

Remarks on Scattering on Scattering Manifolds

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Abstract

In this talk, we discuss scattering theory for a class of manifolds. We consider the asymptotic completeness and the microlocal properties of the scattering matrix. The space we consider is called *scattering manifolds* following R. Melrose, and we construct a time-dependent scattering theory for Schrödinger operators on such manifolds. In particular, we discuss an alternative approach to a theorem by R. Melrose and M. Zworski on the microlocal properties of the *absolute scattering matrix*. This work is partly in progress, and several theorems are preliminary.

Model: We consider an n -dimensional noncompact manifold (without boundary):

$$M = M_0 \cup M_\infty$$

where M_0 is relative compact, and M_∞ is diffeomorphic to $(1, \infty) \times \partial M$, where ∂M is a closed manifold without boundary. We consider ∂M as a boundary of M at infinity. We fix an identification map:

$$I : M_\infty \cong (1, \infty) \times \partial M \ni (r, \theta), \quad r \in (1, \infty), \theta \in \partial M.$$

Let g^∂ be a Riemannian metric on ∂M , and we denote

$$g^\partial = \sum_{i,j} g_{ij}^\partial(\theta) d\theta^i d\theta^j, \quad \theta \in \partial M.$$

Definition: A Riemannian metric g^{cn} on M is called *conic* if it has the following form:

$$g^{cn} = dr^2 + r^2 g^\partial \quad \text{on } M_\infty,$$

where we identify M_∞ with $(1, \infty) \times \partial M$ as above.

Example 0: (Euclidean space) $M = \mathbb{R}^n$, $\partial M = S^{n-1}$, $g^\partial = d\theta^2$ is the surface metric on S^{n-1} . Then $g^{cn} = dr^2 + r^2 d\theta^2$ is the standard flat metric on $\mathbb{R}^n$ in the polar coordinate on $M_\infty = \{x \mid |x| > 1\}$. The identification map is

$$I : r\theta \in M_\infty \mapsto (r, \theta) \in (1, \infty) \times S^{n-1}.$$

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This is a typical example and we should keep this in mind in the following argument.

Example 1: (Conic metric on $\mathbb{R}^n$) Let M and ∂M as in Example 0, but we introduce a different metric on S^{n-1} . Then we have a different conic metric structure on $\mathbb{R}^n$. For example, we can set $g^\partial = \alpha d\theta^2$ with $\alpha > 0$, and we have a different geometric structure.

Definition: A Riemannian metric g on M is called *scattering metric* if

$$g = g^{cn} + m,$$

where g^{cn} is the conic metric, and m is a symmetric 2-form such that

$$m = m^0(r, \theta) dr^2 + r \sum_{j=1}^{n-1} m_j^1(r, \theta) (dr d\theta + d\theta dr) + r^2 \sum_{i,j=1}^{n-1} m_{ij}^2(r, \theta) d\theta^i d\theta^j$$

on M_∞ , and the coefficients satisfy

$$|\partial_r^k \partial_\theta^\alpha m_*^\ell(r, \theta)| \leq C_{k\alpha} r^{-\mu_\ell - k}, \quad (r, \theta) \in M_\infty$$

for any $k, \alpha, \ell = 0, 1, 2$ with $\mu_\ell > 0$.

Scattering metric was defined originally by R. Melrose [2], but here we use an equivalent, but different definition. (This formulation was introduced in [1]). We will assume the metric perturbation m is short-range type in the following sense:

Definition: A metric g on M is called *short-range type* if

$$\mu_0 > 1, \quad \mu_1 > 1/2, \quad \mu_2 > 0.$$

Let Δ_g be the Laplace-Beltrami operator on M corresponding to the Riemannian metric, i.e.,

$$\Delta_g = \frac{1}{\sqrt{G(x)}} \sum_{j,k=1}^n \partial_{x_j} g^{jk}(x) \sqrt{G(x)} \partial_{x_k}$$

where $G(x) = \det(g_{jk}(x))$ and $(g^{jk}) = (g_{jk})^{-1}$.

Definition: A potential function $V \in C^\infty(M; \mathbb{R})$ is called *short-range type* if there is $\mu_3 > 1$ such that for any α and k ,

$$|\partial_r^k \partial_\theta^\alpha V(r, \theta)| \leq C_{k\alpha} r^{-\mu_3 - k}, \quad (r, \theta) \in M_\infty.$$

In the following, we assume g and V are short-range type. We set

$$H = -\Delta_g + V(x) \quad \text{on } \mathcal{H} = L^2(M, \sqrt{G} dx).$$

Proposition 1. H is essentially self-adjoint on $C_0^\infty(M)$. Moreover, $\sigma_{ess}(H) = [0, \infty)$; $\sigma_p(H)$ is discrete with possible accumulation points only at 0; $\sigma_{sc}(H) = \emptyset$; and $\sigma_{ac}(H) = [0, \infty)$.

Idea of Proof. Let $j(r) \in C^\infty(\mathbb{R})$ be a smooth cut-off function such that

$$j(r) = \begin{cases} 1 & (r \geq 3/2), \\ 0 & (r \leq 1). \end{cases}$$

We use the Mourre theory with the conjugate operator:

$$A = \frac{1}{2i} \left(j(r)r \frac{\partial}{\partial r} + \frac{\partial}{\partial r} j(r)r + \frac{1}{2} j(r)r \partial_r (\log G(x)) \right)$$

on M^∞ . Then the rest of the argument is similar to the Euclidean case. $\square$

Scattering Theory: We first construct a *free system*. We might use $-\Delta_{g^{cn}}$ as the free system, but this operator itself is not very easy to handle. So, instead, we set

$$H_{fr} = -\frac{\partial^2}{\partial r^2} \quad \text{on} \quad M_{fr} = \mathbb{R} \times \partial M,$$

$$\mathcal{H}_{fr} = L^2(M_{fr}, dr \cdot \sqrt{g^\partial} d\theta)$$

$$J : \mathcal{H}_{fr} \rightarrow \mathcal{H}, \text{ where } J\varphi(r, \theta) = \begin{cases} 0 & \text{on } M_\infty^c \\ j(r) (\det g^\partial(\theta)/G(r, \theta))^{1/4} \varphi(r, \theta) & \text{on } M_\infty. \end{cases}$$

J is defined so that J is isometry on $L^2([3/2, \infty) \times \partial M)$. Note, asymptotically, $J\varphi \sim r^{-(n-1)/2} \varphi$ as $r \rightarrow \infty$. In fact, for Examples 0 and 1, we have

$$J\varphi(r, \theta) = j(r)r^{-(n-1)/2} \varphi(r, \theta) \quad \text{for } \varphi \in L^2(\mathbb{R} \times S^{n-1}).$$

In this case, if we set $\varphi = e^{ikr}$, a generalized eigenfunction of H_{fr} , then

$$J\varphi = j(r)r^{-(n-1)/2} e^{ikr},$$

which is a spherical wave (generalized eigenfunction of Δ for large r).

We then set the wave operators:

$$W_\pm := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} J e^{-itH_{fr}} : \mathcal{H}_{fr} \rightarrow \mathcal{H}.$$

The existence of $W_\pm$ is easy to show by the standard Cook-Kuroda method. We note g^{jk} has the form:

$$(g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} 1 + a_0 & r^{-1}a_1^t \\ r^{-1}a_1 & r^{-2}g_\partial + r^{-2}a_2 \end{pmatrix}$$

in the (r, θ) coordinate, where $\partial_r^k \partial_\theta^\alpha a_0 = O(r^{-1-\mu-k})$ and $\partial_r^k \partial_\theta^\alpha a_j = O(r^{-\mu-k})$ for $j = 1, 2$ with some $\mu > 0$. Here we denote $g_\partial = (g^\partial)^{-1}$.

We then set

$$\mathcal{H}_{fr, \pm} = \{ \varphi \in \mathcal{H}_{fr} \mid \text{supp } \hat{\varphi} \subset \mathbb{R}_\pm \times \partial M \},$$

where $\hat{\varphi}$ is the Fourier transform of φ in r , i.e.,

$$\hat{\varphi}(\rho, \theta) = (\mathcal{F}\varphi)(\rho, \theta) = \int_{-\infty}^{\infty} e^{-i\rho r} \varphi(r, \theta) dr.$$

Then it is not difficult to see by the stationary phase method that

$$W_{\pm}(\mathcal{H}_{fr,\mp}) = 0,$$

and hence it is natural to consider $W_{\pm} : \mathcal{H}_{fr,\pm} \rightarrow \mathcal{H}$.

Theorem 2. $W_{\pm}$ are isometry from $\mathcal{H}_{fr,\pm}$ to $\mathcal{H}$, and they are complete, i.e., $\text{Ran } W_{\pm} = \mathcal{H}_c(H)$. Hence, in particular, the scattering operator defined by

$$S = W_+^* W_- : \mathcal{H}_{fr,-} \rightarrow \mathcal{H}_{fr,+}$$

is unitary.

Idea of Proof. Let

$$\begin{aligned} H_{\partial} &= -\frac{1}{\sqrt{G(x)}} \sum_{j,k=1}^{n-1} \partial_{\theta_j} j(r) g_{\partial}^{jk}(\theta) \sqrt{G(x)} \partial_{\theta_k} \\ &= -\sum_{j,k=1}^{n-1} \partial_{\theta_j} j(r) g_{\partial}^{jk}(\theta) \partial_{\theta_k} + (\text{lower order terms}). \end{aligned}$$

This operator is, roughly speaking, the pull-back of the Laplace-Beltrami operator on ∂M to M . By the Mourre theory, we can show

$$\langle j(r)r \rangle^{-\alpha} (H - \lambda \pm i0)^{-1} \langle j(r)r \rangle^{-\alpha} \in B(\mathcal{H}), \quad \lambda \in \mathbb{R}_+ \setminus \sigma_p(H), \alpha > 1/2,$$

but these are not sufficient to show the completeness, since perturbation terms: $r^{-1}a_1\partial_r\partial_{\theta}$, $r^{-2}a_2\partial_{\theta}\partial_{\theta}$ are only of $O(r^{-\mu})$, $\mu > 0$, with respect to H . Instead, we show

$$\langle j(r)r \rangle^{-\alpha} (H_{\partial} + 1)(H - \lambda \pm i0)^{-1} (H_{\partial} + 1)^{-1} \langle j(r)r \rangle^{-\alpha} \in B(\mathcal{H}), \quad \lambda \in \mathbb{R}_+ \setminus \sigma_p(H).$$

These estimates are proved by resolvent equations and commutator computations. These imply that

$$(H - \lambda \pm i0)^{-1} : (H_{\partial} + 1)^{-1} \langle j(r)r \rangle^{-\alpha} \mathcal{H} \mapsto (H_{\partial} + 1)^{-1} \langle j(r)r \rangle^{\alpha} \mathcal{H}$$

is bounded, and this is sufficient to show the completeness by using the abstract stationary scattering theory. $\square$

Scattering Matrix: By the intertwining property, we have

$$H_{fr}S = SH_{fr} : \mathcal{H}_{fr,-} \rightarrow \mathcal{H}_{fr,+},$$

and hence

$$\rho^2(\mathcal{F}S\mathcal{F}^{-1}) = (\mathcal{F}S\mathcal{F}^{-1})\rho^2 : \mathcal{H}_{fr,-} \rightarrow \mathcal{H}_{fr,+},$$

where $\mathcal{H}_{fr,\pm} = L^2(\mathbb{R}_{\pm} \times \partial M)$. Thus, $(\mathcal{F}S\mathcal{F}^{-1})$ commutes with multiplication by functions of ρ , and then we learn that $(\mathcal{F}S\mathcal{F}^{-1})$ is decomposed as

$$(\mathcal{F}S\mathcal{F}^{-1})\varphi(\rho, \theta) = (S(\rho)\varphi(-\rho))(\theta), \quad \rho > 0, \varphi \in \mathcal{H}_{fr,+}$$

with $S(\rho) : L^2(\partial M) \rightarrow L^2(\partial M)$, unitary. $S(\rho)$ is called the *scattering matrix*.

Melrose-Zworski Theorem: Let

$$h(\theta, \omega) = \sum_{j,k} g_{\theta}^{jk}(\theta) \omega_j \omega_k \quad \text{for } (\theta, \omega) \in T^*\partial M$$

be the classical Hamiltonian on ∂M , and let $\exp tH_{\sqrt{h}}$ be the Hamilton flow generated by $\sqrt{h}$, which is in fact the geodesic flow. Then we can show

Theorem 3. $S(\rho)$ is an FIO corresponding to the canonical transform $\exp \pi H_{\sqrt{h}}$. In particular,

$$WF(S(\rho)\varphi) = \exp \pi H_{\sqrt{h}}(WF(\varphi)), \quad \varphi \in L^2(\partial M).$$

This result is a generalization of a result by R. Melrose and M. Zworski [3], though they used different definition of the scattering matrix, which is called the *absolute scattering matrix*. The absolute scattering matrix is defined as follows: Let ψ be a generalized eigenfunction of H : $H\psi = \rho^2\psi$. Then ψ has an asymptotic form:

$$\psi(r, \theta) \sim r^{-(n-1)/2} (e^{ir\rho}\varphi_+(\theta) + e^{-ir\rho}\varphi_-(\theta)) \quad \text{as } r \rightarrow \infty$$

with some $\varphi_{\pm} \in L^2(\partial M)$. The map:

$$\tilde{S}(\rho) : \varphi_+ \mapsto \varphi_-$$

is well-defined and $\tilde{S}(\rho)$ is called the absolute scattering matrix since it is defined without using the time-dependent scattering theory. However, we can show

$$\tilde{S}(\rho) = -S(\rho)^{-1}$$

in our notation. As well as the formulation, the proof of Theorem 3 is considerably different from the one by Melrose and Zworski.

Example 0: (revisited) For the Euclidean case, $\exp \pi H_{\sqrt{h}}(\theta, \omega) = (-\theta, -\omega)$. Hence, the singularity of φ is mapped by the scattering matrix to the anti-podal points, which is well-known.

Example 1: (revisited) Let $n = 2$ and we set $g_{\partial} = \alpha g_0$ with $\alpha > 0$ and $g_0 = d\theta^2$, the standard length on S^1 . Then $S(\rho)$ has a different microlocal propagation properties. Namely,

$$WF(S(\rho)\varphi) \subset \{\theta \pm \alpha\pi \mid \theta \in WF(\varphi)\}.$$

Classical Scattering Theory for Conic Metric: In order to understand the meaning of the Melrose-Zworski theorem, let us consider the classical scattering for the conic metric. Let $p(r, \theta, \rho, \omega)$ be the classical Hamiltonian for the conic metric:

$$p(r, \theta, \rho, \omega) = \rho^2 + \frac{1}{\rho^2} \sum_{j,k} g_{\partial}^{jk}(\theta) \omega_j \omega_k, \quad r > 0, \rho \in \mathbb{R}, (\theta, \omega) \in T^* \partial M.$$

Let $(r(t), \theta(t), \rho(t), \omega(t))$ be the solution to the Hamiltonian equation:

$$\dot{r} = \frac{\partial p}{\partial \rho}, \quad \dot{\theta} = \frac{\partial p}{\partial \omega}, \quad \dot{\rho} = -\frac{\partial p}{\partial r}, \quad \dot{\omega} = -\frac{\partial p}{\partial \theta},$$

with $r(0) = r_0$, $\theta(0) = \theta_0$, etc. It is easy to see $h(\theta(t), \omega(t))$ is invariant, i.e., $h(\theta(t), \omega(t)) = h(\theta_0, \omega_0) = h_0$. Then we can solve equation for (r, ρ) easily to obtain

$$r(t) = \sqrt{4E_0 t^2 + 4r_0 \rho_0 t + r_0^2}, \quad E_0 = p(r_0, \theta_0, \rho_0, \omega_0),$$

and (θ, ω) satisfies the equation:

$$\dot{\theta} = \frac{1}{r^2} \frac{\partial h}{\partial \omega}, \quad \dot{\omega} = -\frac{1}{r^2} \frac{\partial h}{\partial \theta}.$$

So, by changing the time variable $t \mapsto \tau(t) = \int_0^t ds / r(s)^2$, we have

$$(\theta(t), \omega(t)) = \exp(\tau(t) H_h)(\theta_0, \omega_0),$$

where $\exp(t H_h)$ is the Hamilton flow generated by h on $T^* \partial M$. As $t \rightarrow \pm\infty$, $\tau(t)$ converges to finite values:

$$\lim_{t \rightarrow \pm\infty} \tau(t) = \tau_{\pm} = \frac{1}{2\sqrt{h_0}} \left(\pm \frac{\pi}{2} - \tan^{-1} \frac{r_0 \rho_0}{\sqrt{h_0}} \right).$$

Hence we have

$$\lim_{t \rightarrow \pm\infty} (\theta(t), \omega(t)) = (\theta_{\pm}, \omega_{\pm}) = \exp(\tau_{\pm} H_h)(\theta_0, \omega_0).$$

Similarly, we can show by straightforward computations,

$$\lim_{t \rightarrow \pm\infty} \rho(t) = \rho_{\pm} = \pm \sqrt{E_0}, \quad \lim_{t \rightarrow \pm\infty} (r(t) - 2t\rho(t)) = r_{\pm} = \pm \frac{r_0 \rho_0}{\sqrt{E_0}}.$$

This gives us the explicit formula for the (inverse) classical scattering operator:

$$(W_{\pm}^{cl})^{-1} : (r_0, \rho_0, \theta_0, \omega_0) \mapsto (t_{\pm}, \rho_{\pm}, \theta_{\pm}, \omega_{\pm}) = \lim_{t \rightarrow \pm\infty} (r(t) - 2t\rho(t), \rho(t), \theta(t), \omega(t)).$$

We note that the corresponding free Hamiltonian is simply given by ρ^2 , which generates the free motion : $(r, \rho, \theta, \omega) \mapsto (r + 2\rho t, \rho, \theta, \omega)$. By the formula, it is easy to show

$$(W_{\pm}^{cl})^{-1} : (\mathbb{R}_+ \times \mathbb{R}) \times (T^* \partial M) \rightarrow (\mathbb{R} \times \mathbb{R}_{\pm}) \times (T^* \partial M)$$

is diffeomorphic, and hence

$$S^{cl} = (W_+^{cl})^{-1} \circ W_-^{cl} : (\mathbb{R}_- \times \mathbb{R}) \times (T^*\partial M) \rightarrow (\mathbb{R}_+ \times \mathbb{R}) \times (T^*\partial M)$$

is also diffeomorphic. In fact, we can easily show

$$S^{cl} : (r, \rho, \theta, \omega) \mapsto (-r, -\rho, \exp((\tau_+ - \tau_-)H_h)(\theta, \omega)),$$

and $\tau_+ - \tau_- = \pi/(2\sqrt{\hbar_0})$. In general, we have $\exp(tH_q) = \exp((2t\sqrt{q})H_{\sqrt{q}})$ for $q \geq 0$, and hence we learn

$$\exp((\tau_+ - \tau_-)H_h)(\theta, \omega) = \exp(\pi H_{\sqrt{\hbar}}).$$

Thus we have

$$S^{cl} = (-I) \otimes \exp(\pi H_{\sqrt{\hbar}}),$$

and we realize that the Melrose-Zworski theorem is a quantization of this observation.

Scattering Calculus: In the proof of Theorem 3, we use the *scattering calculus* following Melrose [2], but again in a quite different formulation. For $a \in C_0^\infty(T^*(\mathbb{R}_+ \times \partial M))$ (or $\in C_0^\infty(T^*(\mathbb{R} \times \partial M))$), we denote the *scattering quantization* by

$$A = a(\hbar r, \theta, D_r, \hbar D_\theta), \quad \hbar > 0.$$

Note the difference of the location of the semiclassical parameter $\hbar > 0$ from the usual semiclassical quantization $a(r, \theta, \hbar D_r, \hbar D_\theta)$. We identify $\mathbb{R}_+ \times \partial M$ with M_∞ , and we consider A as an operator on $L^2(M, \sqrt{G} dx)$. For such an operator A , we consider

$$A(t) = e^{itH_{fr}} J^* e^{-itH} A e^{itH} J e^{-itH_{fr}}, \quad t \in \mathbb{R}.$$

$A(t)$ satisfies the Heisenberg equation:

$$\frac{d}{dt} A(t) = i[T(t), A(t)] + (\text{lower order error terms})$$

where

$$T(t) = e^{itH_{fr}} (HJ - JH_{fr}) e^{-itH_{fr}} \sim j(r - 2tD_r) \cdot \frac{h(\theta, D_\theta)}{(r - 2tD_r)^2}$$

as $r \rightarrow \infty$. We can construct the asymptotic solution to the Heisenberg equation:

$$A(t) = b_\hbar^t(\hbar r, \theta, D_r, \hbar D_\theta) \quad \text{where} \quad b_\hbar^t \sim b_0(\hbar^{-1}t; r, \theta, \rho, \omega) + O(\hbar),$$

and b_0 can be computed explicitly using the classical flow. We let $t \rightarrow \pm\infty$ and we learn

$$\lim_{t \rightarrow \pm\infty} A(t) = W_\pm^* A W_\pm \sim b_\hbar^\pm(\hbar r, \theta, D_r, \hbar D_\theta),$$

where $b_\hbar^\pm \sim (a \circ W_\pm^{cl})(r, \theta, \rho, \omega) + O(\hbar)$. Using this procedure again, we learn

$$SAS^{-1} = c_\hbar(\hbar r, \theta, D_r, \hbar D_\theta), \quad \text{where} \quad c_\hbar \sim a \circ (S^{cl})^{-1} + O(\hbar).$$

If $A = a_1(D_r) a_2(\theta, \hbar D_t h)$, then we learn

$$SAS^{-1} \sim a_1(-D_r)(a_2 \circ \exp(\pi H_{\sqrt{h}}))(\theta, D_\theta),$$

and hence

$$S(\rho)a_x(\theta, \hbar D_\theta)S(\rho)^{-1} \sim (a_2 \circ \exp(\pi H_{\sqrt{h}}))(\theta, D_\theta).$$

Then Theorem 3 follows from an *inverse Egorov theorem*.

Finally we remark that this calculus can also be used to show the propagation properties of the *scattering wave front set* of Melrose, but we omit the detail here.

References

- [1] Ito, K., Nakamura, S.: Singularities of solutions to Schrödinger equation on scattering manifold, 2007, submitted. (<http://jp.arxiv.org/abs/0711.3258>).
- [2] Melrose, R.: Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces, *Spectral and Scattering Theory* (M. Ikawa ed.), 85–130, Marcel Dekker, 1994.
- [3] Melrose, R., Zworski, M.: Scattering metrics and geodesic flow at infinity. *Invent. Math.* **124**, 389–436 (1996).