

Wild ramification and restrictions to curves (research announcement)

By

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Abstract

This is an announcement of the original paper by the author (Internat. J. Math. 29 (2018), no. 8, 1850052), devoted to studying whether wild ramification of an étale sheaf is determined by the restrictions to all curves. In that paper, we proved that this is true for an étale sheaf on a surface.

§ 1. Introduction

Let S be an excellent noetherian scheme, which will play the role of a base scheme. Let U be an S -scheme separated of finite type which is normal and connected. Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be a locally constant constructible sheaf of Λ -modules (resp. Λ' -modules) on $U_{\text{ét}}$. Here, Λ and Λ' are finite fields of characteristics invertible on S . We do not impose the characteristics are the same. For $\mathcal{F}$ and $\mathcal{F}'$, we can consider whether they have the *same wild ramification* over S . We will make the precise definition later.

In the case where S is the spectrum of an algebraically closed field and where the characteristics of Λ and Λ' are the same, the notion “same wild ramification” was introduced by Deligne-Illusie [1]. We now say that $\mathcal{F}$ and $\mathcal{F}'$ have the same wild ramification if the associated representations of the wild inertia group at each point at infinity have the same Brauer traces. They proved that $\mathcal{F}$ and $\mathcal{F}'$ have the same Euler characteristics if they have the same wild ramification [1, Théorème 2.1].

As another interesting choice of the base S , we have the case where S is a henselian trait, i.e., the spectrum of a henselian discrete valuation ring. In this case, again assuming that the characteristics of Λ and Λ' are the same, Vidal formulated the notion

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“same wild ramification” over S , by considering compactification over S [5]. She proved that, if $\mathcal{F}$ and $\mathcal{F}'$ have the same wild ramification over S , the alternating sums of the Swan conductors of the cohomology groups are the same, i.e., we have the equality $\sum_q (-1)^q \text{Sw } H^q(U_{\bar{\eta}}, \mathcal{F}) = \sum_q (-1)^q \text{Sw } H^q(U_{\bar{\eta}}, \mathcal{F}')$, where $\bar{\eta}$ is a geometric generic point of S [5, Corollaire 3.4].

In the case where S is the spectrum of an algebraically closed field but the characteristics of Λ and Λ' are not necessarily the same, the notion “same wild ramification” was formulated by Saito-Yatagawa in a weaker form [4]. They used instead of the Brauer trace the dimensions of the fixed parts of each element. “Having the same wild ramification” in their sense also implies having the same Euler characteristics. Further they proved that it implies having the same characteristic cycles [4, Theorem 0.1].

Inspired by Beilinson’s suggestion that the characteristic cycle be determined by wild ramification of the restrictions to all curves, we consider whether wild ramification of a sheaf is determined by that of the restrictions to all curves:

Conjecture 1.1 ([2, Conjecture 0.1]). *The followings are equivalent.*

- (i) $\mathcal{F}$ and $\mathcal{F}'$ have the same wild ramification over S .
- (ii) $\mathcal{F}$ and $\mathcal{F}'$ have universally the same conductors over S .

Here, as we will make precise later, having *universally the same conductors* means wild ramification of the restrictions to all curves are the same. This definition is due to Beilinson. The implication (i) $\Rightarrow$ (ii) is straightforward.

§ 2. Same wild ramification and same conductors

We introduce the notion “same wild ramification” over a general base. To compare wild ramification of two sheaves we use the dimensions of the fixed parts of each element as Saito-Yatagawa did.

Since we want to consider the action of wild inertia, we regard $\mathcal{F}$ and $\mathcal{F}'$ as representations of $\pi_1(U)$. (In this article, we omit the base points.) Let M and M' be the corresponding representations. Since we work with torsion coefficients, these representations factor through a finite quotient: there exists a finite étale Galois covering V of U such that the action of $\pi_1(U)$ on M and M' factor through the Galois group G of the Galois covering. Then, we consider the wild inertia subgroup at each point at infinity. To do this, we take a normal compactification X of U over S and a point $x \in X$. An inertia group and a wild inertia group at x are defined up to conjugacy as follows: Let Y be the normalization of X in V . Take a point $y \in Y$ lying above x . We define an inertia subgroup I_x at x to be the subgroup of G consisting of elements g which fixes a

geometric point $\bar{y}$ over y : i.e. $I_x = \{g \in G | g\bar{y} = \bar{y}\}$. A wild inertia subgroup P_x at x is defined to be a p_x -Sylow subgroup of I_x , where p_x is the residual characteristic at x .

Definition 2.1 ([2, Definition 2.2]). We say that $\mathcal{F}$ and $\mathcal{F}'$ have the *same wild ramification* over S if there exists a normal compactification X of U over S such that for every point $x \in X$ and for every element $\sigma \in P_x$, we have

$$\dim_{\Lambda} M^{\sigma} = \dim_{\Lambda'} (M')^{\sigma}.$$

Remark.

1. Having the same wild ramification is preserved by pullback.
2. In the case where S is a henselian trait, Vidal defined the wildly ramified part $E_{U/S} \subset \pi_1(U)$. Roughly speaking $E_{U/S}$ is a subset such that $\mathcal{F}$ and $\mathcal{F}'$ have the same wild ramification if and only if for every $\sigma \in E_{U/S}$ we have $\dim_{\Lambda} M^{\sigma} = \dim_{\Lambda'} (M')^{\sigma}$. The wildly ramified part can be defined even in our general setting.
3. There is an interesting choice of the base S which is neither the spectrum of a field nor that of a discrete valuation ring. For example, let X be a normal variety over an algebraically closed field k containing U as a dense open subscheme. Then we can take $S = X$ as a base. If X is proper over k , then having the same wild ramification over X is equivalent to doing so over k . In the case X is not proper, having the same wild ramification over X means wild inertia actions along a compactification which is proper over X are the same in the sense of Definition 2.1.

We introduce the notion “having universally the same conductors” due to Beilinson: An S -curve means an S -scheme separated of finite type which has a compactification over S which is equidimensional and of dimension one. We note that unless S is of dimension zero, an S -curve is NOT a relative curve over S .

Definition 2.2 ([2, Definition 2.5]). We say that $\mathcal{F}$ and $\mathcal{F}'$ have *universally the same conductors* over S if for every morphism $g : C \rightarrow U$ from a regular S -curve and for every closed point v of a regular compactification $\overline{C}$ of C over S , we have an equality $a_v(j!g^*\mathcal{F}) = a_v(j!g^*\mathcal{F}')$, where j is the open immersion $C \rightarrow \overline{C}$ and a_v denotes the Artin conductor at v .

The following lemma is elementary.

Lemma 2.3. $\mathcal{F}$ and $\mathcal{F}'$ have universally the same conductors over S if and only if for every morphism $g : C \rightarrow U$ from a regular S -curve, $g^*\mathcal{F}$ and $g^*\mathcal{F}'$ have the same wild ramification over S .

§ 3. Main results

Theorem 3.1 ([2, Theorem 3.2]). *Assume that the dimension of a compactification of U over S is ≤ 2 (or assume resolution of singularities). Then, Conjecture 1.1 holds.*

Theorem 3.2 ([2, Theorem 0.2, Theorem 0.3]).

1. *When S is the spectrum of an algebraically closed field k , if $\mathcal{F}$ and $\mathcal{F}'$ have universally the same conductors over k , then we have $\chi(U, \mathcal{F}) = \chi(U, \mathcal{F}')$.*
2. *When S is the spectrum of a henselian discrete valuation ring, if $\mathcal{F}$ and $\mathcal{F}'$ have universally the same conductors over S , then we have $\sum_q (-1)^q \mathrm{Sw} H^q(U_{\bar{\eta}}, \mathcal{F}) = \sum_q (-1)^q \mathrm{Sw} H^q(U_{\bar{\eta}}, \mathcal{F}')$, where $\bar{\eta}$ is a generic geometric point of S .*

Remark. In Theorem 3.2, the assumption on dimension or that on resolution of singularities is not needed. The proof of Theorem 3.2 is reduced to the surface case by considering a flat morphism to an S -surface (i.e. a dense open subscheme of a proper S -scheme of dimension two).

§ 4. Outline of the proof of Theorem 3.1

All scheme we consider are assumed to be noetherian and excellent. By virtue of Lemma 2.3, we want to find a curve preserving the inertia subgroup. If we could prove the following conjecture, we would finish the proof, but the author does not know whether it is true or not.

Conjecture 4.1. *Let X be a connected normal scheme containing U as a dense open subscheme. Let $V \rightarrow U$ be a finite étale Galois covering. Then, there exists a blowup $X' \rightarrow X$ with center outside U such that, for every point $x \in X'$, there exists a finite morphism $\bar{C} \rightarrow X'$ from a regular scheme of dimension one and a point $v \in \bar{C}$ which goes to x , such that $I_v = I_x$.*

What we want to know is the action of each element of a wild inertia subgroup. This allows us to reduce the proof to the case where the Galois covering $V \rightarrow U$ trivializing $\mathcal{F}$ is cyclic of p -power order for some prime number p (see [2, Lemma 3.9] for the detail). Thus, it suffices to show the following:

Proposition 4.2 ([2, Lemma 3.5]). *For regular X and cyclic $V \rightarrow U$ of p -power order, Conjecture 4.1 holds (without blowup).*

To prove this proposition, we use Kerz-Schmidt's lemma [3], which states that for normal X and $V \rightarrow U$ which is of order p and ramifies at a point on X of codimension one, there exists a curve which does not kill ramification (again we do not need blowup). We apply this lemma to the covering corresponding to the subgroup of I_x of index p . By the regularity assumption, we can use Zariski-Nagata's purity theorem to assure that this covering is ramified at a codimension one point over x . Then, a curve which does not kill ramification of this covering preserves the inertia subgroup I_x .

Finally, we note that we need resolution of singularity to use the above proposition. The author does not know if we can avoid the use of resolution of singularity.

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