数理解析研究所講究録
595巻 1986年 39-58

Topologically Extremal Real Surfaces in $\mathbb{P}^2\times\mathbb{P}^1$ and $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

by Goo ISHIKAWA (石川 剛郎).

Department of Mathematics, Faculty of Sciences, Nara Women's University, Nara 630, Japan.

From a general viewpoint we illustrate a method of construction of surfaces in $\mathbb{P}^2\times\mathbb{P}^1$ and $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ defined over $\mathbb{R}$ having topologically extremal properties. Precisely we show that for each d, e and r there exists an M-surface A in $\mathbb{P}^2\times\mathbb{P}^1$ (resp. $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$) of degree (d,r) (resp. (d,e,r)) such that the projection $A \longrightarrow \mathbb{P}^1$ has the maximal number of real critical points. The construction of M-surfaces in $\mathbb{P}^3$ by O.Ya.Viro is also made more clear.

0. Introduction.

Harnack [H] pointed out that the number of components in the real locus of a curve in $\mathbb{P}^2$ of degree d defined over $\mathbb{R}$ does not exceed $1+(1/2)(d-1)(d-2)$ and, for each d, there exists a non-singular curve in $\mathbb{P}^2$ of degree d defined over $\mathbb{R}$, the real locus of which has exactly $1+(1/2)(d-1)(d-2)$ components.

Hilbert in his 16th problem proposed to investigate topological restrictions for hypersurfaces in $\mathbb{P}^n$ of fixed degree defined over $\mathbb{R}$.

One may regard an real algebraic function as an one-parameter family of hypersurfaces defined over $\mathbb{R}$, and it is natural to investigete topological restrictions for hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^1$ of fixed degree defined over $\mathbb{R}$.

Let $A \subset \mathbb{P}^n \times \mathbb{P}^1$ be a real hypersurface of degree (d,r), that is, the zero-locus of a polynomial $\sum_{0 \leq i \leq r} F_i(X_0,\dots,X_n)\lambda^{r-i}\mu^i$, where F_i $(0 \leq i \leq r)$ is a real homogeneous polynomial of degree d. Consider the projection $\varphi: A \longrightarrow \mathbb{P}^1$. Our main object is the topology of real locus $A_{\mathbb{R}}$ of A and singularities of the restriction $\varphi_{\mathbb{R}}: A_{\mathbb{R}} \longrightarrow \mathbb{R}P^1$ of φ to $A_{\mathbb{R}}$.

We denote by $P_t(X,K)$ the Poincare series of a space X over a field K with indeterminate t, and by $s(f)$ the number of critical points of a function $f: X \longrightarrow R$ from a n-dimensional manifold to an one-dimensional manifold.

If $A \subset \mathbb{P}^n \times \mathbb{P}^1$ is non-singular, then the diffeomorphism type of A is determined by (d,r). For example,

$$P_1(A,K) = \begin{cases} \chi(A) & (n\text{:even}), \\ 2(n+1) - \chi(A) & (n\text{:odd}), \end{cases} \qquad \text{for any } K,$$

$$\chi(A) = (n+1)(1-d)^n r + 2\left(\frac{(1-d)^{n+1}-1}{d} + n+1\right), \text{ (cf. 1.6).}$$

We call A <u>generic</u> if A is non-singular and $\varphi: A \longrightarrow P^1$ has only non-degenerate critical points.

If A is generic, then $s(\varphi) = (n+1)(d-1)^n r$ (cf. 1.6).

By Harnack-Thom's inequality ([G]), we have an uniform estimate:

$$(0.0)\quad \begin{cases} P_1(A_{\mathbb{R}};\mathbb{Z}/2) \leqq P_1(A;\mathbb{Z}/2), \\ s(\varphi_{\mathbb{R}}) \leqq s(\varphi). \end{cases}$$

In this note from a general viewpoint we show the following

<u>Theorem 0.1</u>. For $n = 1, 2$ and for each (d,r), the estimate (0.0) is sharp, that is, there exists a generic real hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ of degree (d,r) attaining both equalities in (0.0).

Notice that in the case $r = 1$ Theorem 0.1 is proved in [I]. A finer result is obtained in the case $n = 1$. For $A \subset \mathbb{P}^1 \times \mathbb{P}^1$, we denote by $\pi: A \longrightarrow \mathbb{P}^1$ the projection to the first component.

<u>Proposition 0.2</u>. For non-singular real curves $A \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree (d,e) such that both φ, π have only non-degenerate critical points, there exists the sharp estimate:

$$P_1(A_R;Z/2) \leqq 2 + 2(d-1)(e-1),$$

$$s(\varphi_{\mathbb{R}}) \leqq 2(d-1)e, \qquad s(\pi_{\mathbb{R}}) \leqq 2d(e-1).$$

Now let us formulate a general theorem which implies Theorem 0.1.

Let S be a real complex surface (cf. 2.1), $C \subset S$ be a real curve possibly with singularities. A non-singular component E of $C_{\mathbb{R}} \subset S_{\mathbb{R}}$ is an <u>oval</u> (resp. an <u>empty oval</u>) if there exists an embedding $i: D^2 \longrightarrow S_{\mathbb{R}}$ such that $i(\partial D^2) = E$ (and that $i(\operatorname{int} D^2) \cap C_{\mathbb{R}}$ is empty).

Let S be compact, L a real holomorphic line bundle (cf. 2.6), s_0, s_1 M-sections of L (cf. 2.7).

Consider the following condition (*):

(*i) The zero-loci $(s_0)_0$ and $(s_1)_0$ are both connected and of genus g.

(*ii) $(s_0)_0$ and $(s_1)_0$ intersect in $\langle c_1(L)^2,[S]\rangle$ points in $S_{\mathbb{R}}$.

(*iii) The real locus of $(s_0 s_1)_0 = (s_0)_0 \cup (s_1)_0$ has $2g$ empty ovals.

We denote by $\mathbb{P}^1_1$ the real complex curve $(\mathbb{P}^1, \tau_1)$, where τ_1 is the complex conjugation (cf. 2.3). Fix a pair of M-sections λ, μ of $\mathcal{O}_{\mathbb{P}^1_1}(1)$ such that $(\lambda)_0 \neq (\mu)_0$.

Denote by $\psi: S \times \mathbb{P}^1_1 \longrightarrow \mathbb{P}^1_1$, $\xi: S \times \mathbb{P}^1_1 \longrightarrow S$ the projections. For a transverse section s of $\xi^* L \otimes \psi^* \mathcal{O}_{\mathbb{P}^1_1}(r)$ (cf. 1.3), denote by $\varphi: (s)_0 \longrightarrow \mathbb{P}^1_1$, $\pi: (s)_0 \longrightarrow S$ the restrictions of projections. Then, associated to s, there is a natural section of $\operatorname{Hom}(T(s)_0, \varphi^* T\mathbb{P}^1_1)$ defined by the tangent map of φ.

<u>Theorem 0.4.</u> Let S be an M-surface $\underbrace{\text{with}}_{H_1(S;\mathbb{Z}/2)=0}$ connected real part $S_{\mathbb{R}}$, L be a real holomorphic line bundle with a pair s_0, s_1 of M-sections of L satisfying the condition (*). Then, for any r, there exists an M-section s of $\xi^*L\otimes\psi^*\mathcal{O}_{\mathbb{P}^1_1}(r)$ over $S\times\mathbb{P}^1_1$ near $s_0\otimes\lambda^r$, which associates an M-section of $\mathrm{Hom}(T(s)_0,\varphi^*T\mathbb{P}^1_1)$ defied by the projection $\varphi\colon (s)_0\longrightarrow\mathbb{P}^1_1$.

Explicitely, s can be taken in a form

$\sum_{0\leq i\leq r}\varepsilon_i s_i\lambda^i\mu^{r-i}$, where $s_i = s_0$ (i:even), $s_i = s_1$ (i:odd) and $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_r$ are real numbers with $1=\varepsilon_0\gg|\varepsilon_1|\gg\dots\gg|\varepsilon_r|>0$.

Remark 0.5. A sufficient condition for the existence of a pair of M-sections satisfying (*) is given in section 4. Theorem 0.4 with this sufficient condition implies immediately Theorem 0.1 in the case $n = 2$.

Putting $S = \mathbb{P}^1\times\mathbb{P}^1$ $(=\mathbb{P}^1_1\times\mathbb{P}^1_1)$ and $L = \mathcal{O}_{\mathbb{P}^1}(d)\otimes\mathcal{O}_{\mathbb{P}^1}(r)$ over S, we have

<u>Corollary 0.6.</u> For non-singular real surface $A\subset\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ of degree (d,e,r) such that $\varphi\colon A\longrightarrow\mathbb{P}^1$ has only non-degenerate critical points, there exists the sharp estimate:

$$\begin{cases} P_1(A_{\mathbb{R}};\mathbb{Z}/2)\leqq 6der-4de-4er-4rd+4d+4e+4r,\\ s(\varphi_{\mathbb{R}})\leqq (6de-4d-4e+4)r.\end{cases}$$

From Theorem 0.4, it naturally arises the following general problem:

Problem 0.7. Let E be a real holomorphic vector bundle over a real complex manifold. Give a criterion for the existence or the non-existence of M-sections of E.

Lastly we intend to clearfy the construction of M-surfaces in $\mathbb{P}^3$ by Viro [V].

Theorem 0.8.(Viro) For non-singular real surfaces A in $\mathbb{P}^3$ of degree d, there exists the sharp estimate:

$$P_1(A_{\mathbb{R}};\mathbb{Z}/2) \leqq d^3 - 4d^2 + 6d.$$

Let X_0, X_1, X_2, X_3 be homogeneous coordinates of $\mathbb{P}^3$. Put $\mathbb{P}^2 = \{X_3 = 0\}$, $\mathbb{P}^1 = \{X_3 = X_3 = 0\}$ and $\ell = \{X_0 = X_1 = 0\}$. Let $\varphi : \mathbb{P}^3 - \ell \longrightarrow \mathbb{P}^1$ be a projection. Fix a tubular neighborhood U of ℓ in $\mathbb{P}^3$ such that $\overline{U} \cap \mathbb{P}^1$ is empty.

Observe that for each d there exist M-sections $s_0, \ldots, s_d$ of $\mathcal{O}_{\mathbb{P}^2}(0), \ldots, \mathcal{O}_{\mathbb{P}^2}(d)$ near $X_2{}^0, \ldots, X_2{}^d$ respectively such that $(s_i)_0$ and $(s_{i+1})_0$ intersect in $i(i+1)$ points in $\mathbb{RP}^2$, the real locus of $(s_i s_{i+1})_0$ has $(1/2)(i-1)(i-2) + (1/2)i(i-1)$ empty ovals $(0 \leqq i \leqq d-1)$ and $\varphi|(s_i)_0$ has $(i-1)i$ real critical points $(0 \leqq i \leqq d)$. Naturally each s_i is extended to a section $\tilde{s}_i$ of $\mathcal{O}_{\mathbb{P}^3}(i)$ $(0 \leqq i \leqq d)$.

Put $s = \sum_{0 \leqq i \leqq d} \varepsilon_i X_2^{d-i} \tilde{s}_i \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))_{\mathbb{R}}$, and $A = (s)_0$.
Take real numbers $\varepsilon_0, \ldots, \varepsilon_d$ to be $1 = \varepsilon_0 \gg |\varepsilon_1| \gg \ldots \gg |\varepsilon_d| > 0$ and of apropriate signs.

$\varphi_{\mathbb{R}}: A_{\mathbb{R}} - U \longrightarrow \mathbb{R}P^1$ defines a vector field ξ' over $A_{\mathbb{R}} - U$. ξ' is extended to a vector field ξ over $A_{\mathbb{R}}$ with finite singularities.

Denote by $s^+(\xi)$ (resp. $s^-(\xi)$) the sum of positive (resp. negative) indices of singular points of ξ, and put $t_i = \dim H_i(A_{\mathbb{R}}; \mathbb{Z}/2)$ $(i=1,2,3)$. Then we see

$$s^+(\xi) \geqq d + (1/3)d(d-1)(d-2),$$
$$s^-(\xi) \geqq (1/3)(d+1)d(d-1) + (1/3)d(d-1)(d-2).$$

Thus $\chi(A_{\mathbb{R}}) = s^+(\xi) - s^-(\xi) \geqq d - (1/3)(d+1)d(d-1)$. On the other hand $t_0 + t_1 \geqq 2 + (1/3)(d-1)(d-2)(d-3)$. Hence we have

$$\begin{aligned} P_1(A_{\mathbb{R}}; Z/2) &= t_0 + t_1 + t_2 \\ &= 2(t_0 + t_2) - \chi(A_{\mathbb{R}}) \\ &\geqq d^3 - 4d^2 + 6d \quad (= P_1(A; \mathbb{Z}/2)). \end{aligned}$$

By Harnack-Thom's inequality, all equalities are hold.

The author would like to thank J.J.Risler for helpful suggestions, Takashi Matsuoka for informing the existence of [P] and M-h Saito for valuable comments.

1. Preliminary: Complex Topology.

(1.0) Let X be a complex manifold, $\pi: E \longrightarrow X$ a holomorphic vector bundle and $s: X \longrightarrow E$ a holomorphic section. Put $(s)_0 = \{x \in X \mid s(x) = 0\}$.

We call s <u>transverse</u> if s is transverse to the zero section $\zeta \subset E$, that is, for any $s \in (s)_0$, $s_* T_x X \oplus T_{s(x)} \zeta = T_{s(x)} E$

If s is transverse, then $(s)_0$ is a complex submanifold of X.

Denote by H the complex vector space $H^0(X,E)$ of totality of holomorphic sections of E over X, and by PH the projectification of H.

Put $Z = \{(x,[s]) \in X \times PH \mid s(x) = 0\}$ and consider the projection $\Phi: Z \longrightarrow PH$. Then s is transverse if and only if Z is non-singular along $\Phi^{-1}[s]$ and Φ is submersive over $[s]$.

In particular, for transverse sections $s, s' \in H$, $(s)_0$ and $(s')_0$ are diffeomorphic.

(1.1) Let $s \in H^0(X,E)$ be transverse. Put $Z = (s)_0$. Then we have an exact sequence

$$0 \longrightarrow TZ \longrightarrow TX|_Z \longrightarrow E|_Z \longrightarrow 0,$$

of complex vector bundles. Therefore $c_t(TX|_Z) = c_t(TZ)c_t(E|_Z)$ for Chern polynomials. The Chern classes of TZ are calculated by the formula $c_t(TZ) = \dfrac{c_t(TX|_Z)}{c_t(E|_Z)}$ (cf. [F]).

(1.2) Let L be a holomorphic line bundle over a complex manifold V of dimension n. Let Z be the zero-locus of a transverse section of L. Then by (1.1),

$$\chi(Z) = \left\langle \sum_{i+j=n+1} (-1)^j c_i(TV)(c_1(L))^{j+1}, [V]\right\rangle.$$

For example, if $\dim V = 2$, then

$$\chi(Z) = \langle c_1(TV)c_1(L) - c_1(L)^2, [V]\rangle.$$

Furthermore, if Z is connected, then

$$\chi(Z) = 1 + (1/2)\langle c_1(L)^2 - c_1(L)c_1(TV), [V]\rangle.$$

(1.3) Let R be a non-singular curve of genus g. Denote by $\xi: V \times R \longrightarrow V$ and $\psi: V \times R \longrightarrow R$ the projections. Put $L' = \xi^* L \otimes \psi^* \mathcal{O}_R(r)$ over $V \times R$ for each r. Let $A \subset V \times R$ be the zero-locus of a transverse section of L'.

Then $\chi(A) = \langle \rho, [V]\rangle$, where

$$\rho = rc_n(TV) + \sum_{i+j=n, j>0} ((j+1)r+2g-2)c_i(TV)(-c_1(L))^j,$$

as an element of $H^{2n}(V;\mathbb{Z})$.

For example, if $\dim V = 2$, then

$$\chi(A) = \langle rc_2(TV) - (2r+2g-2)c_1(TV)c_1(L) + (3r+2g-2)c_1(L)^2, [V]\rangle.$$

(1.4) Example. Let C, C' and C'' be non-singular curves of genus g, g' and g'' respectively. Put $X = C \times C' \times C''$, and denote projections by p_1, p_2 and p_3 to C, C' and C'' respectively. Let $A \subset X$ be the zero-locus of a transverse section of $L' = p_1^*\mathcal{O}_C(d) \otimes p_2^*\mathcal{O}_{C'}(d') \otimes p_3^*\mathcal{O}_{C''}(d'')$. Then $\chi(A)$ is equal to $6(d-1)(d'-1)(d''-1) + (2+4g'')(d-1)(d'-1) + (2+4g)(d'-1)(d''-1) + (2+4g')(d''-1)(d-1) + (2+4g'g'')(d-1) + (2+4g''g)(d'-1) + (2+4gg')(d''-1) + 6 - 4(g+g'+g'') + 4(gg'+g'g''+g''g)$.

(1.5) In (1.3), denote by $\varphi: A \longrightarrow R$ the projection to R. Put $\xi = \mathrm{Hom}(TA, \varphi^*TR)$. Then $\langle c_n(\xi), [A]\rangle = \langle \eta, [V]\rangle$, where

$$\eta = (-1)^n r \sum_{i+j=n} (j+1)c_i(TV)(-c_1(L))^j,$$

as an element of $H^{2n}(V;Z)$.

For example, if $\dim V = 2$, then

$$\langle c_2(\xi), [A]\rangle = r\langle c_2(TV) - 2c_1(TV)c_1(L) + 3c_1(L)^2, [V]\rangle.$$

(1.6) Let A be a non-singular hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$. of degree (d,r). Then $\chi(A) = \langle c_n(TA), [A]\rangle$ is equal to

$$(n+1)(1-d)^n r + 2\left(\frac{(1-d)^{n+1}-1}{d} + n+1\right).$$

If $\varphi: A \longrightarrow \mathbb{P}^1$ has only isolated critical points, then $s(\varphi) = \langle c_n(\mathrm{Hom}(TA, {}^*T\mathbb{P}^1)), [A]\rangle$ is equal to $(n+1)(d-1)^n r$.

(1.7) Let A be a non-singular irreducible projective variety of dimension n. Then $H_i(A;Z)$ is torsion free for all i, and rank $H_i(A;\mathbb{Z})$ is equal to 0 ($i \neq n$, i:odd), 1 ($i \neq n$, i:even), $n+1-\chi(A)$ ($i=n$, n:odd), $\chi(A)-n$ ($i=n$, n:even).

(1.8) If A is a simply connected compact complex surface, then $P_t(A;K) = P_{-t}(A,K)$, and $P_1(A;K) = P_{-1}(A;K) = \chi(A)$ for any field K.

2. Preliminary: Real Topology.

(2.1) A <u>real structure</u> on a complex manifold X is an anti-holomorphic involution $\tau: X \longrightarrow X$. The pair (X,τ) is called a <u>real complex manifold</u>. Two real complex manifolds (X,τ), (X',τ') are isomorphic if there is an isomorphism $\sigma: X \longrightarrow X'$ of complex manifolds satisfying $\sigma\circ\tau = \tau'\circ\sigma$ (cf. [S]).

(2.2) Let (X,τ) be a real complex manifold. We denote by $X_{\mathbb{R}}$ the space X^{τ} of fixed points of τ in X, and call it the <u>real locus</u> of X (with respect to τ).

(X,τ) is a <u>M-manifold</u> if $P_1(X_{\mathbb{R}};\mathbb{Z}/2) = P_1(X;\mathbb{Z}/2)$ (cf. [G]). A M-manifold (X,τ) of dimension 1 (resp. 2) is called a M-<u>curve</u> (resp. M-<u>surface</u>).

(2.3) Example. The number of equivalence classes of real structures on $\mathbb{P}^n$ is one if n is even and two if n is odd (cf. [F], p.240).

The anti-holomorphic involution $\tau': \mathbb{P}^{2m+1} \longrightarrow \mathbb{P}^{2m+1}$ defined by $\tau'[X_0:X_1:\dots:X_{2i}:X_{2i+1}:\dots:X_{2m}:X_{2m+1}] = [-\overline{X}_1:\overline{X}_0:\dots:-\overline{X}_{2i+1}:\overline{X}_{2i}:\dots:-\overline{X}_{2m+1}:\overline{X}_{2m}]$ gives a real structure not equivalent to the usual real structure defined by the complex conjugation $(\mathbb{P}^{2m+1},\tau_{2m+1})$. We often denote by $\mathbb{P}_0^{2m+1} = (\mathbb{P}^{2m+1},\tau')$, $\mathbb{P}_1^{2m+1} = (\mathbb{P}^{2m+1},\tau_{2m+1})$.

Then $\mathbb{P}^{2m}$ and $\mathbb{P}_1^{2m+1}$ are M-manifolds, but $\mathbb{P}_0^{2m+1}$ is not a M-manifold.

(2.4) From properties of Poicaré series, we see

Lemma. Let (X,τ), (X',τ') be M-manifolds. Then $(X \amalg X', \tau \amalg \tau')$ and $(X \times X', \tau \times \tau')$ are also M-manifolds.

(2.5) Lemma. Let (X,τ) be a M-surface with $H_1(X;\mathbb{Z}/2) = 0$ and $H_0(X_{\mathbb{R}};\mathbb{Z}/2) \cong \mathbb{Z}/2$. Then $\chi(X) + \chi(X_{\mathbb{R}}) = 4$.

Proof. $P_{-1}(X;\mathbb{Z}/2) = P_1(X;\mathbb{Z}/2) = P_1(X_{\mathbb{R}};\mathbb{Z}/2)$.

$$P_1(X_{\mathbb{R}};\mathbb{Z}/2) + P_{-1}(X_{\mathbb{R}};\mathbb{Z}/2) = 2(\dim H_0(X_{\mathbb{R}};\mathbb{Z}/2) + \dim H_2(X_{\mathbb{R}};\mathbb{Z}/2))$$
$$= 4.$$

(2.6) Let $\pi: E \longrightarrow X$ be a holomorphic vector bundle over a real complex manifold (X,τ). A real structure of π is a real structure $T: E \longrightarrow E$ of E as a complex manifold (cf. 2.1) such that $\pi \circ T = \tau \circ \pi$ and the restriction $T_x: E_x \longrightarrow E_{\tau(x)}$ to each fiber $(x \in X)$ is conjugate linear.

We call the triple $E = (\pi;T,\tau)$ a real holomorphic vector bundle (cf. [A]). Notice that the restriction $\pi_{\mathbb{R}}: E_{\mathbb{R}} \longrightarrow X_{\mathbb{R}}$ to the real locus of π is a real vector bundle.

A holomorphic section $s \in H^0(X,E)$ of E is real if $T \circ s \circ \tau^{-1} = s$, that is, $s \in H^0(X,E)_{\mathbb{R}}$ with respect to the anti-holomorphic involution $s \longmapsto T \circ s \circ \tau^{-1}$.

(2.7) <u>Definition.</u> A holomorphic section s of a real holomorphic vector bundle over a real complex manifold (X,τ) is a M-<u>section</u> if s is transverse, real and the zero-locus $(s)_0 \subset X$ with restricted τ is a M-manifold.

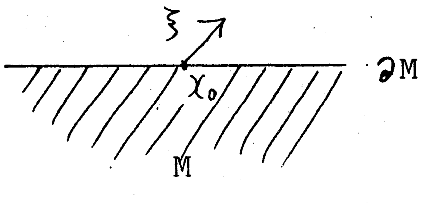
(2.8) Remark. Two real holomorphic vector bundles are isomorphic as real holomorphic vector bundles if and only if they are isomorphic as holomorphic vector bundles.

On $\mathbb{P}^n$, any holomorphic line bundle has a structure of real holomorphic line bundle.

(2.9) Poincaré-Hopf-Pugh formula (cf. [P]).

Let M be a compact C^∞ manifold of dimension n with boundary ∂M.

A tangent vector ξ to M at a point x_0 of M is external if $df_{x_0}(\xi)$ is positive for some C^∞ function f defined in a neighborhood U of x_0 such that $f^{-1}(0) = \partial M \cap U$, f takes negative values in $(M-\partial M)\cap U$ and $df|\partial M \cap U$ does not vanish (figure 1):



external

Let $v\colon \partial M \longrightarrow TM|\partial M$ be a C^∞ section over ∂M to the tangent bundle TM.

Assume that (a): for each $x_0 \in \partial M$, $v(x_0) \neq 0$.

First put $M_0 = M$. Next put

$$M_1' = \{x \in \partial M \ \ v(x)| \text{ is external}\},$$

and put $M_1 = \overline{M_1'}$, and $\partial M_1 = M_1 - M_1'$.

Inductively, if M_k is a C^∞ manifold with boundary ∂M_k ($k \geqq 0$), then put

$$M_{k+1}' = \{x \in \partial M_k \mid (v|\partial M_k)(x) \text{ is external w.r.t. } M_k\},$$

$M_{k+1} = \overline{M_{k+1}'}$ and $\partial M_{k+1} = M_{k+1} - M_{k+1}'$.

Assume that (b): M_k is a C^∞ manifold with boundary ∂M_k, ($k = 1,2,\ldots,n-1$).

Lemma. Let v satisfy two assumptions (a), (b) stated above. Then for any C^∞ extension $w\colon M \longrightarrow TM$ with isolated singularities, we have

$$\text{(c):} \qquad \operatorname{ind} w = \sum_{i=0}^{n} (-1)^i \chi(M_i).$$

Remark. (0) We adopt the following definition of index of a vector field: Let x_0 M be an isolated singular point of w. Take a system of coordinates $x_1,\ldots,x_n$ centered at x_0, and write locally

$$w(x) = a_1(x)(\partial/\partial x_1) + \ldots + a_n(x)(\partial/\partial x_n).$$

Define $\operatorname{ind}_{x_0} w = \deg_0(-a)$, where $a = (a_1,\ldots,a_n)$.

Then put $\operatorname{ind} w = \sum \operatorname{ind}_{x_0} w$, where the sum runs over isolated singular points x_0 of w.

(1) If ∂M is empty, then (c) is the Poincaré-Hopf's formula.

(2) For a C^∞ vector field w over M with only isolated singular points, there exists a non-negative C^∞ function $f: U \longrightarrow \mathbb{R}$ with the following properties:

(i) $f^{-1}(0) = \partial M$. (ii) For any sufficiently small $\varepsilon > 0$, $w|f^{-1}(\varepsilon)$ satisfies two assumptions (a), (b).

3. Non-linear systems of real sections.

In this section we prove Theorem 0.4.

In the situation of Theorem 0.4, put $Z = (s_r)_0 \cong (s_i)_0$ $(0 \leq i \leq r)$, $s^{(r)} = \sum_{0 \leq i \leq r} \varepsilon_i s_i \lambda^i \mu^{r-i}$ and $A^{(r)} = (s^{(r)})_0$. Denote by $s_i^{(r)}$ (resp. $t_i^{(r)}$) $(i=0,1,2)$ the number of real critical points of $\varphi = \psi | A^{(r)}$ of index i (resp. $\dim H_i(A^{(r)}{}_{\mathbb{R}}; \mathbb{Z}/2)$).

Identify $H^4(S;\mathbb{Z})$ with $\mathbb{Z}$ by the fundamental class $[S]$.

(3.1) Proof of Theorem 0.4. By (1.2), $g(Z)$ is equal to $1 + (1/2)(c_1(L)^2 - c_1(L)c_1(TS))$.

Let N be $S_{\mathbb{R}}$ minus the interiors of $2g(Z)$ empty ovals. Put $M = \{(x;\lambda,\mu) \in A^{(r)}{}_{\mathbb{R}} \mid |s^{(r-1)}(x;\lambda,\mu)| \geq \delta,\ x \in N\}$ for a positive number δ with $|\varepsilon_{r-1}| \gg \delta \gg |\varepsilon_r| > 0$. Then M is a C^∞

manifold with boundary such that $\chi(M) = \chi(S_R) - 2g(Z)$.

Set $w = \operatorname{grad}\varphi_{\mathbb{R}}|M$. Then, with respect to w, $\chi(M_1)$ is equal to $c_1(L)^2$ (cf. 2.9) and M_2 is empty. Thus we see

$$\operatorname{index} w = \chi(M) - \chi(M_1) = \chi(S_R) - 2g(Z) - c_1(L)^2.$$

Therefore on M, the number of critical point of $\varphi_{\mathbb{R}}$ of index 1 is not less than $-\operatorname{index} w = c_1(L)^2 + 2g(Z) - \chi(S_{\mathbb{R}})$.

Thus we have

$$s_1^{(r)} - s_1^{(r-1)} \geqq 2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_{\mathbb{R}}) + 2,$$

$$s_0^{(r)}+s_2^{(r)} - (s_0^{(r-1)}+s_2^{(r-1)}) \geqq 2g(Z)$$
$$= c_1(L)^2 - c_1(L)c_1(TS) + 2,$$

$$s_0^{(0)} = s_1^{(0)} = s_2^{(0)} = 0.$$

So we have

$$s_1^{(r)} \geqq r(2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_{\mathbb{R}}) + 2) \quad \dots (1),$$
$$s_0^{(r)} + s_2^{(r)} \geqq r(c_1(L)^2 - c_1(L)c_1(TS) + 2) \quad \dots (2).$$

By (2.5), $\chi(S) + \chi(S_{\mathbb{R}}) = 4$. Hence we have

$$s(\varphi_{\mathbb{R}}) = s^{(r)} + s_1^{(r)} + s_2^{(r)}$$
$$\geqq r(3c_1(L)^2 - 2c_1(L)c_1(TS) + c_2(TS)) \quad \dots (3).$$

By (1.5), equalities in (1), (2) and (3) hold. Thus we have

$$\chi(A_{\mathbb{R}}) = s_0^{(r)} - s_1^{(r)} + s_2^{(r)}$$

$$= r(-c_1(L)^2 - c_2(TS) + 4) \qquad \ldots (4).$$

On the other hand, becanse of the existence of ovals, we have

$$t_0^{(r)} + t_2^{(r)} - (t_0^{(r-1)} + t_2^{(r-1)}) \geqq 2g(Z),$$

$$t_0^{(1)} + t_2^{(1)} \geqq 2.$$

Thus we have

$$t_0^{(r)} + t_2^{(r)} \geqq 2g(Z)(r-1) + 2 \qquad \ldots (5).$$

Therefore, by (4), (5) and (1.3), we have

$$\begin{aligned} P_1(A_R;\mathbb{Z}/2) &= t_0^{(r)} + t_1^{(r)} + t_2^{(r)} \\ &= 2(t_0^{(r)} + t_2^{(r)}) - \chi(A_{\mathbb{R}}) \\ &\geqq (3r-2)c_1(L)^2 - (2r-2)c_1(L)c_1(TS) + rc_2(TS) \\ &= P_1(A;\mathbb{Z}/2) \qquad \ldots (6). \end{aligned}$$

By Harnack-Thom's inequalty $P_1(A_{\mathbb{R}};\mathbb{Z}/2) \leqq P_1(A;\mathbb{Z}/2)$. Hence equalities in (5) and (6) hold. This completes the proof of Theorem 0.4.

(3.2) Example. Let us consider the case $S = \mathbb{P}^2$. Let A be a non-singular surface of $\mathbb{P}^2 \times \mathbb{P}^1$ of degree (d,r). Then $\chi(A) = P_1(A;\mathbb{Z}/2) = 3 + d^2 + 3(d-1)^2(r-1)$.

If $\varphi : A \longrightarrow \mathbb{P}^1$ has only isolated critical points, then $s(\varphi) = \sum_{x \in A} \mu_x(\varphi) = 3(d-1)^2 r$, where $\mu_x(\varphi)$ is the Milnor number of φ at x.

Proposition. Let $A \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a non-singular real surface of degree (d,r) such that $\varphi: A \longrightarrow \mathbb{P}^1$ has only isolated critical points. Then we have the sharp estimate

$$P_1(A_{\mathbb{R}};\mathbb{Z}/2) \leqq 3 + d^2 + 3(d-1)^2(r-1),$$
$$(A_{\mathbb{R}}) \leqq 3(d-1)^2 r.$$

Example. Let $\mathcal{A} = \{\lambda F + \mu G \,/\, [\lambda:\mu] \in \mathbb{P}^1\}$ be a pencil of real plane curves in $\mathbb{P}^2$ of degree d.

$A = (\lambda F + \mu G)_0 \subset \mathbb{P}^2 \times \mathbb{P}^1$ is non-singular if and only if $(F)_0$ and $(G)_0$ intersect transversely in $\mathbb{P}^2$. If A is non-singular, then $A \cong \mathbb{P}^2 \underbrace{\# -\mathbb{P}^2 \# \ldots \# -\mathbb{P}^2}_{d^2}$. In this case, if $(F)_0$ and $(G)_0$ intersect in k points ($0 \leqq k \leqq d^2$, $k \equiv d \pmod{2}$), then $A_{\mathbb{R}} \cong \underset{1+k}{\#} \mathbb{R}P^2$. Thus A is an M-surface if and only if $k = d^2$.

4. Construction of M-curves in a surface.

Let S be a compact real complex surface, L, L' real holomorphic line bundles, s, s' M-sections of L, L' respectively.

Put $C = (s)_0$ and $C' = (s')_0$. Assume that C and C' are both rational and $CC' = \langle c_1(L)c_1(L'),[S]\rangle \geqq 0$. (This assumption for S is rather restrictive (cf. [BPV], Proposition V.4.3).

Consider the following condition:

(**) For any effective divisor α on C of degree CC' with $\operatorname{supp}\alpha \subseteq C_{\mathbb{R}}$, there exists a real section $s'' \in H^0(S,L')_{\mathbb{R}}$ such that $(s'')_0|C = \alpha$.

Theorem 4.0. Under the condition (**), for any natural numbers d and e, $L^{\otimes d} \otimes L'^{\otimes e}$ has an M-section near $s^{\otimes d} \otimes s'^{\otimes e}$ in $H^0(S,L^{\otimes d} \otimes L'^{\otimes e})_{\mathbb{R}}$. Furthermore, if CC' is positive, then $L^{\otimes d} \otimes L'^{\otimes e}$ has a pair of M-sections near $s^{\otimes d} \otimes s'^{\otimes e}$ satisfying (*) (cf. Introduction).

Corollary 4.1. If $C^2 \geqq 0$, then under the condition (**) for $C' = C$, for any natural number d, $L^{\otimes d}$ has an M-section near $s^{\otimes d}$. Furthermore, if C^2 is positive, then $L^{\otimes d}$ has a pair of M-sections near $s^{\otimes d}$ satisfying (*).

(4.2) Example. (1) $S = \mathbb{P}^2$, $L = L' = \mathcal{O}_{\mathbb{P}^2}(1)$ (This corresponds to the Harnack's method).

(2) $S = \mathbb{P}^2$, $L = L' = \mathcal{O}_{\mathbb{P}^2}(2)$, $C = C'$: a real ellipse with $C_{\mathbb{R}} \neq \emptyset$ (This corresponds to the Hilbert's method).

(3) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = L' = p_1{}^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2{}^*\mathcal{O}_{\mathbb{P}^1}(1)$.

(4) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = p_1{}^*\mathcal{O}_{\mathbb{P}^1}(1)$, $L' = p_2{}^*\mathcal{O}_{\mathbb{P}^1}(1)$ (This is used to show Proposition 0.2 and Corollary 0.6).

References.

[A] M.F. Atiyah, K-theory and reality, Quart. J. Math., 17 (1966), 367-386.

[BPV] W. Barth, C. Peters, A. Van de Ven, Compact Complex Surfaces, Ergebnisse der Math., 3-4, Springer-Verlag, 1984.

[F] W. Fulton, Intersection Theory, Ergebnisse der Math., 3-2, Springer-Verlag, 1984.

[G] D.A. Gudkov, The topology of real projective algebraic varieties, Usp. Mat. Nauk 29-4 (1974), 3-79.

[H] A. Harnack, Über die Vielheiligkeit der ebenen algebraischen Curven, Math. Ann., 10 (1876), 189-199.

[I] G. Ishikawa, The number of singular points in a pencil of real plane algebraic curves, Japanese J. Math., 11-2 (1985), 387-401.

[P] C.C. Pugh, A generalized Poincare index formula, Topology 7 (1968), 217-226.

[S] R. Silhol, Real abelian varieties and the theory of Comessatti, Math. Z., 181 (1982), 345-364.

[V] O.Ya. Viro, Construction of M-surfaces, Funkt. Anal. Prilozhen., 13-3 (1979), 71-72.

(March.24.1986).