

Extendability of symplectic torus actions with isolated fixed points

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1 Abstract

I itemise the abstract of my talk in the following.

Problem : When does or doesn't a given effective and symplectic torus group action on a compact connected symplectic manifold with isolated fixed points extend to an effective and symplectic action of a higher dimensional torus group ?

Method : Translating geometrical objects into graphical objects (due to recent works of V. Guillemin and C. Zara [5, 6, 7, 8]) and considering the above problem on the graphical level.

Result : I obtained a necessary condition for the torus action to extend. That is, if the torus action extends, then a certain obstruction must vanish.

2 From geometric data to graphical data

Here I introduce the graphical object obtained from geometric data under suitable assumptions, which is introduced and studied by V. Guillemin and C. Zara.

Consider

(M^{2d}, ω, J) : a cpt. conn. symp. mfd. with a compatible alm. cpx. str.

T^n : an n -dim. torus acting on M effectively and preserving ω and J .

Assume

- (1) M^T discrete, and
- (2) isotropy weights at $\forall p \in M^T$ are pairwise linearly independent.

With this assumption, we obtain the following graphical objects:

(Γ, θ, α) : Goreskey-Kottwitz-MacPherson graph.

(“GKM graph” or “GKM 1-skeleton”)

which consists of three data

Γ : a d-valent graph

θ : a connection on the graph

α : an axial function

Let's describe these three data in detail in the following.

The graph:

$\Gamma = (V_\Gamma, E_\Gamma)$

$V_\Gamma = M^T$ (the fixed point set),

$E_\Gamma \subset V_\Gamma \times \left\{ \begin{array}{l} \text{embedded } \mathbb{CP}^1 \text{'s fixed by} \\ \text{codimension one subtori} \end{array} \right\}$
consisting of pairs (p, Σ) with $p \in \Sigma^T$.

Notation:

$e = (p, \Sigma) = (p, q) \ (\Sigma^T = \{p, q\})$

$\bar{e} = (q, \Sigma) = (q, p)$

$E_p = \{e \in E_\Gamma \mid i(e) = p\}$

The connection:

θ = a collection of bijections $\{\theta_e\}_{e \in E_\Gamma}$:

$\theta_e : E_p \rightarrow E_q \ (e = (p, q))$

satisfying

(1) $\theta_e(e) = \bar{e}$, and (2) $\theta_{\bar{e}} = (\theta_e)^{-1}$

The axial function:

α = a map from E_Γ to $\mathfrak{t}^*$ satisfying

(1) $\alpha(\bar{e}) = -\alpha(e)$, for $\forall e \in E_\Gamma$

(2) $\alpha(\theta_e(f)) = \alpha(f) + c(f, e)\alpha(e)$, for $\forall f \in E_{i(e)}$, $\forall e \in E_\Gamma$.

Moreover, I give the following additional assumption which Guillemin and Zara did not assume in[].

The effectiveness condition:

For any p , let $E_p = \{f_1, \dots, f_d\}$. Then

$$\gcd \left\{ \begin{vmatrix} \alpha(f_{i_1}) \\ \vdots \\ \alpha(f_{i_n}) \end{vmatrix} \mid 1 \leq i_1 < \dots < i_n \leq d \right\} = 1$$

3 Translation of the Extendability

From now on we assume $k = d - n \geq 1$ (this number is called the **complexity** of the torus action).

The T -action extends to a $T \times S^1$ -action.

$\Downarrow$ implies

The axial function α lifts to the axial function (α, m) commuting the following diagram.

$$\begin{array}{ccc} & & t^* \times \mathbb{R} \\ & \nearrow (\alpha, m) & \downarrow pr_1 \\ E_\Gamma & \xrightarrow{\alpha} & t^* \end{array}$$

In this way we obtain an extra function m . Since I will discuss whether the extra function exists or not, I abstract the properties of the function and redefine it as follows.

Definition 1 We call the map $m : E_\Gamma \rightarrow \mathbb{Z} \subset \mathbb{R}$ satisfying the following conditions an **extra weight**:

- (1) $m(\theta_e(f)) = m(f) + c(f, e)m(e)$, for $\forall f \in E_{i(e)}$, $\forall e \in E_\Gamma$, and
- (2) (α, m) satisfies the effectiveness condition.

4 Construction of the Obstruction

Let G_p be the d -dimensional torus acting standardly on $T_p M \cong \mathbb{C}^d$, and let $\mathfrak{g}_p$ be its Lie algebra.

Note that $\text{Map}(E_p, \mathbb{R}) \cong \mathfrak{g}_p$, $\text{Map}(E_p, \mathbb{Z}) \cong L(\mathfrak{g}_p)$, where $L(\mathfrak{g}_p)$ is the lattice of $\mathfrak{g}_p$.

Parallel transport along an edge

For an edge $e = (p, q)$ we define a map $e : L(\mathfrak{g}_p) \rightarrow L(\mathfrak{g}_q)$, $m \mapsto e(m)$ by the equation

$$e(m)(\theta_e(f)) = m(f) + c(f, e)m(e)$$

for $\forall f \in E_p$.

A loop in the graph

We call a **loop** a sequence of edges $e_1 \cdots e_r$ with $i(e_1) = t(e_r) = p$ and $t(e_j) = i(e_{j+1})$ ($j = 1, \dots, r-1$). (denoted also by γ)

A loop is called **trivial** if it is written in the form $e_1 \cdots e_s \bar{e}_s \cdots \bar{e}_1$.

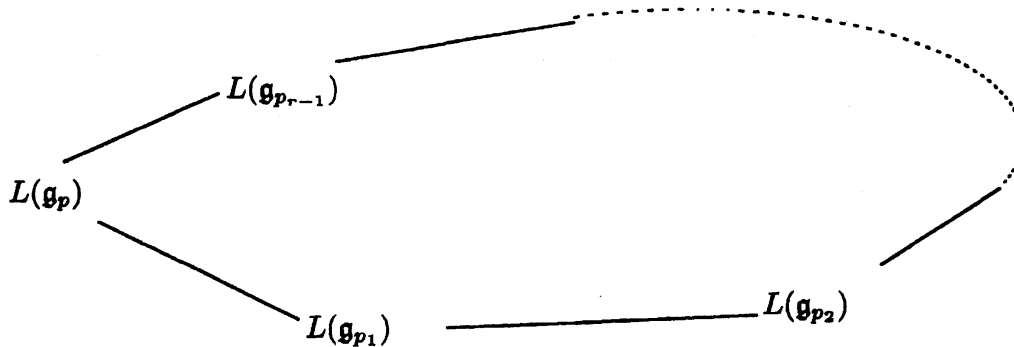
Let $\mathcal{L}(\Gamma, p)$ the set of all loops based at p . $\mathcal{L}(\Gamma, p)$ has a group structure if we quotient it by trivial loops.

Parallel transport along a loop

For a loop $\gamma = e_1 \cdots e_r \in \mathcal{L}(\Gamma, p)$, we define $\gamma : L(\mathfrak{g}_p) \rightarrow L(\mathfrak{g}_p)$ by

$$\gamma(m) = e_r(\cdots (e_1(m)) \cdots).$$

This leads us to the $\mathcal{L}(\Gamma, p)$ -action on $L(\mathfrak{g}_p)$.



The obstruction

The axial function at p , $\alpha_p \in \text{Map}(E_p, \mathfrak{t}^*)$, is considered as a linear map $L(\mathfrak{t}) \xrightarrow{\alpha_p} L(\mathfrak{g}_p)$. Then we have the following short exact sequence

$$0 \longrightarrow L(\mathfrak{t}) \xrightarrow{\alpha_p} L(\mathfrak{g}_p) \xrightarrow{\pi_p} L(\mathfrak{g}_p)/\text{Im}\alpha_p \longrightarrow 0.$$

Here we have the following important lemma.

Lemma 1 For $\Delta \in L(\mathfrak{g}_p)/\text{Im}\alpha_p$ and $m \in (\pi_p)^{-1}(\Delta)$,

1. (α_p, m) is effective $\iff \Delta$ is primitive.
2. For a loop γ based at p , $\gamma(m) - m$ depends only on Δ , does not depend on the choice of m .

Definition 2 We define a map $eo(p, \Delta) : \mathcal{L}(\Gamma, p) \rightarrow L(\mathfrak{g}_p)$ by

$$eo(p, \Delta)(\gamma) = \gamma(m) - m$$

for $m \in (\pi_p)^{-1}(\Delta)$.

From above we have our result:

Theorem 1 *There exists an extra weight for α if and only if there exists a primitive Δ such that $eo(p, \Delta) \equiv 0$.*

5 Some Properties

1. (Calculation rule)

For two loops γ and $\delta \in \mathcal{L}(\Gamma, p)$,

$$eo(p, \Delta)(\gamma\delta) = eo(p, \Delta)(\gamma) + eo(p, \Delta)(\delta) \quad (1)$$

Note that $\mathcal{L}(\Gamma, p)$ acts also on the quotient space $L(\mathfrak{g}_p)/\text{Im}\alpha_p$.

In the case that the complexity is one,

2. The $\mathcal{L}(\Gamma, p)$ -action on $L(\mathfrak{g}_p)/\text{Im}\alpha_p \cong \mathbb{Z}$ is very easy to understand. This action is given by the homomorphism

$$\begin{aligned} \mathcal{L}(\Gamma, p) &\rightarrow \text{Isom}(L(\mathfrak{g}_p)/\text{Im}\alpha_p) \cong \mathbb{Z}_2 \\ \gamma &\mapsto \text{sgn}(\gamma), \end{aligned}$$

where

$$\text{sgn}(\gamma) = \text{sgn}(\theta_\gamma) \times (-1)^{|\gamma|}. \quad (2)$$

θ_γ : the holonomy map $E_p \rightarrow E_p$ along γ w.r.t the connection θ .

$|\gamma|$: the number of the edges of which γ consists. The length of γ .

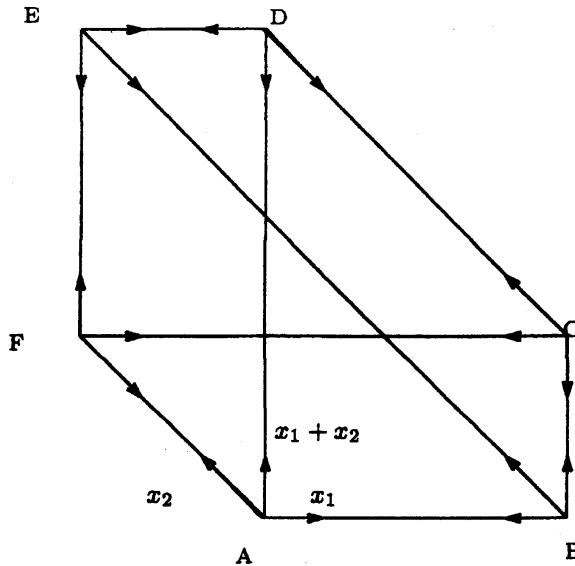
3. (Calculation rule)

$$eo(p, \Delta)(\gamma\delta) = eo(p, \Delta)(\gamma) + \text{sgn}(\gamma)eo(p, \Delta)(\delta) \quad (3)$$

6 Example

1. 3-flag variety $SL(3; \mathbb{C})/B$ with standard T^2 -action

The graph Γ is visualized as follows.



Let the base vertex $p = A$ and let $E_p = \{f_1 = AB, f_2 = AF, f_3 = AD\}$. All loops are of the length even. The holonomy along each loop is trivial. Thus $\text{sgn}(\gamma) = 1$ for all loops γ . Hence the action of $\mathcal{L}(\Gamma, p)$ on $L(\mathfrak{g})_p / \text{Im}(\alpha_p)$ is trivial.

$x_1 = (1, 0)$, $x_2 = (-1, 1)$ and $x_1 + x_2 = (0, 1)$ are weights at p .

$\mathcal{L}(\Gamma, p)$ is generated by four loops, e.g., $ABCD A$, $ADEFA$, $ABCFA$, $ABEFA$. $\text{eo}(p, 1)(\gamma)$ for these generators are written in the following table in which the columns express the values of $\text{eo}(p, 1)(\gamma)$.

	$ABCD A$	$ADEFA$	$ABCFA$	$ABEFA$
f_1	-1	1	-2	-1
f_2	-1	1	1	2
f_3	-2	2	-1	1

By calculation rule (1) or (3), for example, we have

$$\begin{aligned}
 \text{eo}(ABCDEFA) &= \text{eo}(ABCD A \cdot ADEFA) \\
 &= \text{eo}(ABCD A) + \text{eo}(ADEFA) \\
 &= {}^t(-1, -1, -2) + {}^t(1, 1, 2) \\
 &= (0, 0, 0).
 \end{aligned}$$

In this case eo is an additive function.

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