

## Numerical semigroups of toric type of higher dimension

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Let  $k$  be an algebraically closed field of characteristic 0 and  $n$  an integer at least 2. We set  $T = \mathbf{G}_m^n$  where  $\mathbf{G}_m = \text{Spec } k[X, X^{-1}]$  is the multiplicative group. Moreover, we denote by  $M$  (resp.  $N$ ) the group  $\text{Hom}_{\text{Alg.Groups}}(T, \mathbf{G}_m)$  of characters of  $T$  (resp. the group  $\text{Hom}_{\text{Alg.Groups}}(\mathbf{G}_m, T)$  of 1-parameter subgroups of  $T$ ). Then we have a non-singular canonical pairing  $\langle \cdot, \cdot \rangle: M \times N \rightarrow \mathbf{Z}$  where  $\mathbf{Z}$  is the ring of integers. We set  $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$  and  $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$  where  $\mathbf{R}$  is the set of real numbers. Let  $\sigma$  be a strongly convex rational polyhedral cone in  $N_{\mathbf{R}}$ , i.e., there exist a finite number of vectors  $x_i \in N_{\mathbf{R}}$  defined over the ring  $\mathbf{Q}$  of rational numbers such that

$$\sigma = \left\{ \sum_{i=1}^{N'} \lambda_i x_i \mid \lambda_i \geq 0, \text{ all } i \right\} = \sum_{i=1}^{N'} \mathbf{R}_+ x_i$$

and it contains no line through the origin where  $\mathbf{R}_+$  is the set of non-negative real numbers. We set

$$\check{\sigma} = \{r \in M_{\mathbf{R}} \mid \langle r, a \rangle \geq 0, \text{ all } a \in \sigma\}.$$

Then  $\check{\sigma} \cap M$  becomes a subsemigroup of  $M$ . An  $n$ -dimensional affine toric variety is expressed as  $\text{Spec } k[\check{\sigma} \cap M]$ . Let  $\mathbf{M}(\check{\sigma} \cap M)$  be the minimal set of generators for the semigroup  $\check{\sigma} \cap M$ . Then we can embed the affine toric variety  $X_{\sigma} = \text{Spec } k[\check{\sigma} \cap M]$  into the affine  $m$ -space  $\mathbf{A}^m = \text{Spec } k[Y_1, \dots, Y_m]$  using the  $k$ -algebra homomorphism  $k[Y_1, \dots, Y_m] \rightarrow k[\check{\sigma} \cap M]$  which sends  $Y_i$  to  $\mathcal{T}^{b_i}$  where we set  $\mathbf{M}(\check{\sigma} \cap M) = \{b_1, \dots, b_m\}$ .

Let  $H$  be a *numerical semigroup*, i.e., a subsemigroup of the additive semigroup  $\mathbf{N}$  of non-negative integers such that its complement in  $\mathbf{N}$  is finite. We denote by  $g(H)$  the cardinality of  $\mathbf{N} \setminus H$ , which is called the *genus* of  $H$ . We set

$$c(H) = \text{Min}\{c \in \mathbf{N} \mid c + \mathbf{N} \subseteq H\},$$

which is called the *conductor* of  $H$ . Then we get  $c(H) \leq 2g(H)$ . Let  $\mathbf{M}(H)$  be the minimal set of generators for  $H$ . If  $\mathbf{M}(H) = \{a_1, a_2, \dots, a_l\}$ , then we set

$$\alpha_i = \text{Min}\{\alpha \mid \alpha a_i \in \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_l \rangle\}$$

for  $i = 1, \dots, l$ , where for any positive integers  $b_1, \dots, b_l$  we denote by  $\langle b_1, \dots, b_l \rangle$  the subsemigroup of  $\mathbf{N}$  generated by  $b_1, \dots, b_l$ . Moreover,  $H$  is called a *Weierstrass semigroup* if there exist a complete non-singular irreducible algebraic curve  $C$  over  $k$  and its point  $P$  such that

$$H = \{\nu \in \mathbf{N} \mid \text{there is a rational function } f \text{ on } C \text{ such that } (f)_{\infty} = \nu P\}.$$

Let  $\lambda \in \sigma \cap N$  such that  $\langle r, \lambda \rangle > 0$  for any non-zero  $r \in \check{\sigma} \cap M$ . Take a numerical semigroup  $H$  containing the semigroup  $\langle \check{\sigma} \cap M, \lambda \rangle$ . We can define the morphism  $\mathbf{A}^l \rightarrow \mathbf{A}^m$  by the  $k$ -algebra homomorphism

$$k[Y_1, \dots, Y_m] \rightarrow k[X_1, \dots, X_l]$$

which sends  $Y_i$  to  $X^{<b_i, \lambda>} = X_1^{\nu_1} \dots X_l^{\nu_l}$  where  $\mathbf{M}(H) = \{a_1, \dots, a_l\}$  and  $\langle b_i, \lambda \rangle = \nu_1 a_1 + \dots + \nu_l a_l$  for some non-negative integers  $\nu_i$ 's. The above morphism  $\mathbf{A}^l \rightarrow \mathbf{A}^m$  is said to be *induced by*  $\lambda$ . A numerical semigroup  $H$  is *constructed from*  $X_\sigma$  and  $\lambda$  if  $\#\mathbf{M}(H) = \#\mathbf{M}(\check{\sigma} \cap M) - n + 1$  and  $\text{Spec } k[H]$  is isomorphic to the fiber product

$$\mathbf{A}^l \times_{\mathbf{A}^{l+n-1}} \text{Spec } k[\check{\sigma} \cap M]$$

where  $l = \#\mathbf{M}(H)$ ,  $\text{Spec } k[\check{\sigma}_{a,b} \cap M] \rightarrow \mathbf{A}^{l+n-1}$  is the embedding using  $\mathbf{M}(\check{\sigma} \cap M)$ , and  $\mathbf{A}^l \rightarrow \mathbf{A}^{l+n-1}$  is the morphism induced by  $\lambda$ . In this case we also call  $H$  a *numerical semigroup of ( $n$ -dimensional) toric type*. Then we can show that  $H$  is Weierstrass (see Komeda [2]). Here we pose the following problem :

**Problem 1.** Let  $X_\sigma$  be an affine toric variety. Give a numerical semigroup  $H$  which is constructed from  $X_\sigma$  and some  $\lambda \in \sigma \cap N$ .

In the case where  $X_\sigma$  is 2-dimensional we get the following :

**Fact 2.** Let  $X_\sigma$  be a 2-dimensional affine toric variety. Then  $\sigma$  is expressed as  $\sigma = \mathbf{R}_+(1, 0) + \mathbf{R}_+(a, b)$  where  $a$  and  $b$  are integers with  $b > 0$  and  $(a, b) = 1$ . If  $b = 1$ , then we may assume that  $a = 0$ . If  $b > 1$ , then we may assume that  $0 < a < b$ . The above cone  $\sigma$  is denoted by  $\sigma_{a,b}$ . If  $a \leq 9$ , we can give a numerical semigroup  $H_{a,b}$  which is constructed from  $X_{\sigma_{a,b}}$  and  $\lambda = (a^2, (a-1)b)$  (see Komeda [3]).

We would like to consider Problem 1 in a higher dimensional case. This paper is aimed at the following :

**Aim 3.** For any  $n \geq 3$  we give a numerical semigroup  $H$  of  $n$ -dimensional toric type. Namely, we find an  $n$ -dimensional affine toric variety  $X_\sigma$  such that there exists a numerical semigroup  $H$  which is constructed from  $X_\sigma$  and some  $\lambda \in \sigma \cap N$ .

**Example 4.** Consider the 4-dimensional cone

$$\sigma = \mathbf{R}_+(1, 0, 0, 0) + \mathbf{R}_+(0, 0, 0, 1) + \mathbf{R}_+(1, 0, 1, 0) + \mathbf{R}_+(0, 1, 1, 0) + \mathbf{R}_+(0, 1, 0, 1).$$

Let  $X_\sigma = \text{Spec } k[\check{\sigma} \cap M]$  be the 4-dimensional affine toric variety associated to  $\sigma$ . We note that

$$\check{\sigma} \cap M = \langle (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, -1, 0), (0, -1, 1, 1) \rangle.$$

Let  $H$  be a numerical semigroup with  $\mathbf{M}(H) = \{a_1, a_2, a_3\}$ . Assume that  $c(H) < 2g(H)$ , i.e.,  $H$  is *non-symmetric*. Then we have

$$\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3, \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3, \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2,$$

where  $0 < \alpha_{ij} < \alpha_j$ ,  $\alpha_1 = \alpha_{21} + \alpha_{31}$ ,  $\alpha_2 = \alpha_{12} + \alpha_{32}$  and  $\alpha_3 = \alpha_{13} + \alpha_{23}$  (see Herzog [1]). Take  $\lambda = (\alpha_{31}a_1, \alpha_{21}a_1, \alpha_{12}a_2, \alpha_{32}a_2) \in \sigma \cap N$ . Then we can show that the numerical semigroup  $H$  is of 4-dimensional toric type which is constructed from  $X_\sigma$  and  $\lambda$ .

We can generalize the above cone to an  $n$ -dimensional cone  $\sigma$  such that there exists a numerical semigroup which is constructed from  $X_\sigma$  and some  $\lambda \in \sigma \cap N$ .

**Proposition 5.** *Let  $n \geq 4$ . For any  $i$  with  $1 \leq i \leq n$  let  $e_i$  be the vector in  $\mathbf{R}^n$  whose  $j$ -th component is  $\delta_{ij}$  where  $\delta_{ij}$  is Kronecker symbol. We set*

$$\sigma = \mathbf{R}_+ e_1 + \sum_{i=4}^n \mathbf{R}_+ e_i + \mathbf{R}_+(1, 0, 1, 0, \dots, 0) + \mathbf{R}_+(0, 1, 1, 0, \dots, 0) + \mathbf{R}_+(0, 1, 0, 1, \dots, 1).$$

*Consider the  $n$ -dimensional affine toric variety  $X_\sigma = \text{Spec } k[\check{\sigma} \cap M]$ . We note that*

$$\check{\sigma} \cap M = \langle e_i \ (1 \leq i \leq n), (1, 1, -1, 0, \dots, 0), e_{-2,3,j} \ (4 \leq j \leq n) \rangle$$

*where  $e_{-2,3,j}$  is the vector in  $\mathbf{R}^n$  whose second component is  $-1$ , third and  $j$ -th components are 1, and the other components are 0. Let  $H_n$  be a numerical semigroup with*

$$M(H_n) = \{a_1 = n, a_2 = n + 1, a_3 = 2n + 3, a_4 = 2n + 4, \dots, a_{n-1} = 2n + n - 1\}.$$

*Then we have relations*

$$\alpha_1 a_1 = 4a_1 = a_2 + a_{n-1}, \alpha_2 a_2 = 3a_2 = a_1 + a_3, \alpha_3 a_3 = 2a_3 = 2a_2 + a_4,$$

$$\alpha_i a_i = 2a_i = a_{i-1} + a_{i+1} \ (4 \leq i \leq n-2), \alpha_{n-1} a_{n-1} = 2a_{n-1} = 3a_1 + a_{n-2}.$$

*Take  $\lambda = (3a_1, a_1, a_2, 2a_2, a_3, a_4, \dots, a_{n-3}, a_{n-2})$ . Then  $\lambda \in \sigma \cap N$ . We can show that the numerical semigroup  $H_n$  is of  $n$ -dimensional toric type which is constructed from  $X_\sigma$  and  $\lambda$ .*

A desired 3-dimensional affine toric variety is given by the following :

**Example 6.** Let  $\sigma_{1,1,2} = \mathbf{R}_+(1, 0, 0) + \mathbf{R}_+(0, 1, 0) + \mathbf{R}_+(1, 1, 2)$ . Consider the 3-dimensional affine toric variety  $X_\sigma = \text{Spec } k[\check{\sigma} \cap M]$ . We note that

$$\check{\sigma} \cap M = \langle (1, 0, 0), (0, 1, 0), (0, 0, 1), (2, 0, -1), (1, 1, -1), (0, 2, -1) \rangle.$$

For any  $m \in \mathbf{N}$  with  $m \geq 1$ , let  $H$  be a numerical semigroup with

$$M(H) = \{a_1 = 4, a_2 = 4m + 1, a_3 = 4m + 3, a_4 = 4m + 2\}.$$

Then we have relations

$$\alpha_1 a_1 = (2m + 1)a_1 = a_2 + a_3, \alpha_2 a_2 = 2a_2 = ma_1 + a_4$$

$$\alpha_3 a_3 = 2a_3 = (m + 1)a_1 + a_4, \alpha_4 a_4 = 2a_4 = a_2 + a_3.$$

Take  $\lambda = (4m + 1, 4m + 3, 4m + 2)$ . Then  $\lambda \in \sigma \cap N$ . We can show that the numerical semigroup  $H$  is of 3-dimensional toric type which is constructed from  $X_{\sigma_{1,1,2}}$  and  $\lambda$ .

## References

- [1] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*. Manuscripta Math. **3** (1970), 175-193.
- [2] J. Komeda, *On Weierstrass points whose first non-gaps are four*. J. Reine Angew. Math. **341** (1983), 68-86.
- [3] J. Komeda, *Numerical semigroups of 2-dimensional toric type*. In preparation.