}'^{(h^+)}-\text{an}} && (\text{Lemma 4.4.3}). \end{aligned}$$

Then the natural quasi-isomorphism of the lemma is obtained by taking the homotopy colimit of the obtained quasi-isomorphism along $h \rightarrow \infty$. $\square$

Corollary 4.4.5. *Let V be a solid $E_{\blacksquare}[\mathcal{G}_0]$ -module. There is a spectral sequence of solid K -vector spaces*

$$E_2^{i,j} := \text{Ext}_{K_{\blacksquare}[\mathcal{G}_0]}^i(E, H^j(V^{R\text{la}})) \implies H^{i+j}(V^{RG''_0, I-\text{sm}, \text{la}'}).$$

PROOF. This follows from Proposition 4.4.4. $\square$

4.4.2. Completed cohomology of unitary Shimura curves. From here, we use the notation of G in Section 4.2.3.

Let $K_{\wp,0} := \mathrm{GL}_2(\mathcal{O}_{\wp}) \subset \mathrm{GL}_2(F_{\wp})$ and $K_{\wp,m}$ be its congruence subgroup

$$K_{\wp,m} := \left\{ \gamma \in \mathrm{GL}_2(F_{\wp}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varpi^m} \right\}$$

for $m \geq 1$. Let $K_{\wp}$, $K^{\wp} = K_p^{\wp} K^p$ be a compact open subgroup of $\mathrm{GL}_2(F_{\wp})$, $G(\mathbb{A}^{\infty,\wp}) = G(\mathbb{Q}_p^{\wp}) \times G(\mathbb{A}^{\infty,p})$ respectively.

Recall that for an object W in $\mathrm{Rep}_E(G^c)$, we have attached a K_p -equivariant étale (resp. Kummer étale) E -local system $W_{\text{ét}}$ (resp. $W_{\text{két}}$) on $\mathcal{S}h_{\text{ét}}$ (resp. $\mathcal{S}h_{\text{két}}^{\text{tor}}$), base changing to $E/\mathbb{Q}_p$ the argument in Section 4.3.2. Let W_0 be a $\mathcal{O}_E$ -lattice of W which is stable by $K_p^{\wp}$ and $W_{0,\text{ét}}$ (resp. $W_{0,\text{két}}$) be the associated étale (resp. pro-étale) sheaves. Let $j_{K_{\wp},\text{két}}: \mathcal{S}h_{K_{\wp}K^{\wp},\text{ét}} \rightarrow \mathcal{S}h_{K_{\wp}K^{\wp},\text{prokét}}^{\text{tor}}$ be the natural morphism of sites. We have natural isomorphisms $j_{K_{\wp},\text{két},*} W_{0,\text{ét}} \cong W_{0,\text{két}}$ and $j_{K_{\wp},\text{két}}^* W_{0,\text{két}} \cong W_{0,\text{ét}}$ by the construction. Note that we also have a quasi-isomorphism

$$Rj_{K_{\wp},\text{két},*}(W_{0,\text{ét}}/\varpi_E^m) \cong (W_{0,\text{két}}/\varpi_E^m)[0]$$

for $m \geq 1$ by [37, Theorem 4.6.1].

Let us recall the (completed) étale cohomology of unitary Shimura curves. For each integer $i \geq 0$, let (cf. [38, §2.1.1], [77, Definition 6.1.1])

$$H_{?}^i(K^{\wp}, W_0) := \varinjlim_{K_{\wp}} \varprojlim_s H_{\text{ét},?}^i(\mathcal{S}h_{K_{\wp}K^{\wp}}, W_{0,\text{ét}}/\varpi_E^s),$$

$$\tilde{H}_{?}^i(K^{\wp}, W_0) := \varprojlim_s \varinjlim_{K_{\wp}} H_{\text{ét},?}^i(\mathcal{S}h_{K_{\wp}K^{\wp}}, W_{0,\text{ét}}/\varpi_E^s),$$

$$H_{?}^i(K^{\wp}, W_0)_E := H_{?}^i(K^{\wp}, W_0) \otimes_{\mathcal{O}_E} E,$$

$$\tilde{H}_{?}^i(K^{\wp}, W_0)_E := \tilde{H}_{?}^i(K^{\wp}, W_0) \otimes_{\mathcal{O}_E} E,$$

where $? \in \{\emptyset, c\}$ and $K_{\wp}$ runs over compact open subgroups of $\mathrm{GL}_2(F_{\wp})$ which stabilizes W_0 . Let

$$H_{?}^i(K^{\wp}, W) := \varinjlim_{W_0} H_{?}^i(K^{\wp}, W_0)_E,$$

$$\tilde{H}_{?}^i(K^{\wp}, W) := \varinjlim_{W_0} \tilde{H}_{?}^i(K^{\wp}, W_0)_E,$$

where W_0 runs over $\mathcal{O}_E$ -lattices of W stable by $K_p^{\wp}$. The p -adic Lie group $\mathrm{GL}_2(F_{\wp})$ naturally acts on these modules.

For a general $\mathcal{O}_E$ -lattice $W_0 \subset W$, we can similarly define (completed) étale cohomology with W -coefficients $H^i(K^p, W_0)$, $\tilde{H}^i(K^p, W_0)$, and so on, taking towers of Shimura varieties of level $K'_p \subset K_p$, instead of level $K'_{\wp} \subset K_{\wp}$. We may also consider the space of J -locally analytic vectors of the completed cohomology where $J \subset \{z\} \cup \bigsqcup_i \Sigma_{\wp_i}$ and z is the symbol which corresponds to the continuous representation of the component of similitudes $\mathbb{G}_m \subset G(\mathbb{Q}_p)$. If J is of the form $\bigsqcup_{i \in J' \subset [1,s]} \Sigma_{\wp_i}$ (resp. $\{z\} \cup \bigsqcup_{i \in J' \subset [1,s]} \Sigma_{\wp_i}$), we may define the terminology “ J' -locally analytic” (resp. “ $(\{z\} \cup J')$ -locally analytic”) and (derived) J -locally analytic vectors as in Section 4.4.1. Moreover, if $J = \{\sigma\}$ for an element $\sigma \in \{z\} \cup \bigsqcup_{\wp_i | p} \Sigma_{\wp_i}$, we call σ -locally analytic instead of $\{\sigma\}$ -locally analytic. Note that $\tilde{H}^i(K^p, W_0)$ is $(\mathbb{Q}_p)$ -locally analytic if and only if $(\{z\} \cup \bigsqcup_i \Sigma_{\wp_i})$ -locally analytic. For a while, we consider completed cohomology $\tilde{H}_{?}^i(K^p, -)$ rather than $\tilde{H}_{?}^i(K^{\wp}, -)$. The advantage of using them is

that they can be embedded in the completed cohomology of Siegel modular varieties, this argument is treated in Section 4.5.

By Corollary 4.4.5 and the proof of [38, Theorem 1.1.13] (see also [73, Corollary 2.2.11]), we have the analogous assertion for admissible Banach representations.

Proposition 4.4.6. *For a subset $J \subset \{z\} \cup \bigsqcup_{\wp_i|p} \Sigma_{\wp_i}$ and an admissible Banach representation V of G over E , we have*

$$V^{RJ-\text{la}} = R\Gamma(\mathfrak{g}^J, V^{R\text{la}})$$

Definition 4.4.7. Let $j: \mathcal{Sh}_{K^p} \hookrightarrow \mathcal{Sh}_{K^p}^{\text{tor}}$ be an open immersion of perfectoid Shimura varieties. For a pro-Kummer étale $\mathcal{O}_E$ -local system Λ on $\mathcal{Sh}_{K^p, \text{proét}}$, let $j_!\Lambda := R\varprojlim_s j_!(\Lambda/\varpi_E^s)$ be the (derived) p -adic completion of the extension by zero of the pro-Kummer étale local systems Λ/ϖ_E^s over $\mathcal{Sh}_{K^p, \text{proét}}$.

Proposition 4.4.8. *Let $\Lambda = W_{0, \text{prokét}}$ or $W_{0, \text{prokét}}/\varpi_E^m$. The objects in the derived category of $\mathcal{O}_E$ -local systems*

$$R\Gamma_{\text{prokét}}(\mathcal{Sh}_{K^p}^{\text{tor}}, \Lambda) \quad \text{and} \quad R\Gamma_{\text{prokét}}(\mathcal{Sh}_{K^p}^{\text{tor}}, j_!\Lambda)$$

are represented by bounded complexes of admissible $K_\wp$ -representations.

In particular,

$$H_{\text{prokét}}^i(\mathcal{Sh}_{K^p}^{\text{tor}}, \Lambda) \quad \text{and} \quad H_{\text{prokét}}^i(\mathcal{Sh}_{K^p}^{\text{tor}}, j_!\Lambda)$$

are admissible K_p -representations for $i \geq 0$.

PROOF. The pullback of the pro-Kummer étale sheaf Λ by the map of sites $\mathcal{Sh}_{K^p, \text{prokét}}^{\text{tor}} \rightarrow \mathcal{Sh}_{K_p K^p, \text{prokét}}^{\text{tor}}$ is the constant pro-Kummer étale sheaf $\underline{W}_0$ associated with W_0 , which is equipped with the G -action. So we have a quasi-isomorphism

$$R\Gamma_{\text{prokét}}(\mathcal{Sh}_{K^p}^{\text{tor}}, \Lambda) \xrightarrow{\sim} \varinjlim_{K'_p} R\Gamma_{\text{két}}(\mathcal{Sh}_{K'_p K^p}^{\text{tor}}, \underline{W}_0)$$

by [37, Proposition 5.1.6]. Then, the proof is similar to [77, Theorem 6.1.6]. $\square$

Corollary 4.4.9. *For $i \geq 0$, we have natural K_p -equivariant isomorphisms*

$$\begin{aligned} H_{\text{prokét}}^i(\mathcal{Sh}_{K^p}^{\text{tor}}, W_{0, \text{prokét}}) &\cong \tilde{H}^i(K^p, W_{0, \text{prokét}}), \\ H_{\text{prokét}}^i(\mathcal{Sh}_{K^p}^{\text{tor}}, j_! W_{0, \text{prokét}}) &\cong \tilde{H}_c^i(K^p, W_{0, \text{prokét}}). \end{aligned}$$

PROOF. By Proposition 4.4.8, the proof is similar to [77, Corollary 6.1.7]. $\square$

4.4.3. Comparison between J -locally analytic completed cohomology and the cohomology of perfectoid unitary Shimura curves. In this subsection and the next section, we will use the terminology of geometric Sen theory in [76]. In this subsection, we suppose one of the following two conditions on (G, X) , E , and J :

- Condition 4.4.10.** (1) (G, X) is a Shimura data as in Section 4.2.3, $E/\mathbb{Q}_p$ is a finite extension such that $|\text{Hom}_{\mathbb{Q}_p}(F_{\wp_i}, E)| = [F_{\wp_i} : \mathbb{Q}_p]$ for each $\wp_i|p$, and $J \in \{z\} \cup \bigsqcup_{\wp_i|p} \Sigma_{\wp_i}$.
 (2) (G, X) is a Shimura data of Hodge type (Section 4.2.2), $E = \mathbb{Q}_p$, and $J = \mathbb{Q}_p$.

The following argument is based on [77, §6.2] (in which the case of general Shimura varieties and its $\mathbb{Q}_p$ -locally analytic completed cohomology is treated, considering the pro-Kummer étale sites on infinite level Shimura varieties).

Let W be an object in $\text{Rep}_E(G^c)$ and W_0 be its $\mathcal{O}_E$ -lattice. As in the previous section, for each compact open subgroup $K_p K^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}^{\infty,p})$, let $\mathcal{S}h = \mathcal{S}h_{K_p K^p}$ denote the rigidification of the Shimura variety over $\mathbb{C}_p$ associated with the datum (G, X) . Also let $\mathcal{S}h_{K_p K^p}^{\text{tor}}$ be a toroidal compactification of $\mathcal{S}h_{K_p K^p}$ over $\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Note that in the case of unitary Shimura curves, they are compact and so we can assume that $\mathcal{S}h_{K_p K^p} = \mathcal{S}h_{K_p K^p}^{\text{tor}}$.

Let $D := \mathcal{S}h_{K_p K^p}^{\text{tor}} \setminus \mathcal{S}h_{K_p K^p}$ be the normal crossing divisor in $\mathcal{S}h_{K_p K^p}^{\text{tor}}$. We can write $D = \bigcup_{a \in I} D_{K_p, a}$ for a finite set I such that each $D_{K_p, a}$ is irreducible and $D_{K_p, I'} := \bigcap_{a \in I'} D_{K_p, a}$ is a smooth divisor for each subset $I' \subset I$. For a level $K'_p \subset K_p$, let $D_{K'_p, I'} := D_{K_p, I'} \times_{\mathcal{S}h_{K_p K^p}^{\text{tor}}} \mathcal{S}h_{K'_p K^p}^{\text{tor}}$ and let $\iota_{K'_p, I'} : D_{K'_p, I'} \hookrightarrow \mathcal{S}h_{K'_p K^p}^{\text{tor}}$ denote the closed immersion. Write $\widehat{\mathcal{O}}_{I'}$ for the pro-Kummer étale sheaf of completed (bounded) functions on $D_{K_p, I'}$, endowed with the induced log structure. Let $\widehat{\mathcal{I}}_D$ be the kernel of the map of pro-Kummer étale sheaves

$$\widehat{\mathcal{I}}_{\mathcal{S}h} := \ker \left(\widehat{\mathcal{O}}_{\mathcal{S}h} \rightarrow \bigoplus_{a \in I} \iota_{K_p, a, *} \widehat{\mathcal{O}}_a \right).$$

Let $\pi_{K_p} : \mathcal{S}h_{K_p} \rightarrow \mathcal{S}h_{K_p K^p}$ denote the natural projection. The map π_{K_p} is a pro-étale Galois covering with Galois group

$$\widetilde{K}_p := K_p K^p / (K_p K^p \cap \overline{Z(\mathbb{Q})} \cdot K^p).$$

Similarly let $\pi_{K_p}^{\text{tor}} : \mathcal{S}h_{K_p}^{\text{tor}} \rightarrow \mathcal{S}h_{K_p K^p}^{\text{tor}}$ be the natural projection. Set $d := \dim \mathcal{S}h_{K_p K^p}$. For $n \geq 1$ an integer, let

$$\mathbb{T}_n := \text{Spa}(\mathbb{C}_p \langle T^{\pm \frac{1}{n}} \rangle, \mathcal{O}_{\mathbb{C}_p} \langle T^{\pm \frac{1}{n}} \rangle) \quad \text{and} \quad \mathbb{D} := \text{Spa}(\mathbb{C}_p \langle S^{\frac{1}{n}} \rangle, \mathcal{O}_{\mathbb{C}_p} \langle S^{\frac{1}{n}} \rangle).$$

and $\mathbb{T} := \mathbb{T}_1$ and $\mathbb{D} := \mathbb{D}_1$. Write $\mathbb{T}_{\infty} := \varprojlim_n \mathbb{T}_n$ and $\mathbb{D}_{\infty} := \varprojlim_n \mathbb{D}_n$.

We assume the following conditions on (U, V) :

Condition 4.4.11. For open subspaces $U \subset \mathcal{S}h_{K_p K^p}^{\text{tor}}$ and $V \subset \mathcal{F}\ell_{\mu}$,

- (1) The $\mathcal{O}_{\mathcal{F}\ell_{\mu}}(V)$ -sheaves $\mathfrak{n}_{\mu}^0$ and $\mathfrak{g}^{c,0}/\mathfrak{n}_{\mu}^0$, where $\mathfrak{g}^c := \text{Lie } G_{\mathbb{Q}_p}^c$, and $\mathfrak{g}^{c,0}, \mathfrak{n}_{\mu}^0 = \mathfrak{n}_{\mu}^{c,0}$ are constructed as in Section 4.3.2, are finite free.
- (2) The compact open subgroup $K_p \subset G(\mathbb{Q}_p)$ stabilizes V .
- (3) $\pi_{K_p}^{\text{tor}, -1}(U) \subset \pi_{\text{HT}}^{\text{tor}, -1}(V)$ is an affinoid perfectoid space.
- (4) U admits a toric chart, i.e., there is a map $U \rightarrow \mathbb{T}^e \times \mathbb{D}^{d-e}$ for some $e \in [0, d]$ that factors as a composite of a finite étale map and a rational localization.
- (5) U has the log structure obtained from $U \rightarrow \mathbb{T}^e \times \mathbb{D}^{d-e}$ by pullback.

Note that such U covers the whole $\mathcal{S}h_{K_p K^p}^{\text{tor}}$ by Proposition 4.3.2 (2) and [37, Corollary 3.11]. For an object U' in $U_{\text{prokét}}$, let U'_I be the space $D_{I'} \times_{\mathcal{S}h} U'$ and $\widetilde{U}' := \pi_{K_p}^{\text{tor}, -1}(U) \times_U U'$ with the induced log structures. Let $U_n := U \times_{(\mathbb{T}^e \times \mathbb{D}^{d-e})} (\mathbb{T}_n^e \times \mathbb{D}_n^{d-e})$, $U_{\infty} := \varprojlim_n U_n$, and $U_{p^{\infty}} := \varprojlim_n U_{p^n}$. Let $\Gamma \cong \widehat{\mathbb{Z}}^d$ (resp. $\Gamma_J \cong \widehat{\mathbb{Z}}^{|J \cap \{e+1, \dots, d\}|}$) be the pro-Kummer étale Galois group of $\mathbb{T}_{\infty}^e \times \mathbb{D}_{\infty}^{d-e} \rightarrow \mathbb{T}^e \times \mathbb{D}^{d-e}$ (resp. $U_{\infty, I'} \rightarrow U_{I'}$). Write $\Gamma := \Gamma_p \times \Gamma^p$ and $\Gamma_{I'} := \Gamma_{I', p} \times \Gamma_{I'}^p$ as a product of its prime-to- p part and its pro- p -Sylow subgroup. Note that $U_{p^{\infty}}, \widetilde{U}_{p^{\infty}}, U_{\infty}$, and $\widetilde{U}_{\infty}$ are log affinoid perfectoid in the sense of [37, Definition 5.3.1].

Let $\widetilde{\mathfrak{g}} := \text{Lie } G^c$ with its dimension $g := \dim_{\mathbb{Q}_p} \widetilde{\mathfrak{g}}$.

Lemma 4.4.12. For $i \geq 0$, $I' \subset I$ and a pro-Kummer étale $\widehat{\mathcal{O}}_{I'}$ -module $\mathcal{F}$ which is a colimit of relative locally analytic ON Banach sheaves, there is a natural isomorphism

$$H_{\text{prokét}}^i(U_{I'}, \widehat{\mathcal{O}}_{I'} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}) \cong H^i(\mathfrak{n}_{\mu}^0, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{p^{\infty}, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F})^{\widetilde{K}_p \times \Gamma_{I', p}},$$

where $\mathfrak{n}_\mu^0$ acts on J -locally analytic pro-Kummer étale E -local systems on $\widehat{\mathcal{O}}_{Sh^{\text{tor}}}$ via the natural embedding $\pi_{HT}^{\text{tor},*}(\mathfrak{n}_\mu^0) \hookrightarrow \widetilde{\mathfrak{g}}_{\text{prokét}}|_{Sh_{K^p}^{\text{tor}}}$ of the pullback of the vector bundle $\mathfrak{n}_\mu^0$ on $\mathcal{F}\ell_\mu$ into the constant sheaf $\widetilde{\mathfrak{g}}_{\text{prokét}}|_{Sh_{K^p}^{\text{tor}}}$.

PROOF. We follow the argument of Step 1 and Step 2 of the proof in [77, Proposition 6.2.8]. There is an equivalence of topoi $\widetilde{U}_{\infty, I', \text{proét}} \cong \widetilde{U}_{\infty, I', \text{prokét}}$ by [37, Lemma 5.3.8]. So by [82, Lemma 4.12] and the definition of ON relative locally analytic pro-Kummer étale $\widehat{\mathcal{O}}_{I'}$ -module, we have

$$R\Gamma_{\text{prokét}}(\widetilde{U}_{\infty, I'}, \mathcal{F}) = \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}.$$

By the above equality and that $\widetilde{U}_{\infty, I'} \rightarrow U_{I'}$ is a pro-Kummer étale $\widetilde{K}_p \times \Gamma_{I'}$ -torsor, there is a quasi-isomorphism

$$R\Gamma_{\text{prokét}}(U_{I'}, \widehat{\mathcal{O}}_{I'} \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}) \cong R\Gamma_{\text{cont}}(\widetilde{K}_p \times \Gamma_{I'}, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}(\widetilde{U}_{\infty, I'})).$$

Since $\Gamma_{I'}^p$ is a prime-to- p profinite group, we have quasi-isomorphisms

$$\begin{aligned} R\Gamma_{\text{cont}}(\widetilde{K}_p \times \Gamma_{I'}, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}(\widetilde{U}_{\infty, I'})) \\ \cong R\Gamma_{\text{cont}}(\widetilde{K}_p \times \Gamma_{I', p}, R\Gamma_{\text{cont}}(\Gamma_{I'}^p, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}(\widetilde{U}_{\infty, I'}))) \\ \cong R\Gamma_{\text{cont}}(\widetilde{K}_p \times \Gamma_{I', p}, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{p^\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}). \end{aligned}$$

We use geometric Sen theory to translate cohomology of the obtained complex into Sen operator cohomology. The triple $(\widehat{\mathcal{O}}_{I'}(\widetilde{U}_{p^\infty, I'}), \widetilde{K}_p \times \Gamma_{I', p}, \text{pr}_2)$, where $\text{pr}_2 := \widetilde{K}_p \times \Gamma_{I', p} \rightarrow \Gamma_{I', p}$ is the projection, is a (strongly decomposable) Sen theory in the sense of [76, Definition 2.2.6].

By the assumption, we can attach a (geometric) Sen operator

$$\theta_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathbb{Q}_p} \text{Lie}(\widetilde{K}_p \times \Gamma_{I', p})^\vee$$

for $\mathcal{F}$ ([76, Definition 2.5.3]) and the geometric Sen operator of Sh_{K^p} ([76, Proposition 3.2.6])

$$\theta_{Sh} := \theta_{Sh_{K^p}}: \widehat{\mathcal{O}}_{Sh} \otimes (\text{Lie } G)_{\text{prokét}}^\vee \rightarrow \widehat{\mathcal{O}}_{Sh}(-1) \otimes_{\mathcal{O}_{Sh}} \Omega_{Sh}^1(\log).$$

Then for $i \geq 0$, we have

$$\begin{aligned} H^i(\widetilde{K}_p \times \Gamma_{I', p}, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{p^\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F}) &\cong H^i(\theta_{\mathcal{F}}, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{p^\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F})^{\widetilde{K}_p \times \Gamma_{I', p}} \\ &\cong H^i(\theta_{Sh}, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{p^\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F})^{\widetilde{K}_p \times \Gamma_{I', p}} \\ &\cong H^i(\mathfrak{n}_\mu^0, \widehat{\mathcal{O}}_{I'}(\widetilde{U}_{p^\infty, I'}) \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{F})^{\widetilde{K}_p \times \Gamma_{I', p}}, \end{aligned}$$

where the first isomorphism is from [76, Proposition 2.5.4, Remark 2.5.8], the second isomorphism is from [76, Theorem 3.3.2 (5)], and the last isomorphism is from the computation of θ_{Sh} ([77, Theorem 5.2.5]). $\square$

We recall [75, Definition 2.1.2]. Let $\mathfrak{h}_1, \dots, \mathfrak{h}_g$ be a basis of $\widetilde{\mathfrak{g}}$ over $\mathbb{Q}_p$. For $h \gg 0$, the exponential of the $\mathbb{Z}_p$ -lattice $\mathfrak{K}_h$ generated by $\{p^h \mathfrak{h}_k\}_k$ defines an affinoid group $\mathbb{G}_h$ isomorphic to a closed polydisc of dimension g , and let $\mathring{\mathbb{G}}_h := \bigcup_{h' > h} \mathbb{G}_{h'}$. Let $\mathbb{G}^{(h)}$ be the rigid analytic group containing $\widetilde{K}_p$ as defined in loc. cit. (applying in the case of $G = \widetilde{K}_p$) for $h \in \mathbb{Q}_{>0}$. For $\mathbb{G} = \mathbb{G}_h, \mathring{\mathbb{G}}_h, \mathbb{G}^{(h)}, \widetilde{K}_p$, the spaces $C^{J\text{-la}}(\mathbb{G}, B)$ of J -locally analytic functions $\mathbb{G} \rightarrow B$ for an E -Banach space B is defined as in Section 4.4.1. Let

$C^{J-\text{la}}(\mathbb{G}, E)_{\text{prokét}} := \overline{C^{J-\text{la}}(\mathbb{G}, E)}$ be the pro-Kummer étale sheaf. Let $C^{J-\text{la}}(\tilde{\mathbf{g}}, E)_{\text{prokét}} := \varinjlim_{K_p} C^{J-\text{la}}(\tilde{K}_p, E)_{\text{prokét}}$.

Lemma 4.4.13. *For $I' \subset I$, the sheaf $\widehat{\mathcal{O}}_{I'} \widehat{\otimes}_{\mathbb{Q}_p} C^{J-\text{la}}(\tilde{K}_p, E)_{\text{prokét}}$ is a colimit of ON relative locally analytic pro-Kummer étale Banach $\widehat{\mathcal{O}}_{I'}$ -modules. In particular, this sheaf satisfies the conditions of Lemma 4.4.12.*

PROOF. We can write as $C^{J-\text{la}}(\tilde{K}_p, E)_{\text{prokét}} = \varinjlim_h C^{J-\text{an}}(\mathbb{G}^{(h)}, E)_{\text{prokét}}$ (see [75, Definition 2.1.10]). For each $h > 0$, the inclusion $C^{\mathbb{Q}_p-\text{an}}(\mathbb{G}^{(h)}, E) \subset C^{\mathbb{Q}_p-\text{an}}(\mathbb{G}^{(h)}, E)_{\text{prokét}}$ is a closed embedding of ON relative locally analytic Banach spaces. $\square$

Set $A := \widehat{\mathcal{O}}_{I'}(\tilde{U}_{p^\infty, I'})$. We write

$$A \widehat{\otimes}_{\mathbb{Q}_p} C^{J-\text{la}}(\tilde{\mathbf{g}}, E)_{\text{prokét}} := \varinjlim_{K'_p \subset K_p} (A \widehat{\otimes}_{\mathbb{Q}_p} \varinjlim_{K_p} C^{J-\text{la}}(\tilde{K}_p, E)_{\text{prokét}})^{\tilde{K}'_p - \text{sm}},$$

where K'_p is a compact open subgroup of K_p , for the subspace of smooth vectors.

Lemma 4.4.14. *There is a quasi-isomorphism*

$$R\Gamma(\mathfrak{n}_\mu^0, A \widehat{\otimes}_{\mathbb{Q}_p} C^{J-\text{la}}(\tilde{\mathbf{g}}, E)_{\text{prokét}}) \cong \varinjlim_{r \rightarrow \infty} (A \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathbb{X}'_r) \otimes_{\mathbb{Q}_p} E),$$

where $\mathcal{O}(\mathbb{X}'_r)$ and $\{\mathbb{X}'_r\}_{r \gg 0}$ is a shrinking system of closed polydiscs of the same dimension over V .

PROOF. We follow the argument of Step 3 and Step 4 of the proof of [77, Proposition 6.2.8]. The case of (1) in Condition 4.4.10, the lemma follows from loc. cit., so we may assume the condition (2). Recall that

$$(4.4.3) \quad \mathfrak{g}_E := \mathfrak{g} \otimes E \cong \mathfrak{z}_E \times \prod_{\sigma \in \bigsqcup_{\wp_i|p} \Sigma_{\wp_i}} \mathfrak{gl}_{2,E,\sigma} = \prod_{\sigma \in \{z\} \cup \bigsqcup_{\wp_i|p} \Sigma_{\wp_i}} \mathfrak{g}_{E,\sigma},$$

where $\mathfrak{z}_E := \text{Lie } \mathbb{G}_{m,E}$, so we can write

$$\tilde{\mathfrak{g}}_E = \prod_{\sigma \in \{z\} \cup \bigsqcup_{\wp_i|p} \Sigma_{\wp_i}} \tilde{\mathfrak{g}}_{E,\sigma},$$

where $\tilde{\mathfrak{g}}_{E,\sigma}$ is a quotient of $\mathfrak{g}_{E,\sigma}$ and set

$$\tilde{\mathfrak{g}}_{J,E} := \prod_{\sigma \in J} \tilde{\mathfrak{g}}_{E,\sigma}.$$

We also let

$$\mathfrak{n}_{\mu,E} := \mathfrak{n}_\mu \otimes E \cong \mathfrak{n}_{E,\tau},$$

where $\tau \in \Sigma$ is the fixed place and $\mathfrak{n}_{E,\tau} \subset \tilde{\mathfrak{g}}_{E,\tau}$ is a one dimensional unipotent subalgebra, by Proposition 4.3.4 and its following argument. Let $\mathfrak{N}$ be a generator of $\mathfrak{n}_{E,\tau}$. Also let $\tilde{\mathfrak{g}}^0 := \mathcal{O}_{\mathcal{F}\ell_\mu} \otimes \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_E^0 := \mathcal{O}_{\mathcal{F}\ell_\mu} \otimes \tilde{\mathfrak{g}}_E$.

Recall that each element of the set $\{z\} \cup \bigsqcup_{\wp_i|p} \Sigma_{\wp_i}$ corresponds to each component of the direct product decomposition of $\tilde{\mathfrak{g}}_E^0 = \prod_{\sigma} \tilde{\mathfrak{g}}_{E,\sigma}^0$ in (4.4.3). For each $\sigma \in J \setminus \{\tau\}$, let $\{\mathfrak{Y}_{1,\sigma}, \dots, \mathfrak{Y}_{g_\sigma,\sigma}\}$, where each $g_\sigma \geq 0$ is an integer, be a basis of $\tilde{\mathfrak{g}}_{E,\sigma}^0$ over the fixed $V \subset \mathcal{F}\ell_\mu$. If $\tau \in J$, for an integer $g_\tau \geq 0$, we also let $\{\mathfrak{Y}_{1,\tau}, \dots, \mathfrak{Y}_{g_\tau,\tau}\}$ be a set of elements in $\tilde{\mathfrak{g}}_{E,\tau}^0$ such that $\mathfrak{Y}_{1,\tau}, \dots, \mathfrak{Y}_{g_\tau,\tau}$, and $\mathfrak{N}$ generates $\tilde{\mathfrak{g}}_{E,\tau}^0$.

For $r \gg 0$, we can construct an affinoid rigid space $\mathbb{X}_{J,r}$ over V_E isomorphic to a polydisc of dimension g from the basis

$$\begin{cases} \{p^r \mathfrak{Y}_{s,\sigma}, p^r \mathfrak{N}\}_{\sigma \in J \setminus \{\tau\}, 1 \leq s \leq g_\sigma} & (\tau \in J) \\ \{p^r \mathfrak{Y}_{s,\sigma}\}_{\sigma \in J, 1 \leq s \leq g_\sigma} & (\tau \notin J) \end{cases}$$

by the exponential map (see also Step 3. in the proof of [77, Proposition 6.2.8] for the detail).

Let $\mathring{\mathbb{X}}_{J,r} := \bigcup_{r' < r} \mathbb{X}_{J,r'}$ be the Stein space. For each $r \gg 0$, we fix a compact open subgroup $K_p(r) \subset K_p$ such that $\mathbb{X}_{J,r}$ and $\mathring{\mathbb{X}}_{J,r}$ (also $\mathcal{O}(\mathbb{X}_{J,r})$ and $\mathcal{O}(\mathring{\mathbb{X}}_{J,r})$) are stable by the (continuous) actions of left multiplications of $K_p(r)$. Let $\mathbb{X}'_{J,r}$ be the rigid space constructed from $\{p^r \mathfrak{Y}_{s,\sigma}, p^r\}_{\sigma \in J \setminus \{\tau\}, 1 \leq s \leq g_\sigma}$ as above. Note that $\dim \mathbb{X}'_{J,r} + 1 = \dim \mathbb{X}_{J,r}$ if $\tau \in J$ and otherwise $\mathbb{X}'_{J,r} = \mathbb{X}_{J,r}$.

By the construction and [86, Example 2.3], we can write

$$(4.4.4) \quad A \widehat{\otimes}_{\mathbb{Q}_p} C^{J-\text{la}}(\mathfrak{g}, E)_{\text{prokét}} = \varinjlim_{r \rightarrow \infty} A \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathbb{X}_{J,r}) \otimes_{\mathbb{Q}_p} E = \varinjlim_{r \rightarrow \infty} A \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathring{\mathbb{X}}_{J,r}) \otimes_{\mathbb{Q}_p} E,$$

where the tensor product of the third term is projective tensor product of Fréchet spaces.

Then we have a quasi-isomorphism

$$R\Gamma(\mathfrak{n}_\mu^0, A \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathring{\mathbb{X}}_{J,r}) \otimes_{\mathbb{Q}_p} E) \cong A \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathring{\mathbb{X}}'_{J,r}) \otimes_{\mathbb{Q}_p} E$$

by the same argument of [77, Lemma 6.2.9]. The quasi-isomorphism and the equation (4.4.4) proves the lemma. $\square$

- Definition 4.4.15.** (1) Let $\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}$ denote the analytic site of the perfectoid space $\mathcal{S}h_{K^p}^{\text{tor}}$. We call an open subspace $U \subset \mathcal{S}h_{K^p}^{\text{tor}}$ *rational* if it is the pullback of a rational open subspace at finite level. We may similarly define an analytic site $D_{\infty, I', \text{an}}$ of the adic space $D_{\infty, I'} := D_{K_p, I'} \times_{\mathcal{S}h} \mathcal{S}h_{K^p}^{\text{tor}}$ (This space is a perfectoid space such that $D_{\infty, I'} \sim \varprojlim_{K_p} D_{K_p, I'}$ by the last part of the proof of [54, Proposition 6.1]) and its rational open subspaces.
- (2) Write $\widehat{\mathcal{O}}_{\mathcal{S}h, \text{an}}$ for the sheaf of topological rings restricting the sheaf $\widehat{\mathcal{O}}_{\mathcal{S}h}$ on the site $\mathcal{S}h_{\text{prokét}}$ to the site $\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}$. We may similarly define sheaves $\widehat{\mathcal{I}}_{\mathcal{S}h, \text{an}}$ on $\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}$ and $\widehat{\mathcal{O}}_{I', \text{an}}$ on $D_{\infty, I', \text{an}}$.
- (3) Let $\mathcal{O}_{\mathcal{S}h, E}^{J-\text{la}} \subset \widehat{\mathcal{O}}_{\mathcal{S}h, E, \text{an}} := \widehat{\mathcal{O}}_{\mathcal{S}h, \text{an}} \otimes_{\mathbb{Q}_p} E$ denote a sub-(pre)sheaf on $\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}$ such that for each rational open subspace $U \subset \mathcal{S}h_{K^p}^{\text{tor}}$,

$$\mathcal{O}_{\mathcal{S}h, E}^{J-\text{la}}(U) = \varinjlim_{K_{p,U}} \widehat{\mathcal{O}}_{\mathcal{S}h, E, \text{an}}(U)^{K_{p,U}, J-\text{an}}$$

where $K_{p,U} = \prod_{\sigma \in \{z\} \cup \bigcup_{\varphi_i|p} \Sigma_{\varphi_i}} K_{p,U,\sigma}$ runs over open subgroups of K_p (as a p -adic Lie subgroup) which stabilizes U . We may similarly define sub-(pre)sheaves $\mathcal{O}_{I', E}^{J-\text{la}} \subset \widehat{\mathcal{O}}_{I', E, \text{an}} := \widehat{\mathcal{O}}_{I', \text{an}} \otimes_{\mathbb{Q}_p} E$ on $D_{\infty, I', \text{an}}$ and $\mathcal{I}_{\mathcal{S}h, E}^{J-\text{la}} \subset \widehat{\mathcal{I}}_{\mathcal{S}h, E, \text{an}} := \widehat{\mathcal{I}}_{\mathcal{S}h, \text{an}} \otimes_{\mathbb{Q}_p} E$ on $\mathcal{S}h_{K^p, \text{an}}^{\text{tor}}$.

Since taking J -locally analytic vectors is left exact, $\mathcal{O}_{\mathcal{S}h, E}^{J-\text{la}}$, $\mathcal{O}_{I', E}^{J-\text{la}}$, and $\mathcal{I}_{\mathcal{S}h, E}^{J-\text{la}}$ satisfies the gluing condition on rational open subspaces (cf. [77, Lemma 6.2.2]). In the following, we let $\mathcal{O}_{\mathcal{S}h, E}^{J-\text{la}}$, $\mathcal{O}_{I', E}^{J-\text{la}}$ denote the associated sheaves.

Proposition 4.4.16. *For a sufficient small compact open subgroup $K_p \subset G(\mathbb{Q}_p)$ and each open subspace $U \subset \mathcal{Sh}_{K_p K_p}^{\text{tor}}$ in Condition 4.4.11, there are natural quasi-isomorphisms*

$$R\Gamma_{\text{prokét}}(U, \iota_{K_p, I'}^* \widehat{\mathcal{O}}_{I'} \widehat{\otimes}_{\mathbb{Q}_p} C^{J-\text{la}}(\widetilde{K}_p, E)_{\text{prokét}}) \cong \mathcal{O}_{I', E}^{J-\text{la}}(\pi_{K_p}^{\text{tor}, -1}(U) \cap D_{\infty, I'})$$

and a long exact sequence

$$0 \rightarrow \mathcal{I}_{\mathcal{Sh}, E}^{J-\text{la}}(\pi_{K_p}^{\text{tor}, -1}(U)) \rightarrow \mathcal{O}_{\mathcal{Sh}, E}^{J-\text{la}}(\pi_{K_p}^{\text{tor}, -1}(U)) \rightarrow \cdots \rightarrow \mathcal{O}_{\mathcal{Sh}, E}^{J-\text{la}}(\pi_{K_p}^{\text{tor}, -1}(U) \cap D_{\infty, I}) \rightarrow 0.$$

PROOF. This follows by Lemma 4.4.12, Lemma 4.4.13, Lemma 4.4.14, and the remaining part of the proof of [77, Proposition 6.2.8]. $\square$

Proposition 4.4.17. *There are the following natural $G(\mathbb{Q}_p)$ -equivariant quasi-isomorphisms of derived solid $\mathbb{C}_p$ -vector spaces*

$$(1) (R\Gamma_{\text{prokét}}(\mathcal{Sh}_{K^p}^{\text{tor}}, \mathcal{O}_E) \widehat{\otimes}_{\mathbb{Z}_p}^L \mathbb{C}_p)^{RJ-\text{la}} = R\Gamma_{\text{an}}(\mathcal{Sh}_{K^p}^{\text{tor}}, \mathcal{O}_{\mathcal{Sh}, E}^{J-\text{la}}),$$

$$(2) (R\Gamma_{\text{prokét}}(\mathcal{Sh}_{K^p}^{\text{tor}}, j_! \mathcal{O}_E) \widehat{\otimes}_{\mathbb{Z}_p}^L \mathbb{C}_p)^{RJ-\text{la}} = R\Gamma_{\text{an}}(\mathcal{Sh}_{K^p}^{\text{tor}}, \mathcal{I}_{\mathcal{Sh}, E}^{J-\text{la}}),$$

where $R\Gamma_{\text{an}}$ is the right derived functors of taking global sections in the analytic topology.

PROOF. This follows by Proposition 4.4.16 and the proof of [77, Theorem 6.2.6]. $\square$

Write $H_{\text{an}}^i := H^i R\Gamma_{\text{an}}$.

Corollary 4.4.18. *For $i \geq 0$, there are natural $G(\mathbb{Q}_p)$ -equivariant isomorphisms*

$$(\widetilde{H}^i(K^p, E) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\text{la}} \cong H_{\text{an}}^i(\mathcal{Sh}_{K^p}^{\text{tor}}, \mathcal{O}_{\mathcal{Sh}, E}^{\text{la}}),$$

$$(\widetilde{H}_c^i(K^p, E) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\text{la}} \cong H_{\text{an}}^i(\mathcal{Sh}_{K^p}^{\text{tor}}, \mathcal{I}_{\mathcal{Sh}, E}^{\text{la}}).$$

PROOF. This follows by Theorem 4.4.2, Corollary 4.4.9, and Proposition 4.4.17, and the construction of $\widetilde{H}_?^i(K^p, W)$ for $? \in \{\emptyset, c\}$. $\square$

4.4.4. The case of $\widetilde{H}^1$ of unitary Shimura curves. In this subsection, we assume Condition 4.4.10 (1) on (G, X) , E , and J .

Proposition 4.4.19. *For any subset $J \in \{z\} \cup \bigsqcup_{\wp_i | p} \Sigma_{\wp_i}$, there is a natural $G(\mathbb{Q}_p)$ -equivariant isomorphism*

$$(\widetilde{H}^1(K^p, E) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{J-\text{la}} \cong H_{\text{an}}^1(\mathcal{Sh}_{K^p}, \mathcal{O}_{\mathcal{Sh}, E}^{J-\text{la}}).$$

PROOF. By Proposition 4.4.17 (1), it suffices to show an isomorphism

$$H^1((R\Gamma_{\text{prokét}}(\mathcal{Sh}_{K^p}, E) \widehat{\otimes}_{\mathbb{Q}_p}^L \mathbb{C}_p)^{RJ-\text{la}}) \cong (\widetilde{H}^1(K^p, E) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{J-\text{la}}.$$

By spectral sequence, we may reduce to show

$$H^2(\widetilde{H}^0(K^p, W) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{RJ-\text{la}} = 0.$$

Since the representation $\widetilde{H}^0(K^p, E)$ of $G(\mathbb{Q}_p)$ is admissible by [38, Theorem 2.1.5], it suffices to show

$$H_{\text{cont}}^0(\mathfrak{g}^J, H^2(\widetilde{H}^0(K^p, E) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{R\text{la}}) = 0, \quad \text{and}$$

$$H_{\text{cont}}^2(\mathfrak{g}^J, H^1(\widetilde{H}^0(K^p, E) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{R\text{la}}) = 0$$

by Proposition 4.4.6. Then the two vanishings of the cohomology follow by Theorem 4.4.2. $\square$

For a subset $\mathcal{J} \subset \{z\} \cup [1, s]$, let

$$G_{\mathcal{J}} := \prod_{k \in \mathcal{J}} G_k,$$

where $G_z := \mathbb{G}_{m, \mathbb{Q}_p}$ and $G_i := \text{Res}_{F_{\wp_i}/\mathbb{Q}_p} \text{GL}_2(F_{\wp_i})$ for $i \in [1, s]$. Let $\mathcal{J}' := (\{z\} \cup [1, s]) \setminus \mathcal{J}$. Let $\mathfrak{z} := \text{Lie } Z$, $\mathfrak{g}_k := \text{Lie}(G_k)$ for $k \in \{z\} \cup [1, s]$, and $\mathfrak{g}_p^{\wp} := \text{Lie}(G(\mathbb{Q}_p^{\wp}))$ be the Lie algebras over $\mathbb{Q}_p$. We characterize $\tilde{H}^1(K^{\wp}, E)$ as the subspace of $\mathcal{Q}$ -locally analytic vectors in $\tilde{H}^1(K^p, E)$ by the method of [38].

Lemma 4.4.20. *For each object W in $\text{Rep}_E(G^c)$, we have*

$$\text{Ext}_{\mathfrak{g}_p^{\wp}}^2(W^{\vee}, (\tilde{H}^0(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\wp^{-1a}}) = 0.$$

PROOF. Recall that $\wp_1 = \wp$. We may assume that W is a finite dimensional simple algebraic representation of G_E because any finite dimensional algebraic representation of a connected reductive group over a field of characteristic 0 is semisimple. By the classification of simple algebraic representations of reductive groups (see e.g., [49] and also [86, §2.2]), we write

$$W = \bigotimes_{k \in \{z\} \cup [1, s]} W_k,$$

where W_k is a simple algebraic representation of $G_{k,E}$. Write $\mathfrak{z}_{k,E}$ for the center of $\mathfrak{g}_{k,E}$. Let $\mathfrak{z}_E := \bigoplus_{k \in \{z\} \cup [1, s]} \mathfrak{z}_{k,E}$ be the center of $\mathfrak{g}_E$ and $\mathfrak{s} := \bigoplus_{k \in [1, s]} \mathfrak{sl}_{2,E}$. Since the 0-th completed cohomology $\tilde{H}^0(K^p, E)$ is the E -vector space of the profinite set of connected component of infinite level unitary Shimura curves isomorphic to the tensor product of a finite dimensional trivial representation of $G(E)$ and $C_{\text{cont}}(\mathbb{Z}_p^{\times}, E)$ where G acts on the profinite set via the reduced norm, the subgroup $\prod_{k \in [1, s]} \text{SL}_2(E) \subset G(E)$ trivially acts on $\tilde{H}^0(K^p, E)$.

Then by Künneth formula (cf. the proof of [38, Proposition 4.3.1]), for an integer i , we have an isomorphism

$$\begin{aligned} \bigoplus_{a+b=i} (\text{Ext}_{\mathfrak{z}}^a(E(\chi^{-1}), (\tilde{H}^0(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\wp^{-1a}}) \otimes_E \text{Ext}_{\mathfrak{s}}^b(W^{\vee}, E)) \\ \xrightarrow{\sim} \text{Ext}_{\mathfrak{g}_E}^i(W^{\vee}, (\tilde{H}^0(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\wp^{-1a}}) \end{aligned}$$

where χ_E is the central character of the representation W . The module

$$\text{Ext}_{\mathfrak{z}}^2(E(\chi^{-1}), (\tilde{H}^0(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\wp^{-1a}})$$

vanishes by that $\mathfrak{z}$ is a direct sum of one dimensional Lie algebras and Hochschild-Serre spectral sequence.

Then by Künneth formula again, we have

$$\bigoplus_{a_1 + \dots + a_s = i} \bigotimes_{k \in [1, s]} \text{Ext}_{\mathfrak{sl}_{2,E}}^{a_k}(W_k^{\vee}, E) \xrightarrow{\sim} \text{Ext}_{\mathfrak{s}}^i(W^{\vee}, E).$$

Thus we may reduce to show for each $k \in [2, s]$ and $i = 1$ or 2 ,

$$(4.4.5) \quad \text{Ext}_{\mathfrak{gl}_{2,E}}^i(W_k^{\vee}, (\tilde{H}^0(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\wp^{-1a}}) = 0.$$

The vanishings of (4.4.5) follows by the similar argument of [38, Proposition 4.3.1, Corollary 4.3.2]. $\square$

Proposition 4.4.21. *There is a natural $G(\mathbb{A}^\infty)$ -equivariant isomorphism*

$$(\tilde{H}^1(K^p, W) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\wp-\text{la}} \cong \varinjlim_{K_p^\wp} (\tilde{H}^1(K^p K_p^\wp, W) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\text{la}},$$

where $K_p^\wp$ runs over compact open subgroups of $G(\mathbb{Q}_p^\wp)$.

PROOF. By the same argument of [38, Corollary 2.2.18], we have a spectral sequence

$$E_2^{i,j} = \text{Ext}_{\mathfrak{g}_p^\wp}^i(W^\vee, (\tilde{H}^j(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\wp-\text{la}}) \implies \varinjlim_{K_p^\wp} (\tilde{H}^{i+j}(K^p K_p^\wp, W) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\text{la}}.$$

Then the proposition follows from Lemma 4.4.20 in the case of $W = E$. $\square$

Corollary 4.4.22. *There is a natural $G(\mathbb{A}^\infty)$ -equivariant isomorphism*

$$(\tilde{H}^1(K^p, W) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\tau-\text{la}} \cong \varinjlim_{K_p^\wp} (\tilde{H}^1(K^p K_p^\wp, W) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\tau-\text{la}},$$

where $K_p^\wp$ runs over compact open subgroups of $G(\mathbb{Q}_p^\wp)$.

PROOF. This follows by taking $\tau - \text{la}$ of the isomorphism in Proposition 4.4.21. $\square$

4.5. Sheaves of locally analytic sections

In this section, we will show that locally analytic vectors for the action of $G(\mathbb{Q}_p)$ (not $\text{GL}_2(F_\wp)$) in completed cohomology of unitary Shimura curves comes from locally analytic completed cohomology of Siegel modular varieties. As a consequence, we give a description of completed cohomology of $K^\wp$ -level in terms of cohomology of sheaves of locally analytic sections on $K^\wp$ -level unitary Shimura curve. Let $(\text{GSp}_{2g}, D_{\text{GSp}_{2g}})$ be a Shimura data which gives a Siegel modular variety in Section 4.2.2. For a compact open subgroup $K = K_p K^p \subset \text{GSp}_{2g}(\mathbb{A}^\infty)$, let $\mathcal{Y}_K$ be the associated Siegel modular variety with $(\text{GSp}_{2g}, D_{\text{GSp}_{2g}})$.

4.5.1. Surjectivity of the maps of locally analytic sections. We recall the construction of Hodge-Tate period map in Proposition 4.3.2, in the case of Siegel modular varieties (cf. [83, §III.3]). Let $f: \mathcal{A}_K \rightarrow \mathcal{Y}_K$ be the universal abelian variety as for the moduli interpretation of $\mathcal{X}$ (cf. [40, §1]). The standard representation $\text{GSp}_{2g, \mathbb{Q}_p} \rightarrow \text{GL}(V_{\text{std}}) \cong \text{GL}_{2n, \mathbb{Q}_p}$ defines a $\mathbb{Q}_p$ -local system $V_{\text{std}, \text{proét}}$ on the pro-étale site $\mathcal{Y}_{K, \text{proét}}$. The local system is naturally isomorphic to $R^1 f_* \mathbb{Q}_p$. On the other hand, V_{dR} is naturally isomorphic to the relative de Rham cohomology of $\mathcal{A}_K / \mathcal{Y}_K$. This is a rank $2g$ vector bundle on $\mathcal{Y}_K$ equipped with a (decreasing) Hodge filtration $\text{Fil}^\bullet$ and Gauss-Manin connection ∇ .

Let $\mathcal{Y}_K^{\text{tor}}$ be a toroidal compactification of $\mathcal{Y}_K$. Similarly, we have a $\mathbb{Q}_p$ -local system $V_{\text{std}, \text{prokét}}$ on $\mathcal{Y}_{K, \text{prokét}}^{\text{tor}}$ and the corresponding vector bundle $V_{\text{dR}, \log}$ on $\mathcal{Y}_K^{\text{tor}}$ equipped with a logarithmic connection $\nabla_{\log}$. The graded components of $V_{\text{dR}, \log}$ vanish except in degree 0 or 1, and we have an exact sequence

$$(4.5.1) \quad 0 \rightarrow \text{gr}^0(V_{\text{dR}, \log}) \otimes_{\mathcal{O}_{\mathcal{Y}_K^{\text{tor}}}} \hat{\mathcal{O}}_{\mathcal{Y}_K^{\text{tor}}} \rightarrow V_{\text{std}, \text{prokét}} \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathcal{O}}_{\mathcal{Y}_K^{\text{tor}}} \rightarrow \text{gr}^1(V_{\text{dR}, \log}) \otimes_{\mathcal{O}_{\mathcal{Y}_K^{\text{tor}}}} \hat{\mathcal{O}}_{\mathcal{Y}_K^{\text{tor}}}(-1) \rightarrow 0$$

by [82, Proposition 7.9].

Let $\mathcal{Y}_{K^p}^{\text{tor}} \sim \varprojlim_{K_p} \mathcal{Y}_{K_p K^p}^{\text{tor}}$ be a perfectoid space constructed by Theorem 4.3.1. Since $V_{\text{std}, \text{prokét}}$ is trivialized on $\mathcal{Y}_{K^p}^{\text{tor}}$, we have an exact sequence

$$(4.5.2) \quad 0 \rightarrow \text{gr}^0(V_{\text{dR}, \log}) \otimes_{\mathcal{O}_{\mathcal{Y}_K^{\text{tor}}}} \hat{\mathcal{O}}_{\mathcal{Y}_{K^p}^{\text{tor}}} \rightarrow \underline{V}_{\text{std}} \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathcal{O}}_{\mathcal{Y}_{K^p}^{\text{tor}}} \rightarrow \text{gr}^1(V_{\text{dR}, \log}) \otimes_{\mathcal{O}_{\mathcal{Y}_K^{\text{tor}}}} \hat{\mathcal{O}}_{\mathcal{Y}_{K^p}^{\text{tor}}}(-1) \rightarrow 0$$

of vector bundles from (4.5.1). Fix a Hodge cocharacter $\tilde{\mu}: \mathbb{G}_m \rightarrow \mathrm{GSp}_{2g, \mathbb{C}}$ defined over $\mathbb{Q}_p$ attached to an element $h \in X$ and let $\mathcal{F}\ell_{\tilde{\mu}}$ be the associated rigid space with $\mathrm{Fl}_{\mathbb{C}_p} := G/P_{\tilde{\mu}} \times_{\mathrm{Spec} \mathbb{Q}_p} \mathrm{Spec} \mathbb{C}_p$. The position

$$\mathrm{gr}^0(V_{\mathrm{dR}, \log}) \otimes_{\mathcal{O}_{\mathcal{Y}^{\mathrm{tor}}}} \widehat{\mathcal{O}}_{\mathcal{Y}_{K^p}^{\mathrm{tor}}} \subset \underline{V}_{\mathrm{std}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_{\mathcal{Y}_{K^p}^{\mathrm{tor}}}$$

in the exact sequence (4.5.2) induces the $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$ -equivariant Hodge-Tate period map

$$\pi_{\mathrm{HT}}^{\mathrm{tor}}: \mathcal{Y}_{K^p}^{\mathrm{tor}} \rightarrow \mathcal{F}\ell_{\tilde{\mu}}$$

as constructed in Proposition 4.3.2.

Fix a $\mathbb{Q}_p$ -valued point $\infty \in \mathcal{F}\ell_{\tilde{\mu}}$, a point $\widetilde{\infty} \in \mathcal{Y}_{K^p}^{\mathrm{tor}}$ which is mapped to ∞ by $\pi_{\mathrm{HT}}^{\mathrm{tor}}$, and a symplectic basis $\{e_1, \dots, e_{2g}\} \in V_{\mathrm{std}}$ such that $\{e_1 \otimes 1, \dots, e_g \otimes 1\}$ generates $\mathrm{gr}^0(V_{\mathrm{dR}, \log}) \otimes \{\widetilde{\infty}\}$, where $\{\widetilde{\infty}\} \hookrightarrow \mathcal{Y}_{K^p}^{\mathrm{tor}}$ is the closed immersion of perfectoid spaces. Taking global sections of (4.5.2), we have an exact sequence

$$(4.5.3) \quad \begin{aligned} 0 \rightarrow H^0(\mathcal{Y}_{K^p}^{\mathrm{tor}}, \mathrm{gr}^0(V_{\mathrm{dR}, \log}) \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{tor}}}) &\rightarrow H^0(\mathcal{Y}_{K^p}^{\mathrm{tor}}, \underline{V}_{\mathrm{std}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{tor}}}) \\ &\rightarrow H^0(\mathcal{Y}_{K^p}^{\mathrm{tor}}, \mathrm{gr}^1(V_{\mathrm{dR}, \log}) \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{tor}}}) \rightarrow 0 \end{aligned}$$

since $H^1(\mathcal{Y}_{K^p}^{\mathrm{tor}}, \mathrm{gr}^0(V_{\mathrm{dR}, \log}) \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathcal{O}}_{\mathcal{Y}^{\mathrm{tor}}}) = 0$ by the almost purity theorem [37, Theorem 5.4.3]. We also write $e_1, \dots, e_{2g}$ for the images of $e_1 \otimes 1, \dots, e_{2g} \otimes 1$ by the third map of (4.5.3).

Let (G, X) be as in Section 4.2.2 and fix a Hodge cocharacter $\tilde{\mu}: \mathbb{G}_m \rightarrow G_{\mathbb{C}}$ defined over $\mathcal{F}$. Let $\iota: (G, X) \hookrightarrow (\mathrm{GSp}_{2g}, D_{\mathrm{GSp}_{2g}})$ be a closed embedding of Shimura data such that the Hodge cocharacters μ and $\tilde{\mu}$ are compatible by the embedding. Let $K^p \subset G(\mathbb{A}^{\infty, p}) \subset G(\mathbb{A}^{\infty})$ be a neat compact open subgroup which satisfies the condition of Theorem 4.3.1. Let $i: \mathcal{F}\ell_{\mu} \hookrightarrow \mathcal{F}\ell_{\tilde{\mu}}$ be the closed immersion of the flag varieties induced by ι .

We recall the construction of perfectoid Shimura varieties of Hodge type $\mathcal{S}h_{K^p}^{\mathrm{tor}}$. Let $K'_p K'^p \subset \mathrm{GSp}_{2g}(\mathbb{A}^{\infty})$ be a sufficiently small compact open subgroup such that $K_p K^p = K'_p K'^p \cap G(\mathbb{A}^{\infty})$. Let $\mathcal{Y} := \mathcal{Y}_{K'_p K'^p}^{\mathrm{tor}}$ and $U' \subset \mathcal{Y}$ be an open subspace such that there exist an open subspace $V' \subset \mathcal{F}\ell_{\tilde{\mu}}$ where (U', V') satisfies Condition 4.4.11, $U' \subset \mathfrak{B}$ where $\mathfrak{B}$ is as in Proposition 4.3.2 and $V := i^{-1}(V') \subset \mathcal{F}\ell_{\tilde{\mu}}$ is non-empty. Let $\pi_{K'_p K'^p}^{\mathrm{tor}}: \mathcal{Y}_{K'^p}^{\mathrm{tor}} \rightarrow \mathcal{Y}_{K'_p K'^p}^{\mathrm{tor}}$ and $\pi_{K_p K^p}^{\mathrm{tor}}: \mathcal{S}h_{K^p}^{\mathrm{tor}} \rightarrow \mathcal{S}h_{K_p K^p}^{\mathrm{tor}}$ be the natural projections. For any open subgroup $K^{p'} \subset \mathrm{GSp}_{2g}(\mathbb{A}^{\infty, p})$ which satisfies the condition of Theorem 4.3.1, the open subspace $\mathcal{Y}_{K^{p'}}^{\mathrm{tor}}(U') := \pi_{K_p K^p}^{\mathrm{tor}, -1}(U')$ is affinoid perfectoid (cf. [83, Theorem III.3.18]) and we write $\mathcal{Y}_{K^{p'}}^{\mathrm{tor}}(U') = \mathrm{Spa}(S_{K^{p'}}, S_{K^{p'}}^+)$. It follows that

$$(4.5.4) \quad \mathcal{Y}_{K^p}^{\mathrm{tor}}(U') := \varprojlim_{K^p \subset K^{p'} \subset \mathrm{GSp}_{2g}(\mathbb{A}^{\infty})} \mathcal{Y}_{K^{p'}}^{\mathrm{tor}}(U') = \mathrm{Spa}(S_{K^p}, S_{K^p}^+),$$

where $S_{K^p}^+$ is the p -adic completion of $\varinjlim_{K^p \subset K^{p'}} S_{K^{p'}}^+$, is affinoid perfectoid.

For any sufficient small compact open subgroup $K'_p K'^p \subset \mathrm{GSp}_{2g}(\mathbb{A}^{\infty})$ with $K_p K^p = K'_p K'^p \cap G(\mathbb{A}^{\infty})$, a closed immersion

$$(\mathcal{Y}_{K^p}^{\mathrm{tor}} \times_{\mathcal{Y}_{K'_p K'^p}^{\mathrm{tor}}} \mathcal{S}h_{K_p K^p}^{\mathrm{tor}})(U') := \mathcal{Y}_{K^p}^{\mathrm{tor}}(U') \times_{\mathcal{Y}_{K'_p K'^p}^{\mathrm{tor}}} \mathcal{S}h_{K_p K^p}^{\mathrm{tor}} \subset \mathcal{Y}_{K^p}^{\mathrm{tor}}(U')$$

obtained by the base change of the closed immersion $i_{K'_p K'^p}: \mathcal{S}h_{K_p K^p}^{\mathrm{tor}} \hookrightarrow \mathcal{Y}_{K'_p K'^p}^{\mathrm{tor}}$ is defined by some ideal $I \subset S_{K^p}$ by [33, Proposition 1.15], and is affinoid perfectoid by [83, Lemma II.2.2]. We write

$$(4.5.5) \quad (\mathcal{Y}_{K^p}^{\mathrm{tor}} \times_{\mathcal{Y}_{K'_p K'^p}^{\mathrm{tor}}} \mathcal{S}h_{K_p K^p}^{\mathrm{tor}})(U') = \mathrm{Spa}(R_{K^p, K'_p K'^p}, R_{K^p, K'_p K'^p}^+).$$

Then we can write

$$(4.5.6) \quad \mathcal{S}h_{K^p}^{\text{tor}}(U') := \varprojlim_{K'_p, K^p \subset K^{p'}} (\mathcal{Y}_{K^p}^{\text{tor}} \times_{\mathcal{Y}_{K^p}^{\text{tor}}} \mathcal{S}h_{K^p K^p}^{\text{tor}})(U') = \text{Spa}(R_{K^p}, R_{K^p}^+) \subset \mathcal{S}h_{K^p}^{\text{tor}}$$

as an affinoid open perfectoid subspace, where $R_{K^p}^+$ is the p -adic completion of

$$\varinjlim_{K'_p, K^p \subset K^{p'}} R_{K^p, K'_p K^{p'}}.$$

The open subspaces $\mathcal{S}h_{K^p}^{\text{tor}}(U')$ covers the whole $\mathcal{S}h_{K^p}^{\text{tor}}$. Then we have the closed immersions of perfectoid spaces $i_\infty: \mathcal{S}h_{K^p}^{\text{tor}} \hookrightarrow \mathcal{Y}_{K^p}^{\text{tor}}$ and $i_{\infty, U'}: \mathcal{S}h_{K^p}^{\text{tor}}(U') \hookrightarrow \mathcal{Y}_{K^p}^{\text{tor}}(U')$. We note that the construction of $\mathcal{S}h_{K^p}^{\text{tor}}$ is compatible with the method of [77] by the last part of the proof of [54, Proposition 6.1].

Let $i: \mathcal{F}\ell_\mu \hookrightarrow \mathcal{F}\ell_{\tilde{\mu}}$ be the closed immersion of the flag varieties induced by $\iota, \mu, \tilde{\mu}$. The map i is compatible with the Hodge-Tate period maps by Proposition 4.3.2.

For each compact open subgroup $K'_p K^{p'} \subset \text{GSp}_{2g}(\mathbb{A}^\infty)$ which gives a closed immersion $i_{K'_p K^{p'}}: \mathcal{S}h_{K^p K^p}^{\text{tor}} \hookrightarrow \mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}$, let $j_{K'_p K^{p'}}: \mathcal{Y}_{K'_p K^{p'}}^{\text{tor}} \setminus \mathcal{S}h_{K^p K^p}^{\text{tor}} \hookrightarrow \mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}$ be the open immersion. We have an exact sequence of Kummer étale sheaves

$$0 \rightarrow j_{K'_p K^{p'}}! \mathbb{Z}/p^s \rightarrow \mathbb{Z}/p^s \rightarrow i_{K'_p K^{p'},*} \mathbb{Z}/p^s \rightarrow 0$$

for each integer $s > 0$ by [55, Lemma 2.1.5] and that the cone decompositions which define the toroidal compactifications are compatibly taken. Taking a derived limit, we have an exact sequence of Kummer étale sheaves

$$0 \rightarrow j_{K'_p K^{p'}}! \mathbb{Z}_p \rightarrow \mathbb{Z}_p \rightarrow i_{K'_p K^{p'},*} \mathbb{Z}_p \rightarrow 0.$$

Tensoring with $\mathcal{O}_{\mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}}/p^s$, we have an exact sequence of Kummer étale sheaves

$$0 \rightarrow j_{K'_p K^{p'}}! \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}}/p^s \rightarrow \mathcal{O}_{\mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}}/p^s \rightarrow i_{K'_p K^{p'},*} \mathcal{O}_{\mathcal{S}h_{K^p K^p}^{\text{tor}}}/p^s \rightarrow 0$$

by [37, Lemma 4.5.7]. Taking a derived limit and inverting p , we have an exact sequence of pro-Kummer étale sheaves

$$(4.5.7) \quad 0 \rightarrow \widehat{\mathcal{I}}_{\mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}} \rightarrow \widehat{\mathcal{O}}_{\mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}} \rightarrow i_{K'_p K^{p'},*} \widehat{\mathcal{O}}_{\mathcal{S}h_{K^p K^p}^{\text{tor}}} \rightarrow 0$$

where $\widehat{\mathcal{I}}_{\mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}} := \varprojlim_s j_{K'_p K^{p'}}! \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{Y}_{K'_p K^{p'}}^{\text{tor}}}/p^s$. We let

$$\begin{aligned} \mathcal{O}_{\mathcal{Y}}^{\text{GSp}_{2g}(\mathbb{Q}_p)-\text{la}}(\mathcal{Y}_{K^p}^{\text{tor}}(U')) &:= \widehat{\mathcal{O}}_{\mathcal{Y}}(\mathcal{Y}_{K^p}^{\text{tor}}(U'))^{\text{GSp}_{2g}(\mathbb{Q}_p)-\text{la}}, \\ \mathcal{O}_{\mathcal{S}h}^{G(\mathbb{Q}_p)-\text{la}}(\mathcal{S}h_{K^p}^{\text{tor}}(U')) &:= \widehat{\mathcal{O}}_{\mathcal{S}h}(\mathcal{S}h_{K^p}^{\text{tor}}(U'))^{G(\mathbb{Q}_p)-\text{la}}. \end{aligned}$$

Proposition 4.5.1. *The $G(\mathbb{Q}_p)$ -equivariant map*

$$\mathcal{O}_{\mathcal{Y}}^{\text{GSp}_{2g}(\mathbb{Q}_p)-\text{la}}(\mathcal{Y}_{K^p}^{\text{tor}}(U')) \rightarrow \mathcal{O}_{\mathcal{S}h}^{G(\mathbb{Q}_p)-\text{la}}(\mathcal{S}h_{K^p}^{\text{tor}}(U'))$$

induced from taking locally analytic vectors of the completed colimit of third maps of the exact sequence (4.5.7) is surjective. The map remains to be surjective when the source is restricted to $\mathcal{O}_{\mathcal{Y}}^{\text{GSp}_{2g}(\mathbb{Q}_p)-\text{la}}(\mathcal{Y}_{K^p}^{\text{tor}}(U'))$.

Moreover, for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, if an element in the source is $\mathfrak{h}$ -smooth, its image in the target is $\mathfrak{h}$ -smooth.

PROOF. We rewrite $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{Y}_{K'_p K^{p''}}$ and let $K'_p K^{p'} \subset K''_p K^{p''} \subset \mathrm{GSp}_{2g}(\mathbb{A}^\infty)$ be a compact open subgroup as in (4.5.6). Note that for the open subspace $V := i^{-1}(V') \subset \mathcal{F}\ell_{\mu, I}$, the pair $(i_{K''_p K^{p''}}^{-1}(U'), V)$ also satisfies Condition 4.4.11. We use the argument of the proof of part (2) of [77, Proposition 6.2.8].

Let $U'_{K'_p K^{p'}} := U' \times_{\mathcal{Y}} \mathcal{Y}_{K'_p K^{p'}}$. We have an injection and an isomorphism

$$(4.5.8) \quad \begin{aligned} H^0(U'_{K'_p K^{p'}}, \widehat{\mathcal{O}}_{\mathcal{Y}_{K'_p K^{p'}}} \widehat{\otimes}_{\mathbb{Q}_p} C^{\mathrm{GSp}_{2g}(\mathbb{Q}_p) - \mathrm{la}}(\tilde{K}'_p, E)_{\mathrm{prok\acute{e}t}}) &\hookrightarrow \mathcal{O}_{\mathcal{Y}}^{\mathrm{GSp}_{2g}(\mathbb{Q}_p) - \mathrm{la}}(\mathcal{Y}_{K^p}^{\mathrm{tor}}(U')), \\ H^0(i_{K''_p K^{p''}}^{-1}(U'), \widehat{\mathcal{O}}_{Sh} \widehat{\otimes}_{\mathbb{Q}_p} C^{G(\mathbb{Q}_p) - \mathrm{la}}(\tilde{K}_p, E)_{\mathrm{prok\acute{e}t}}) &\cong \mathcal{O}_{Sh}^{G(\mathbb{Q}_p) - \mathrm{la}}(Sh_{K^p}^{\mathrm{tor}}(U')), \end{aligned}$$

where $\tilde{K}_p$ is a certain quotient of K_p and $\tilde{K}'_p = K'_p$, by Proposition 4.4.16 and the compatibility of i, i_∞ with the Hodge-Tate period maps.

Retake toric charts of $i_{K'_p K^{p'}}^{-1}(U')$ and U' such that the following diagram commutes:

$$(4.5.9) \quad \begin{array}{ccc} i_{K'_p K^{p'}}^{-1}(U'_{K'_p K^{p'}}) & \longrightarrow & \mathbb{T}^{e'} \times \mathbb{D}^{d'-e'} \\ \downarrow & & \downarrow \\ U'_{K'_p K^{p'}} & \longrightarrow & \mathbb{T}^e \times \mathbb{D}^{d-e}, \end{array}$$

where $d' \geq d, e' \geq e$, and the right vertical map is the product of the natural projections to first e (resp. $d - e$) coordinates. We may assume to take toric charts compatibly with choices of $K'_p K^{p'}$. We note that $i_{K''_p K^{p''}}^{-1}(U') = i_{K'_p K^{p'}}^{-1}(U'_{K'_p K^{p'}})$

Then we can construct affinoid perfectoid covers $\widetilde{U}'_{K'_p K^{p'}, p^\infty}$ and $\widetilde{i_{K''_p K^{p''}}^{-1}(U')}_{p^\infty}$ of $U'_{K'_p K^{p'}}$ and $i_{K''_p K^{p''}}^{-1}(U')$ with Galois groups $\tilde{K}'_p \times \Gamma'_p$ and $\tilde{K}_p \times \Gamma_p$ respectively as in Section 4.4.3, which are compatible with the map $i_{K'_p K^{p'}}$.

We have an injection and an isomorphism

$$(4.5.10) \quad \begin{aligned} \varinjlim_{r \rightarrow \infty} (\widehat{\mathcal{O}}_{\mathcal{Y}}(\widetilde{U}'_{K'_p K^{p'}, p^\infty}) \widehat{\otimes}_{\mathcal{O}(V')} \mathcal{O}(\mathbb{X}'_r))_{\tilde{K}'_p - \mathrm{sm}, \Gamma'_p} &\hookrightarrow \mathcal{O}_{\mathcal{Y}}^{\mathrm{GSp}_{2g}(\mathbb{Q}_p) - \mathrm{la}}(\mathcal{Y}_{K^p}^{\mathrm{tor}}(U')), \\ \varinjlim_{r \rightarrow \infty} (\widehat{\mathcal{O}}_{Sh}(\widetilde{i_{K''_p K^{p''}}^{-1}(U')}_{p^\infty}) \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathbb{X}_r))_{\tilde{K}_p - \mathrm{sm}, \Gamma_p} &\cong \mathcal{O}_{Sh}^{G(\mathbb{Q}_p) - \mathrm{la}}(Sh_{K^p}^{\mathrm{tor}}(U')), \end{aligned}$$

where $\{\mathbb{X}'_r\}_{r \gg 0}$ and $\{\mathbb{X}_r\}_{r \gg 0}$ are shrinking system of closed polydiscs over V' and V respectively (We consider the case of $\mathbb{Q}_p$ -coefficient and $\mathbb{Q}_p$ -locally analytic subspaces in this proposition, nevertheless the same proof of Lemma 4.4.14 works) by Lemma 4.4.12, Lemma 4.4.13, Lemma 4.4.14, and the isomorphisms (4.5.8).

By the detailed construction of $\{\mathbb{X}'_r\}_{r \gg 0}$ and $\{\mathbb{X}_r\}_{r \gg 0}$ in the proof of Lemma 4.4.14, we have a natural surjection $\mathcal{O}(\mathbb{X}'_r) \rightarrow \mathcal{O}(\mathbb{X}_r)$, so we obtain a surjection for $r \gg 0$

$$\left(\varinjlim_{K'_p K^{p'}} \widehat{\mathcal{O}}_{\mathcal{Y}}(\widetilde{U}'_{K'_p K^{p'}, p^\infty}) \right)^\wedge \widehat{\otimes}_{\mathcal{O}(V')} \mathcal{O}(\mathbb{X}'_r) \rightarrow \widehat{\mathcal{O}}_{Sh}(\widetilde{i_{K''_p K^{p''}}^{-1}(U')}_{p^\infty}) \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathbb{X}_r),$$

where $(-)^\wedge$ denotes taking p -adic completions. Then it suffices to show that the restriction for any (or, some) $K'_p K^{p'}$,

$$\widehat{\mathcal{O}}_{\mathcal{Y}}(\widetilde{U}'_{K'_p K^{p'}, p^\infty}) \widehat{\otimes}_{\mathcal{O}(V')} \mathcal{O}(\mathbb{X}'_r) \rightarrow \widehat{\mathcal{O}}_{Sh}(\widetilde{i_{K''_p K^{p''}}^{-1}(U')}_{p^\infty}) \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathbb{X}_r)$$

amounts to be surjective after taking locally analytic vectors.

Let $\tilde{A}_{K'_p K^{p'}, \infty} := \widehat{\mathcal{O}}_{\mathcal{Y}}(\tilde{U}'_{K'_p K^{p'}, p^\infty})$ and $\tilde{B}_\infty := \widehat{\mathcal{O}}_{Sh}(i_{K''_p K^{p''}}^{-1}(U')_{p^\infty})$. Write $A_{K'_p K^{p'}, H', n} := \tilde{A}_{K'_p K^{p'}, \infty}^{H' \times p^n \Gamma'_p}$ and $B_{H, n} := \tilde{B}_\infty^{H \times p^n \Gamma_p}$ for a positive integer n and compact open subgroups $H' \subset K'_p$ (or $\tilde{K}'_p$) and $H \subset K_p$ (or $\tilde{K}_p$). Let

$$\begin{aligned}\mathcal{F}'_{K'_p K^{p'}, r} &:= \widehat{\mathcal{O}}_{\mathcal{Y}}(\tilde{U}'_{K'_p K^{p'}, p^\infty}) \widehat{\otimes}_{\mathcal{O}(V')} \mathcal{O}(\mathbb{X}'_r), \\ \mathcal{F}''_r &:= \widehat{\mathcal{O}}_{Sh}(i_{K''_p K^{p''}}^{-1}(U')_{p^\infty}) \widehat{\otimes}_{\mathcal{O}(V')} \mathcal{O}(\mathbb{X}'_r), \\ \mathcal{F}_r &:= \widehat{\mathcal{O}}_{Sh}(i_{K''_p K^{p''}}^{-1}(U')_{p^\infty}) \widehat{\otimes}_{\mathcal{O}(V)} \mathcal{O}(\mathbb{X}_r).\end{aligned}$$

For each $r \gg 0$, there exist open compact subgroups $\tilde{K}'_p(r) \subset \tilde{K}'_p$ and $\tilde{K}_p(r) := \tilde{K}'_p(r) \cap G(\mathbb{Q}_p) \subset \tilde{K}_p$ such that $\mathcal{F}'_{K'_p K^{p'}, r}$ is $\tilde{K}'_p(r)$ -stable and $\mathcal{F}_r$ is $\tilde{K}_p(r)$ -stable.

We can take $\mathbb{Q}_p$ -ON bases $\{v'_{k'}\}_{k'}$ of $\mathcal{O}(\mathbb{X}'_r)$, and $\{v_k\}_k$ of $\mathcal{O}(\mathbb{X}_r)$ which contains $\{v'_{k'}\}_{k'}$ as a subset. Then by [76, Theorem 2.4.3, Remark 2.5.2], there is a compact open subgroup $H'(r) \times p^n \Gamma'_p \subset \tilde{K}'_p(r) \times \Gamma'_p$ such that $\mathcal{F}'_{K'_p K^{p'}, r}$ admits a unique decompletion (in [76, Definition 2.4.1], it is called Sen module) to an ON Banach $\tilde{K}'_p(r) \times \Gamma'_p$ -representation $S_{H'(r), n}(\mathcal{F}'_{K'_p K^{p'}, r})$ over $A_{K'_p K^{p'}, H'(r), n}$ and $\mathcal{F}''_r$ and $\mathcal{F}_r$ admit unique decompletions to an ON Banach $H(r) \times p^n \Gamma_p$ -representations $S_{H(r), n}(\mathcal{F}''_r)$ and $S_{H(r), n}(\mathcal{F}_r)$ over $B_{H(r), n}$ where $H(r) := H'(r) \cap \tilde{K}_p(r)$. Note that the maps $A_{K'_p K^{p'}, H'(r), n} \rightarrow B_{H(r), n}$ is surjective by the construction.

The uniqueness of the decompletions ([76, Theorem 2.4.3]) yields

$$\begin{aligned}S_{H(r), n}(\mathcal{F}''_r) &\cong S_{H'(r), n}(\mathcal{F}'_{K'_p K^{p'}, r}) \widehat{\otimes}_{A_{H'(r), n}} B_{H(r), n} \\ &= S_{H'(r), n}(\mathcal{F}'_{K'_p K^{p'}, r}) \otimes_{A_{H'(r), n}} B_{H(r), n}\end{aligned}$$

and then we have a surjection $S_{H'(r), n}(\mathcal{F}'_r) \rightarrow S_{H(r), n}(\mathcal{F}''_r)$. We also have a surjection $S_{H(r), n}(\mathcal{F}''_r) \rightarrow S_{H(r), n}(\mathcal{F}_r)$ since the decompletions preserves surjectivity by [76, Proposition 2.5.4]. So we have a surjection

$$(4.5.11) \quad S_{H'(r), n}(\mathcal{F}'_{K'_p K^{p'}, r}) \rightarrow S_{H(r), n}(\mathcal{F}_r)$$

as the composite of the two surjections. Since the Sen operators of $\mathcal{F}'$ and $\mathcal{F}$ are trivial by the construction, the action of $\tilde{K}'_p(r) \times \Gamma'_p$ (resp. $\tilde{K}_p(r) \times \Gamma_p$) on $S_{H'(r), n}(\mathcal{F}'_{K'_p K^{p'}, r})$ (resp. $S_{H(r), n}(\mathcal{F}_r)$) is smooth. Write $\Gamma'_p = \Gamma_p \times \Gamma''_p$ by the compatibility of toric charts (4.5.9), and extend the Γ_p -action on $S_{H(r), n}(\mathcal{F}_r)$ to a Γ'_p -action by the trivial action of Γ''_p . The group Γ'_p acts on the surjective map (4.5.11) via a finite quotient. Thus, taking Γ'_p invariant of (4.5.11) and the colimit along $r \rightarrow \infty$, we have a surjection

$$\varinjlim_{r \rightarrow \infty} S_{H'(r), n}(\mathcal{F}'_{K'_p K^{p'}, r})^{\Gamma'_p} \twoheadrightarrow \varinjlim_{r \rightarrow \infty} S_{H(r), n}(\mathcal{F}_r)^{\Gamma_p}$$

By the definition of decompletions and the equations (4.5.10), the colimit along $n \rightarrow \infty$ and $H(r) \rightarrow 1$ of the surjection is the same as that induced from the third map of the exact sequence (4.5.7). The last claim follows from the construction. $\square$

Let $\mathfrak{g}$, $\mathfrak{p}_\mu$, $\mathfrak{p}_{\tilde{\mu}}$, $\mathfrak{n}_\mu$, $\mathfrak{n}_{\tilde{\mu}}$ is the Lie algebras as defined in Section 4.3.2. Let $\tilde{\mathfrak{p}}_\mu := \text{Lie}(P_\mu^c)$.

Since the locus of points $x \in \mathcal{F}\ell_\mu$ such that $\mathfrak{n}_{\mu, x}^0 \cap \tilde{\mathfrak{p}}_{\mu, x}^0 \neq 0$ is non-empty Zariski closed in $\mathcal{F}\ell_\mu$, we need to argue with the localized Sen modules for proving the following proposition by the method of the proof of Proposition 4.5.1.

Proposition 4.5.2. *The $G(\mathbb{Q}_p)$ -equivariant map*

$$(4.5.12) \quad \mathcal{O}_y^{\mathrm{GSp}_{2g}(\mathbb{Q}_p)-\mathrm{la}}(\mathcal{Y}_{K^p}^{\mathrm{tor}}(U'))^{\mathfrak{p}_{\tilde{\mu}}} \rightarrow \mathcal{O}_{Sh}^{G(\mathbb{Q}_p)-\mathrm{la}}(\mathcal{Sh}_{K^p}^{\mathrm{tor}}(U'))^{\mathfrak{p}_{\mu}}$$

induced from the third map of the exact sequence (4.5.7) is surjective. Moreover, for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, if an element in the source is $\mathfrak{h}$ -smooth, its image in the target is $\mathfrak{h}$ -smooth.

PROOF. The proof is similar to Lemma 4.4.14 and Proposition 4.5.1. We just explain different points of the proof.

Instead of the whole space of locally analytic functions $C^{\mathrm{la}}(\widetilde{\mathrm{gsp}}_{2g}, \mathbb{Q}_p)$, we use the subspace of $\mathfrak{p}_{\tilde{\mu}}$ -smooth functions (of $\mathbb{Q}_p$ -coefficients)

$$C^{\mathrm{la}, \mathfrak{p}_{\tilde{\mu}}-\mathrm{sm}}(\widetilde{\mathrm{gsp}}_{2g}, \mathbb{Q}_p) := \lim_{K'_p \subset \mathrm{GSp}_{2g}(\mathbb{Q}_p)} C^{\mathrm{la}, P_{\tilde{\mu}} \cap \tilde{K}'_p - \mathrm{sm}}(\tilde{K}'_p, \mathbb{Q}_p) \subset C^{\mathrm{la}}(\widetilde{\mathrm{gsp}}_{2g}, \mathbb{Q}_p).$$

We let (LHS) and (RHS) be the source and the target of the map (4.5.12) respectively and set

$$\begin{aligned} \mathcal{H}' &:= H^0(\mathfrak{n}_{\tilde{\mu}}^0, \mathcal{O}_y^{\mathrm{GSp}_{2g}(\mathbb{Q}_p)-\mathrm{la}}(\mathcal{Y}_{K^p}^{\mathrm{tor}}(U')) \hat{\otimes}_{\mathbb{Q}_p} C^{\mathrm{la}, \mathfrak{p}_{\tilde{\mu}}-\mathrm{sm}}(\widetilde{\mathrm{gsp}}_{2g}, \mathbb{Q}_p)), \\ \mathcal{H} &:= H^0(\mathfrak{n}_{\mu}^0, \mathcal{O}_{Sh}^{\mathrm{GSp}_{2g}(\mathbb{Q}_p)-\mathrm{la}}(\mathcal{Sh}_{K^p}^{\mathrm{tor}}(U')) \hat{\otimes}_{\mathbb{Q}_p} C^{\mathrm{la}, \mathfrak{p}_{\mu}-\mathrm{sm}}(\widetilde{\mathrm{gsp}}_{2g}, \mathbb{Q}_p)). \end{aligned}$$

By Lemma 4.4.12, we have isomorphisms (LHS) $\cong \mathcal{H}'^{\tilde{K}'_p \times \Gamma'_p}$ and (RHS) $\cong \mathcal{H}^{\tilde{K}_p \times \Gamma_p}$. Since (LHS) and (RHS) are $\mathcal{O}_{\mathcal{F}\ell_{\tilde{\mu}}}(V')$ -module, it suffices to show for each classical point $x \in V'$, the localized map (LHS) $_x \rightarrow$ (RHS) $_x$ is surjective.

Note that localizations commute with taking (Lie algebra) cohomology. We obtain the analog of Lemma 4.4.14 for the localizations of the $\mathfrak{p}_{\tilde{\mu}}$ -(or $\mathfrak{p}_{\mu}$ -) invariant subspaces in the same way and it follows that the localized maps $\mathcal{H}'_x \rightarrow \mathcal{H}_x$ are surjective. So we can follow the argument of Proposition 4.5.1. $\square$

Corollary 4.5.3. *Suppose that $\mathfrak{n}_{\mu,x}^0 \cap \tilde{\mathfrak{p}}_{\mu,x}^0 = 0$ for each point $x \in V := i^{-1}(V')$. If an element $f \in \mathcal{O}_{Sh}^{G(\mathbb{Q}_p)-\mathrm{la}}(\mathcal{Sh}_{K^p}^{\mathrm{tor}}(U'))$ is annihilated by $\mathfrak{p}_{\mu}$, then f is annihilated by $\mathfrak{g}$, that is, f is fixed by an open subgroup of $G(\mathbb{Q}_p)$.*

PROOF. First we consider the case of Siegel modular varieties. In this case, the argument of [69, Remark 4.3.4] works.

By using the action of the element $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GSp}_{2g}(\mathbb{Q}_p)$, where each block is of size $g \times g$, we may assume that $\pi_{K'_p}^{\mathrm{tor}, -1}(U')$ is a rational open subset of some ϵ -neighborhood of the anticanonical locus $\mathcal{X}_{\Gamma(p^\infty)}^*(\epsilon)_a$ in [83, §III.2] for some $\epsilon \in (0, 1/2)$. The element f is annihilated by an open congruence subgroup Γ'_0 of the Siegel parabolic subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$ by the assumption. We may assume $\Gamma_0(p^m) \subset K'_p$ and let $U'_{\Gamma'_0}$ be the preimage of $U'_{K'_p K'^p}$ in $\mathcal{X}_{\Gamma'_0}^*(\epsilon)_a$. By [83, Corollary III.2.23], there is a Tate's normalized trace from $\mathcal{O}_{\mathcal{X}_{\Gamma'_0}^*(\epsilon)_a}(U'_{\Gamma'_0})$ to sections of finite levels. Then f is fixed by some open subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$ by the proof of [11, Théorème 3.2] or [69, Lemma 3.2.5].

In the case of general Shimura varieties of Hodge type, the claim follows from the Siegel case and Proposition 4.5.2. $\square$

Remark 4.5.4. Corollary 4.5.3 can be also obtained by geometric Sen theory of Shimura varieties. By [77, Theorem 5.2.5], the image of the geometric Sen operator is generated by $\mathfrak{n}_{\mu}^0$. Then by [76, Corollary 3.4.6], the action of $\mathfrak{n}_{\mu}^0$ on $\mathcal{O}_{Sh}^{G(\mathbb{Q}_p)-\mathrm{la}}$ vanishes. So

for each (U, V) in our settings, $\mathfrak{p}_\mu \otimes \mathcal{O}_{\mathcal{F}\ell_\mu}$ and $\mathfrak{n}_\mu^0$ generates $\mathfrak{g} \otimes \mathcal{O}_{\mathcal{F}\ell_\mu}$, and thus the corollary follows.

Proposition 4.5.5. *For each element $s \in \mathcal{O}_y^{\mathrm{GSp}_{2g}(\mathbb{Q}_p)^{-\mathrm{la}}}(\mathcal{Y}_{K^p}^{\mathrm{tor}}(U'))$, there exists a sufficiently small compact open subgroup $K'_p \subset K_p$ and $e'_1, \dots, e'_l \in \mathcal{O}_y^{\mathrm{GSp}_{2g}(\mathbb{Q}_p)^{-\mathrm{la}}}(\mathcal{Y}_{K^p}^{\mathrm{tor}}(U'))$, where l is the dimension $g^2 + \frac{g(g+1)}{2} + 1$ of the Siegel parabolic subgroup of GSp_{2g} such that*

$$s = \sum_{k_1, \dots, k_l \in \mathbb{Z}_{\geq 0}} c(k_1, \dots, k_l) \prod_{i=1}^l e_i'^{k_i}$$

with $c(k_1, \dots, k_l) \in H^0(U', \mathcal{O}_{\mathcal{Y}_{K^p}^{\mathrm{tor}}})$ and $c(k_1, \dots, k_l) \rightarrow 0$ as $\sum_{i=1}^l k_i \rightarrow \infty$.

PROOF. We may assume $\mathfrak{n}_\mu^0 \cap \tilde{\mathfrak{p}}_\mu^0 = 0$ on U' by the $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$ -action on $\mathcal{F}\ell_\mu$ and shrinking U' if necessary. We use the method of [71, §4] and [73, Theorem 3.4.5] and the notation in the proof of Proposition 4.5.1.

Let $\mathcal{G}_n$ for $n \in \mathbb{Z}_{\geq 0}$ be Lie subgroups constructed as in the beginning of Section 4.4.1. Write $\mathcal{P}_n := P_\mu \cap \mathcal{G}_n$.

By the assumption, we can see the module $H^0(\mathfrak{n}_\mu^0, \mathcal{O}(\mathbb{X}'_r))$ as a $\tilde{K}'_p(r)$ -module for each $r \gg 0$. By the construction of $\mathbb{X}'_r$, we may assume that there is a natural isomorphism

$$(4.5.13) \quad \varinjlim_n C^{\mathrm{an}}(\mathcal{P}_n, \mathbb{Q}_p) \cong \varinjlim_r \mathcal{O}(\mathbb{X}'_r).$$

Let $l := g^2 + \frac{g(g+1)}{2} + 1$. We can choose functions $e_1, \dots, e_l \in C^{\mathrm{an}}(P_\mu, \mathbb{Q}_p)$ such that $C^{\mathrm{an}}(P_\mu, \mathbb{Q}_p) = \mathbb{Q}_p \langle e_1, \dots, e_l \rangle$ and $\{e_i\}_{1 \leq i \leq l}$ generates a finite dimensional representation of P_μ as in the proof of [73, Theorem 3.4.5].

Let $\tilde{A}_\infty := \hat{\mathcal{O}}_y(\tilde{U}'_{p^\infty})$ and $A_{H',n} := \tilde{A}_\infty^{H' \times p^n \Gamma'_p}$ for each open subgroup $H'(r) \times p^n \Gamma'_p \subset \tilde{K}'_p \times \Gamma'_p$. Let $\mathcal{F}'_r := \hat{\mathcal{O}}_y(\tilde{U}'_{p^\infty}) \hat{\otimes}_{\mathcal{O}(V')} \mathcal{O}(\mathbb{X}'_r)$ and $S_{H'(r),n}(\mathcal{F}'_r)$ be the unique decompletion of $\mathcal{F}'_r$ for a sufficiently small open subgroup $H'(r) \times p^n \Gamma'_p \subset \tilde{K}'_p \times \Gamma'_p$. Note that we have

$$(4.5.14) \quad \mathcal{F}'_r = S_{H'(r),n}(\mathcal{F}'_r) \hat{\otimes}_{A_{H'(r),n}} \tilde{A}_{p^\infty}.$$

For $i \in [1, l]$, the image of e_i by the composite of the natural map $C^{\mathrm{an}}(P_\mu, \mathbb{Q}_p) \rightarrow \varinjlim_n C^{\mathrm{an}}(\mathcal{P}_n, \mathbb{Q}_p)$ and the isomorphism (4.5.13) is in the image of the natural injection $\mathcal{O}(\mathbb{X}'_r) \hookrightarrow \varinjlim_r \mathcal{O}(\mathbb{X}'_r)$ by the construction. Write $e_{i,r}$ for the corresponding element in $\mathcal{O}(\mathbb{X}'_r)$ to e_i .

Write $1 \otimes e_{i,r} = \sum_j s_{i,j,r} \otimes f_{i,j,r}$ where $s_{i,j,r} \in S_{H'(r),n}(\mathcal{F}'_r)$ and $f_{i,j,r} \in \tilde{A}_{p^\infty}$ by the equation (4.5.14). Since $\{e_{i,r}\}_{1 \leq i \leq l}$ generates a finite dimensional representation of $H'(r) \times p^n \Gamma'_p$, the summation $\sum_j s_{i,j,r} \otimes f_{i,j,r}$ is of finite term.

Choose $f'_{i,j,r} \in \varinjlim_{H',m} A_{H',m}$ which is sufficiently close to $\tilde{A}_{p^\infty}$. Then the element $e'_{i,r} := \sum_j s_{i,j,r} \otimes f'_{i,j,r}$ is sufficiently closed to $\sum_j s_{i,j,r} \otimes f_{i,j,r}$. Thus we have

$$\mathcal{F}'_r = \tilde{A}_{p^\infty} \langle e_1, \dots, e_l \rangle = \tilde{A}_{p^\infty} \langle e'_1, \dots, e'_l \rangle$$

Taking $H'(r) \times p^n \Gamma'_p$ -invariant, we have

$$(4.5.15) \quad A_{H'(r),n} \langle e'_1, \dots, e'_l \rangle \cong S_{H'(r),n}(\mathcal{F}'_r)$$

by the construction of $e'_1, \dots, e'_l$.

Since the Sen operator of $\mathcal{F}'_r$ is trivial, we have

$$\mathcal{F}'_r{}^{H', \text{Lie}(\Gamma'_p) - \text{sm}} = \varinjlim_m S_{H', m}(\mathcal{F}'_r)$$

for each open subgroup $H' \subset \tilde{K}'_p$. Taking $\tilde{K}'_p \times \Gamma'_p$ -invariant, we have an identification

$$(4.5.16) \quad \mathcal{O}_y^{\text{GSp}_{2g}(\mathbb{Q}_p) - \text{la}}(\mathcal{Y}_{K^p}^{\text{tor}}(U')) = \varinjlim_r \mathcal{F}'_r{}^{\tilde{K}'_p \times \Gamma'_p} = \varinjlim_m S_{H', m}(\mathcal{F}'_r)^{\tilde{K}'_p \times \Gamma'_p}.$$

The proposition follows from (4.5.15) and (4.5.16). $\square$

4.5.2. Locally analyticity of perfectoid Shimura varieties. In this subsection, we introduce an ad hoc definition of Faltings extensions for Shimura varieties of Hodge type and locally analyticity of their perfectoid covers. We will show that any Shimura varieties of Hodge type admits a cover by open affinoid subspaces over which the perfectoid cover is locally analytic. This is a generalization of [69, §3.5, §3.6, §4.2].

We use the notation in the previous section. Let $\tilde{B} := \hat{\mathcal{O}}_{Sh}(\tilde{U})$ and $\tilde{B}_\infty := \hat{\mathcal{O}}_{Sh}(\tilde{U}_{p^\infty})$. Fix an isomorphism $\Gamma_p \cong \mathbb{Z}_p^d$. We consider the following unipotent representation of Γ_p on $W = \mathbb{Q}_p^{d+1}$:

$$\Gamma_p \cong \mathbb{Z}_p^d \rightarrow \text{GL}_{d+1}(\mathbb{Q}_p), \quad \gamma \mapsto A_\gamma = (a_{\gamma,1}, \dots, a_{\gamma,d}) \mapsto \begin{pmatrix} 1 & A_\gamma \\ 0 & 1 \end{pmatrix}.$$

The representation defines the exact sequence

$$0 \rightarrow \mathbb{Q}_p \rightarrow W \rightarrow \mathbb{Q}_p^d \rightarrow 0.$$

Tensoring the exact sequence with $\tilde{B}_\infty$ and take Γ_p -invariants with respect to the diagonal actions, we have an exact sequence

$$(4.5.17) \quad 0 \rightarrow \tilde{B} \rightarrow (\tilde{B}_\infty \otimes_{\mathbb{Q}_p} W)^{\Gamma_p} \rightarrow \tilde{B}^d \rightarrow 0.$$

For $i \in [1, d]$, let $\gamma_i \in \Gamma_p$ be the corresponding element to the i -th fundamental vector $e_i \in \mathbb{Z}_p^d$. By the similar argument of [69, Proposition 3.5.3], we have

Proposition 4.5.6. *The following conditions are equivalent:*

- (1) *The sequence (4.5.17) remains exact after taking $\mathcal{G}_0$ -analytic vectors for some uniform pro- p subgroup $\mathcal{G}_0$ of $G(\mathbb{Q}_p)$.*
- (2) *The sequence (4.5.17) remains exact after taking $G(\mathbb{Q}_p)$ -locally analytic vectors of each term.*
- (3) *There exist $G(\mathbb{Q}_p)$ -locally analytic vectors $z_1, \dots, z_d \in \tilde{B}_\infty$ such that $\gamma_i(z_i) = z_i - 1$ and $\gamma_i(z_j) = z_j$ if $i \neq j$.*

We say that $\tilde{U}$ is a *locally analytic covering* of U if one of the conditions in Proposition 4.5.6 holds.

Proposition 4.5.7. *The topological G -module $\tilde{B}$ is $\mathfrak{LA}$ -acyclic if and only if $\tilde{U}$ is a locally analytic covering of U .*

PROOF. The “only if” part is clear. The argument of the “if” part is similar to [69, §3.6]. First we have a natural isomorphism

$$(4.5.18) \quad H_{\text{cont}}^i(\mathcal{G}_n, \tilde{B} \hat{\otimes}_{\mathbb{Q}_p} C^{\text{an}}(\mathcal{G}_n, \mathbb{Q}_p)) \cong H_{\text{cont}}^i(\Gamma_p, (\tilde{B}_\infty)^{\mathcal{G}_n - \text{an}})$$

for $i, n \geq 0$ by the almost purity theorem (cf. [69, Lemma 3.6.4]) and we have an isomorphism (cf. [69, §3.6.5])

$$R^i \mathfrak{L}\mathfrak{A}(\tilde{B}) \cong \varinjlim_{n,k} H_{\text{cont}}^i(\Gamma_p, (\tilde{B}_\infty)^{(\mathcal{G}_n \times p^k \Gamma_p) - \text{an}}).$$

We can take locally analytic vectors $z_1, \dots, z_d \in \tilde{B}_\infty$ as in Proposition 4.5.6 (3) by the assumption. Note that each z_j is in $\tilde{B}_{\infty,j}$. Let $\tilde{B}_\infty^{\text{la,fin}} := \varinjlim_{n,k} \tilde{B}_\infty^{(\mathcal{G}_n \times p^k \Gamma_p) - \text{an}}$. For each $j \in [1, d]$, consider the complex

$$C_j := (\tilde{B}_\infty^{\text{la,fin}} \xrightarrow{\gamma_j - 1} \tilde{B}_\infty^{\text{la,fin}}),$$

where the first term is of degree 0. Then each C_j has trivial H^1 by the argument of [69, §3.6.7].

The cohomology group (4.5.18) can be computed using the tensor product complex $\bigotimes_{j=1}^d C_j$, which is the Koszul complex attached to the Γ_p -action on $\tilde{B}_\infty^{\text{la,fin}}$. Then its vanishing for $i \geq 1$ follows from each $H^1(C_j) = 0$ and degenerating of Künneth spectral sequence. $\square$

Proposition 4.5.8. *The pro-Kummer étale cover $\tilde{U}$ is a locally analytic covering of U .*

PROOF. Let $f: U \rightarrow \mathbb{T}^e \times \mathbb{D}^{d-e}$ denote the fixed toric chart of U and T_i be the i -th coordinate of $\mathbb{T}^e \times \mathbb{D}^{d-e}$ for $i \in [1, d]$. By [36, Proposition 2.3.15, Corollary 2.4.2], identifying $\tilde{B}^d$ with $\tilde{B} \otimes_B \Omega_{U/\mathbb{C}_p}^1(\log)$ sending e_i to $f^*(\frac{dT_i}{T_i})$ for $i \in [1, e]$ and e_i to $f^*(d \log T_i)$ for $i \in [e+1, d]$, the exact sequence (4.5.17) is naturally isomorphic to the section on $\tilde{U}$ (which is also exact by the almost purity theorem) of the exact sequence

$$(4.5.19) \quad 0 \rightarrow \hat{\mathcal{O}}_{Sh} \rightarrow \text{gr}^1 \mathcal{O}_{\mathbb{B}_{\text{dR}, \log, Sh}}(-1) \xrightarrow{\bar{\nabla}_{\log}} \hat{\mathcal{O}}_{Sh}(-1) \otimes_{\mathcal{O}_{Sh}} \Omega_{Sh}^1(\log) \rightarrow 0,$$

where $\bar{\nabla}_{\log}$ is induced from the graded quotient of the log derivation

$$\nabla_{\log}: \mathcal{O}_{\mathbb{B}_{\text{dR}, Sh}} \rightarrow \mathcal{O}_{\mathbb{B}_{\text{dR}, Sh}} \otimes_{\mathcal{O}_{Sh}} \Omega_{Sh}^1(\log)$$

by Griffiths transversality.

Let $G\text{-QCoh}(\text{Fl}_{\mu, \mathbb{C}_p})$ be the category of quasi-coherent sheaves on $\text{Fl}_{\mu, \mathbb{C}_p}$ and $G\text{-}\mathcal{O}_{\mathcal{F}\ell_\mu}\text{-mod}$ be the category of G -equivariant $\mathcal{O}_{\mathcal{F}\ell_\mu}$ -modules. Let $\pi: G_{\mathbb{C}_p} \rightarrow \text{Fl}_{\mu, \mathbb{C}_p}$ be the natural quotient map and the functor (cf. [77, Proposition 3.1.1])

$$\mathcal{W}: \text{Rep}_{\mathbb{C}_p}(P_{\mu, \mathbb{C}_p}) \rightarrow G\text{-QCoh}(\text{Fl}_{\mu, \mathbb{C}_p}), \quad W \mapsto (\mathcal{V} \mapsto (\mathcal{O}(\pi^{-1}(\mathcal{V})) \otimes_{\mathbb{C}_p} W)^{P_{\mu, \mathbb{C}_p}}),$$

where $P_{\mu, \mathbb{C}_p}$ acts on $\mathcal{O}(\pi^{-1}(\mathcal{V}))$ as the right regular representation and $\mathcal{V} \subset \text{Fl}_{\mu, \mathbb{C}_p}$ is an open subset. The algebraic group $G_{\mathbb{Q}_p}$ acts on each $\mathcal{W}(W)$ as the left regular representation on (the translations of) $\mathcal{O}(\pi^{-1}(\mathcal{V}))$. We write $\mathcal{W}^{\text{ad}}$ for the composite of $\mathcal{W}$ with the natural functor $G\text{-QCoh}(\text{Fl}_{\mu, \mathbb{C}_p}) \rightarrow G\text{-}\mathcal{O}_{\mathcal{F}\ell_\mu}\text{-mod}$.

By [77, Theorem 5.1.4], the exact sequence (4.5.19) is naturally isomorphic to

$$(4.5.20) \quad 0 \rightarrow \hat{\mathcal{O}}_{Sh} \rightarrow \pi_{\text{HT}}^{\text{tor},*}(\mathcal{W}^{\text{ad}}(\mathcal{O}(N_\mu^c)^{\leq 1})) \rightarrow \pi_{\text{HT}}^{\text{tor},*}(\mathfrak{n}_\mu^{c,0,\vee}) \rightarrow 0,$$

where the notation $\mathcal{O}(N_\mu^c)^{\leq 1}$ is given in [77, §3.3], in particular, it is a finite dimensional algebraic representation of $P_{\mu, \mathbb{C}_p}$, and the third map is obtained from taking $(\pi_{\text{HT}}^{\text{tor},*} \circ \mathcal{W}^{\text{ad}})(-)$ on the quotient map $\mathcal{O}(N_\mu^c)^{\leq 1} \rightarrow \text{gr}_1(\mathcal{O}(N_\mu^c))$.

Recall that an open subset $V \subset \mathcal{F}\ell_\mu$ can be taken such that Condition 4.4.11 is satisfied. We may assume that V is contained in the subspace of $\mathcal{F}\ell_\mu$ which comes from

an affine open subspace $\mathcal{V} \subset \mathrm{Fl}_{\mu, \mathbb{C}_p}$. Here $\mathcal{O}(\pi^{-1}(\mathcal{V})) \otimes_{\mathbb{C}_p} \mathcal{O}(N_\mu^c)^{\leq 1}$, so $\mathcal{W}(\mathcal{O}(N_\mu^c)^{\leq 1})(\mathcal{V}) = (\mathcal{O}(\pi^{-1}(\mathcal{V})) \otimes_{\mathbb{C}_p} \mathcal{O}(N_\mu^c)^{\leq 1})^{P_{\mu, \mathbb{C}_p}}$ is an algebraic (and so analytic) representation of an open subgroup of $G(\mathbb{Q}_p)$. The module $\mathcal{W}(\mathcal{O}(N_\mu^c)^{\leq 1})(\mathcal{V})$ is also finite projective over $\mathcal{O}(\mathcal{V})$ by that the map $\pi: G_{\mathbb{C}_p} \rightarrow \mathrm{Fl}_{\mu, \mathbb{C}_p}$ is faithfully flat and descent of finite projective modules via the map π .

Then we have equalities

$$(4.5.21) \quad \begin{aligned} \pi_{\mathrm{HT},*}^{\mathrm{tor},*}(\mathcal{W}^{\mathrm{ad}}(\mathcal{O}(N_\mu^c)^{\leq 1}))(\tilde{U}) &= \mathcal{W}(\mathcal{O}(N_\mu^c)^{\leq 1})(\mathcal{V}) \hat{\otimes}_{\mathcal{O}(\mathcal{V})} \mathcal{O}(\tilde{U}) \\ &= \mathcal{W}(\mathcal{O}(N_\mu^c)^{\leq 1})(\mathcal{V}) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{O}(\tilde{U}). \end{aligned}$$

Since the functor $\mathcal{W}$ is exact (cf. [77, Proposition 3.1.1]), the map of algebraic representations

$$(4.5.22) \quad \mathcal{W}(\mathcal{O}(N_\mu^c)^{\leq 1})(\mathcal{V}) \rightarrow \mathfrak{n}_\mu^{c,0,\vee}(\mathcal{V})$$

obtained from taking $\mathcal{W}$ on the quotient map $\mathcal{O}(N_\mu^c)^{\leq 1} \rightarrow \mathrm{gr}_1(\mathcal{O}(N_\mu^c))$ is surjective. So the exact sequence gained by taking sections on $\tilde{U}$ of (4.5.20) remains exact by taking locally analytic vectors since tensoring the surjective map (4.5.22) with $\mathcal{O}(\tilde{U})^{\mathrm{la}}$ remains to be surjective. Thus $\tilde{U}$ is a locally analytic covering of U . $\square$

4.5.3. Proof of the main theorem. We prove the main theorem using the results obtained in the section. For a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$, let $\mathcal{O}_{K^p} := \pi_{\mathrm{HT},*}^{\mathrm{tor}}(\hat{\mathcal{O}}_{\mathrm{Sh}}|_{\mathrm{Sh}_{K^p}})$ and $\mathcal{I}_{K^p} := \pi_{\mathrm{HT},*}^{\mathrm{tor}}(\hat{\mathcal{I}}_{\mathrm{Sh}}|_{\mathrm{Sh}_{K^p}})$ be sheaves of topological algebras on $\mathcal{F}\ell_\mu$ in analytic topology. Let $\mathcal{O}_{K^p}^{\mathrm{la}} \subset \mathcal{O}_{K^p}$ and $\mathcal{I}_{K^p}^{\mathrm{la}} \subset \mathcal{I}_{K^p}$ be subsheaves of locally analytic sections as in Definition 4.4.15.

Theorem 4.5.9. *For any $i \geq 0$ and a neat compact open subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$, there are natural $G(\mathbb{Q}_p)$ -equivariant isomorphisms*

$$\begin{aligned} (\tilde{H}^i(K^p, \mathbb{Q}_p) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{la}} &\cong H^i(\mathcal{F}\ell_\mu, \mathcal{O}_{K^p}^{\mathrm{la}}), \\ (\tilde{H}_c^i(K^p, \mathbb{Q}_p) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{la}} &\cong H^i(\mathcal{F}\ell_\mu, \mathcal{I}_{K^p}^{\mathrm{la}}) \end{aligned}$$

PROOF. This follows from Proposition 4.5.1, Proposition 4.5.5, Proposition 4.5.7, Proposition 4.5.8, [83, Theorem IV.2.1], and the argument of the proof of [69, Theorem 4.4.6]. $\square$

4.5.4. The case of unitary Shimura curves. In the last subsection, we observe τ -locally analytic vectors in completed cohomology of unitary Shimura curves using the method which has been developed until the previous subsection.

Let (G, X) be a Shimura data which gives a unitary Shimura curve as in Section 4.2.3. Since unitary Shimura curves are compact and we want to focus on the unitary Shimura curves, write $U := i_{K_p' K^p}^{-1}(U')$, $\mathrm{Sh}_{K^p} = \mathrm{Sh}_{K^p}^{\mathrm{tor}}$, and $\tilde{U} := \mathrm{Sh}_{K^p}^{\mathrm{tor}}(U')$. We now explicitly write down the sections

$$\mathcal{O}_{\mathrm{Sh}, E}^{\tau-\mathrm{la}}(\mathrm{Sh}_{K^p}(U)) = H^0(\mathrm{Sh}_{K^p}(U), \mathcal{O}_{\mathrm{Sh}, E}^{\tau-\mathrm{la}}).$$

by the method of expansions along Borel subalgebras as in [69, §4.3]. Recall the exact sequence of vector bundles on $\mathrm{Sh} = \mathrm{Sh}_K = \mathrm{Sh}_{K_p K^p}$

$$(4.5.23) \quad 0 \rightarrow (\varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^1)^{-,1,\mathrm{rig}}_{\wp, \tau} \rightarrow (R^1 \varepsilon_* \Omega_{\mathcal{A}/(\mathrm{Sh}_K)_{\tau, E}}^\bullet)^{-,1,\mathrm{rig}}_{\wp, \tau} \rightarrow (R^1 \varepsilon_* \mathcal{O}_{\mathcal{A}})^{-,1,\mathrm{rig}}_{\wp, \tau} \rightarrow 0,$$

which is the rigidified version of the exact sequence in Proposition 4.3.4 (2). Taking the colimit along $K_p \rightarrow 1$ and the completion, we have an exact sequence of vector bundles on $\mathcal{S}h_{K^p, \text{proét}}$

$$(4.5.24) \quad 0 \rightarrow \mathcal{V}^0 \rightarrow \underline{V_{\varphi, \tau}^{-,1}} \otimes_{\widehat{\mathbb{Q}_p}} \widehat{\mathcal{O}}_{\mathcal{S}h_{K^p}} \rightarrow \mathcal{V}^1 \rightarrow 0,$$

which is naturally embedded in the exact sequence (4.5.2). Note that $V_{\varphi, \tau}^{-,1}$ is isomorphic to the standard representation of $\text{GL}_2(F_\varphi)$ (and fix such an isomorphism), and $\mathcal{V}^0$ and $\mathcal{V}^1$ are vector bundles defined over $\mathcal{S}h = \mathcal{S}h_{K^p K^p}$ and of rank 1.

Let $\{e_1, e_2\}$ denote the basis of the standard representation $V_{\varphi, \tau}^{-,1}$ of $\text{GL}_2(F_\varphi)$. Taking global sections of (4.5.24) on $\mathcal{S}h_{K^p}$, we have an exact sequence

$$(4.5.25) \quad 0 \rightarrow H^0(\mathcal{S}h_{K^p}, \mathcal{V}^0) \rightarrow H^0(\mathcal{S}h_{K^p}, \underline{V_{\varphi, \tau}^{-,1}} \otimes_{\widehat{\mathbb{Q}_p}} \widehat{\mathcal{O}}_{\mathcal{S}h_{K^p, E}}) \rightarrow H^0(\mathcal{S}h_{K^p}, \mathcal{V}^1) \rightarrow 0$$

by the almost purity theorem [37, Theorem 5.4.3]. Fix a generator $v \in H^0(\mathcal{S}h_{K^p}, \mathcal{V}^1)$ and the induced isomorphism $H^0(\mathcal{S}h_{K^p}, \mathcal{O}_{\mathcal{S}h_{K^p, E}}) \cong H^0(\mathcal{S}h_{K^p}, \mathcal{V}^1)$ by v . We equip with the induced norm in $H^0(\mathcal{S}h_{K^p}, \mathcal{V}^1)$ from the Banach algebra $H^0(\mathcal{S}h_{K^p}, \mathcal{O}_{\mathcal{S}h_{K^p, E}})$ by the above isomorphism. Recall that the flag variety $\mathcal{F}\ell_\mu$ associated with a Hodge cocharacter attached to (G, X) is given by the filtration defined by the exact sequence (4.5.25) as observed in Section 4.3.3. Fix a $\mathbb{C}_p$ -point $\infty \in \mathcal{S}h_{K^p}$ such that the point $\pi_{\text{HT}}(\infty) \subset \mathcal{F}\ell_\mu$ corresponds to the filtration $\langle e_1 \rangle \subset V_{\varphi, \tau}^{-,1}$.

We also write e_1, e_2 for the image of $e_1 \otimes 1, e_2 \otimes 1$ by the third map of (4.5.25). Since the Hodge-Tate period map $\pi_{\text{HT}}: \mathcal{S}h_{K^p} \rightarrow \mathcal{F}\ell_\mu$ is $\text{GL}_2(F_\varphi)$ -equivariant, we may assume that e_1 generates $H^0(\mathcal{S}h_{K^p}, \mathcal{V}^1)$ over $\mathcal{S}h_{K^p}(U)$ by the action of some element in $\text{GL}_2(F_\varphi)$. Let $x := \frac{e_2}{e_1} \in \widehat{\mathcal{O}}_{\mathcal{S}h}(\mathcal{S}h_{K^p}(U))$.

By the decomposition of $G_{\mathbb{Q}_p}$ (4.2.1) and the argument of connected components of Shimura varieties,

$$\widetilde{H}^0(K^p, E) \xrightarrow{\sim} C(\mathcal{O}_\varphi^\times, E)^{\oplus |I_K|} \otimes_E C^\wp$$

where I_K be the ideal class group of $\mathcal{O}_K$ and $C^\wp$ is an admissible representation of G , on which $\text{GL}_2(F_\varphi)$ acts trivially (see also [38, §4.2] for the case of the modular curve). Let $t \in H^0(\mathcal{S}h_{K^p}, \widehat{\mathcal{O}}_{\mathcal{S}h, E}) = \widetilde{H}^0(K^p, E) \widehat{\otimes}_{\widehat{\mathbb{Q}_p}} \mathbb{C}_p$ be an element given by $\omega_\tau^{\oplus |I_K|} \otimes 1 \in \widetilde{H}^0(K^p, E)$ where $\omega_\tau := \tau|_{\mathcal{O}_\varphi^\times}: \mathcal{O}_\varphi^\times \hookrightarrow E$ is the restriction of the embedding $\tau \in \Sigma_\varphi$. Note that e_1, e_2, x, t are τ -locally analytic sections by the construction.

Recall that U is taken to be in $\mathfrak{B}$ in Proposition 4.3.2. Fix a uniform pro- p group $\mathcal{G}_0 \subset K_p$ and let $\mathcal{G}_n \subset \mathcal{G}_0$ be a subgroup defined in Section 4.4.1. For each integer $n \geq 0$, let $U_{\mathcal{G}_n} \subset \mathcal{S}h_{\mathcal{G}_n K^p}$ be the preimage of U . Let $r(-1) = 0$ and we can find

- (1) An integer $r(n) > r(n-1)$.
- (2) An element $x_n \in H^0(U_{\mathcal{G}_{r(n)}}, \mathcal{O}_{\mathcal{S}h_{\mathcal{G}_{r(n)} K^p}})$ such that $\|x - x_n\|_{\mathcal{G}_{r(n)}} \leq p^{-n}$ in $H^0(\widetilde{U}, \mathcal{O}_{\mathcal{S}h_{K^p}})$.
- (3) An element $t_n \in H^0(\mathcal{S}h_{\mathcal{G}_{r(n)} K^p}, \mathcal{O}_{\mathcal{S}h_{\mathcal{G}_{r(n)} K^p, E}})$ such that $\|t - t_n\|_{\mathcal{G}_{r(n)}} \leq p^{-n}$ in $H^0(\mathcal{S}h_{K^p}, \mathcal{O}_{\mathcal{S}h_{K^p}})$.
- (4) An element $e_{1,n} \in H^0(U_{\mathcal{G}_{r(n)}}, \mathcal{V}_1)$ such that $\|e_1 - e_{1,n}\|_{\mathcal{G}_{r(n)}} \leq p^{-n}$ in $H^0(\widetilde{U}, \mathcal{V}_1)$.

for each integer $n \geq 0$ by [69, Lemma 2.1.5].

We define $\mathcal{O}_E^n(U) \{x, e_1, t\} \subset \mathcal{O}_{\mathcal{S}h, E}^{\tau-\text{la}}(\mathcal{S}h_{K^p}(U))$ to be the subset of elements f which can be written as

$$f = \sum_{i,j,k \geq 0} c_{i,j,k}^{(n)} (x - x_n)^i \left(\log \left(\frac{e_1}{e_{1,n}} \right) \right)^j \left(\log \left(\frac{t_1}{t_{1,n}} \right) \right)^k$$

such that $c_{i,j,k}^{(n)} \in H^0(U_{\mathcal{G}_{r(n)}}, \mathcal{O}_{Sh_{\mathcal{G}_{r(n)}K^p, E}})$ for integers $i, j, k \geq 0$ and $\|c_{i,j,k}\| \leq p^{(n-1)(i+j+k)} C'$ holds for some uniform constant C' . It is a Banach algebra over $\mathbb{C}_p$ with norm

$$\|f\|_n := \sup_{i,j,k \geq 0} \|c_{i,j,k}^{(n)}(f) p^{(n-1)(i+j+k)}\|.$$

Proposition 4.5.10. *For any positive integer $m > 0$, there exists a positive integer n and a natural continuous embeddings of Banach spaces*

$$\mathcal{O}_{Sh, E}^{\tau-\text{la}}(\mathcal{S}h_{K^p}(U))^{\mathcal{G}_m-\text{an}} \subset \mathcal{O}_E^n(U)\{x, e_1, t\} \subset \mathcal{O}_{Sh, E}^{\tau-\text{la}}(\mathcal{S}h_{K^p}(U))^{\mathcal{G}_{r(n)}-\text{an}}.$$

In particular, there is a natural isomorphism of topological vector spaces

$$\varinjlim_n \mathcal{O}_{Sh, E}^{\tau-\text{la}}(\mathcal{S}h_{K^p}(U))^{\mathcal{G}_n-\text{an}} \cong \varinjlim_n \mathcal{O}_E^n(U)\{x, e_1, t\}.$$

PROOF. This follows from Corollary 4.5.3 and the argument of [69, §4.3]. $\square$

Let $K^\varphi := K^p K_p^\varphi$ for some compact open subgroup $K_p^\varphi \subset G(\mathbb{Q}_p^\varphi)$. For an object W in $\text{Rep}_E(G^c)$, let $\mathcal{O}_{K^p, W} := \pi_{\text{HT},*} \mathcal{O}_{Sh_{K^p, W}}$. In the case of unitary Shimura curves, there exists perfectoid unitary Shimura curves of level K^φ .

Theorem 4.5.11 ([50, Theorem 3.3.3] (and also e.g. [73, Theorem 3.2.1])). *If K^φ is sufficiently small, there exists a unique perfectoid space $\mathcal{S}h_{K^\varphi}$ over $\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ such that*

$$\mathcal{S}h_{K^\varphi} \sim \varprojlim_{K_\varphi} \mathcal{S}h_{K^\varphi K_\varphi},$$

where K_φ runs over all compact open subgroup of $G(\mathbb{Q}_\varphi)$. The space $\mathcal{S}h_{K^\varphi}$ is a pro-étale K_φ -torsor over $\mathcal{S}h_{K^\varphi K_\varphi}$. Moreover, there exists a Hodge-Tate period map

$$\pi_{\text{HT}, \varphi}: \mathcal{S}h_{K^\varphi} \rightarrow \mathcal{F}\ell_\mu$$

which is compatible with π_{HT} and satisfies the similar properties in Proposition 4.3.2.

We define $\mathcal{O}_{K^\varphi, W} := \pi_{\text{HT},*} \mathcal{O}_{Sh_{K^\varphi, W}}$. Note that the analog of Proposition 4.5.10 in the case of K^φ -level perfectoid unitary Shimura curves can be verified in the same way. For a subset $J \subset \{z\} \cup \bigsqcup_{\varphi_i | p} \Sigma_{\varphi_i}$, we can define J -locally analytic subsheaves $\mathcal{O}_{K^\varphi, W}^{J-\text{la}} \subset \mathcal{O}_{K^\varphi, W}$ and $\mathcal{O}_{K^\varphi, W}^{J-\text{la}} \subset \mathcal{O}_{K^\varphi, W}$ as in Definition 4.4.15.

Theorem 4.5.12. *There is natural $G(\mathbb{Q}_p)$ -equivariant isomorphisms*

$$(\tilde{H}^1(K^\varphi, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\tau-\text{la}} \cong H^1(\mathcal{F}\ell_\mu, \mathcal{O}_{K^p, E}^{\tau-\text{la}})^{K_p^\varphi} \cong H^1(\mathcal{F}\ell_\mu, \mathcal{O}_{K^\varphi, E}^{\tau-\text{la}}).$$

PROOF. Initially, we show the first isomorphism. We claim that there is a natural $G(\mathbb{Q}_p)$ -equivariant isomorphism

$$H_{\text{an}}^i(\mathcal{S}h_{K^p}, \mathcal{O}_{Sh, E}^{\tau-\text{la}}) \cong H^i(\mathcal{F}\ell_\mu, \mathcal{O}_{K^p, E}^{\tau-\text{la}})$$

for $i \geq 0$. It follows that there exists an affinoid open cover $\{U_i\}_i$ of $\mathcal{S}h$ such that each $\pi_{K^p}^{-1}(U_i)$ is a locally analytic covering of U_i in the sense of [69, Definition 3.5.4] by the argument of [69, Corollary 4.2.3] and the Kodaira-Spencer isomorphisms for unitary Shimura curves (cf. [34, §4], [35, §3.3.1.3]). Then the claim follows by Proposition 4.5.10 the first part of the proof of [69, Theorem 4.4.6].

By Proposition 4.4.19, we have a natural $G(\mathbb{Q}_p)$ -equivariant isomorphism

$$(\tilde{H}^1(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\tau-\text{la}} \cong H_{\text{an}}^1(\mathcal{S}h_{K^p}, \mathcal{O}_{Sh, E}^{\tau-\text{la}}).$$

Then it suffices to construct natural $G(\mathbb{Q}_p)$ -equivariant isomorphisms

$$(\tilde{H}^1(K^p, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{K_p^\varphi, \tau-\text{la}} \cong (\tilde{H}^1(K^\varphi, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\tau-\text{la}}.$$

The first isomorphism follows from Corollary 4.4.22 and that

$$\varinjlim_{K_p^\varphi} (\tilde{H}^1(K^p K_p^\varphi, E) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{\tau-\text{la}}$$

is a smooth $\text{GL}_2(F_\varphi)$ -representation by the degeneration of spectral sequence.

Next we show the second isomorphism. Take an open cover $\bigcup_i V_i = \mathcal{F}\ell_\mu$ in analytic topology, where Then each section $\mathcal{O}_{K^p, E}^{\tau-\text{la}}(V_i)$ is a smooth K_p^φ -representation by Proposition 4.5.10. So there is a spectral sequence

$$E_2^{p,q} = H^p(K_p^\varphi, H^q(\mathcal{F}\ell_\mu, \mathcal{O}_{K^p, E}^{\tau-\text{la}})) \implies H^{p+q}(\mathcal{F}\ell_\mu, \mathcal{O}_{K^\varphi, E}^{\tau-\text{la}})$$

degenerating at $E_2^{0,1}$ -term and the second isomorphism follows from this. $\square$

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