

One-sided phase constraint $x(t) \geq 0$ forms an envelope

相条件 $x(t) \geq 0$ は包絡線を生成する¹

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Introduction

In this paper, we are concerned with the following max-type function:

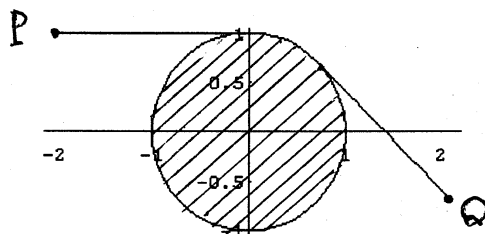
$$S(x) := \max_{t \in T} G(x(t), t) \quad x \in X, \quad (1)$$

where T is a compact metric space, X is a subspace of the set of all n -dimensional vector-valued continuous functions $C(T)^n$ equipped with the uniform norm. We denote by G_x and G_{xx} the gradient (row) vector and the Hesse matrix of f w.r.t. x , respectively, and assume them to be continuous on $R^n \times T$. This max-type function is induced from a phase constraint

$$G(x(t), t) \leq 0 \quad \forall t \in T,$$

which appears in variational problems and optimal control problems. For instance, a variational problem to find the shortest path in R^2 joining two given points P and Q that does not transverse the unit ball is formulated as follows:

$$\begin{aligned} & \text{Minimize} \quad \int_0^1 \sqrt{\dot{x}_1^2 + \dot{x}_2^2} dt \\ & \text{subject to} \quad (x_1(0), x_2(0)) = P, \quad (x_1(1), x_2(1)) = Q, \\ & \quad \quad \quad 1 - x_1^2(t) - x_2^2(t) \leq 0 \quad \forall t \in [0, 1]. \end{aligned}$$



There are two aims in this paper. First one is to give formulae for first- and second-order directional derivatives of $S(x)$. Second one is to show that one-sided phase constraint $x(t) \geq a(t)$, where $a(t)$ is a given continuous function, always forms an envelope except two trivial cases:

$$\begin{aligned} & x(t) \equiv a(t), \\ & x(t) > a(t) \quad \text{for every } t. \end{aligned}$$

¹This research is partially supported by the Grant-in-Aid for General Scientific Research from the Ministry of Education, Science and Culture, No. 08640294

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By the way, there are a lot of papers that dealt with another max-type function:

$$S_0(x) := \max_{t \in T} G(x, t) \quad x \in R^n, \quad (2)$$

Clarke[1], Correa and Seeger[2], Danskin [3], Dem'yanov and Malozemov[4] Demyanov and Zabrodin[5], Hettich and Jongen[6], Ioffe[7], Kawasaki[8][9] [10][11][13], Shiraishi[17], Seeger[16], Wetterling[18]. We encounter this max-type function, for example, in Tchebycheff approximation. The latter max-type function $S_0(x)$ is a special case of $S(x)$. Indeed, if we take as $X \{x \in C(T)^n \mid x(t) \equiv \text{constant vector} \in R^n\}$, then $S(x)$ reduces to $S_0(x)$. So $S(x)$ inherits a lot of properties from $S_0(x)$.

論文の概要

次の max-型関数の 1 次と 2 次の方向微分について考察する。

$$S(x) := \max_{t \in T} G(x(t), t) \quad x \in X, \quad (3)$$

ただし T はコンパクト距離空間, X は n 次元ベクトル値連続関数全体 $C(T)^n$ の部分空間とする。この max-型関数は変分問題や最適制御問題の相条件

$$G(x(t), t) \leq 0 \quad \forall t \in T$$

を考察するとき出会う。本論文では、この相条件から包絡線が生成されるかどうかを調べるために、max-型関数 $S(x)$ の 2 次の方向微分を表す公式を与える。

ところで、従来よく研究されてきた max-型関数は次の関数である。

$$S_0(x) := \max_{t \in T} G(x, t) \quad x \in R^n, \quad (4)$$

この関数はチェビシェフ近似問題と密接に関係する。さらに、集合 T が x に依存してよいとすれば、 $S_0(x)$ の最小化問題はパラメトリック最適化問題になる。 $S(x)$ が $S_0(x)$ と本質的に異なる点は、後者は x と t が独立に動けるのに対し、前者は x が t に依存することである。しかしながら、 $S_0(x)$ は $S(x)$ のスペシャルケースと見なすこともできる。つまり、 X として n 次元ベクトル値定数関数全体 $\{x(t) \equiv a \mid a \in R^n\}$ をとればよい。従って、 $S(x)$ は $S_0(x)$ の多くの性質を受け継ぐことになる。その結果、相条件も包絡線を生成する。より正確に言えば、片側相制約 $x(t) \geq a(t)$ について、二つの自明なケース：

$$\bar{x}(t) \equiv a(t),$$

$$\bar{x}(t) > a(t) \quad \text{for every } t.$$

を除いて、点 $\bar{x}$ において包絡線を生成する方向 y を選ぶ事が出来る。

In the following, we denote by $T(x)$ the set of all extreme points $G(x(\cdot), \cdot)$, that is,

$$T(x) := \{t \in T ; G(x(t), t) = S(x)\}, \quad x \in C(T)^n.$$

THEOREM 1 *The function $S(x)$ is continuous.*

THEOREM 2 *The function $S(x)$ is directionally differentiable in any direction $y \in X$, and its directional derivative is given by*

$$S'(x; y) = \max\{G_x(x(t), t)y(t); t \in T(x)\}. \quad (5)$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 2, we get Danskin's formula.

COROLLARY 1 (*Danskin[3]*) *The function $S_0(x)$ is directionally differentiable in any direction $y \in R^n$ and its directional derivative is given by*

$$S'_0(x; y) = \max\{G_x(x, t)y; t \in T(x)\}. \quad (6)$$

Next, we consider a second-order directional derivative of $S(x)$.

DEFINITION 1 *The upper second-order directional derivative of $S(x)$ at x in the direction y is defined by*

$$\bar{S}''(x; y) = \limsup_{\varepsilon \rightarrow +0} \frac{S(x + \varepsilon y) - S(x) - \varepsilon S'(x; y)}{\varepsilon^2} \quad (7)$$

DEFINITION 2 (*[9]*) *For any functions $u, v \in C(T)$ satisfying*

$$\begin{cases} u(t) \geq 0 & \forall t \in T, \\ v(t) \geq 0 & \text{if } u(t) = 0, \end{cases} \quad (8)$$

we define a function $E : T \rightarrow [-\infty, +\infty]$ by

$$E(t) := \begin{cases} \sup \left\{ \limsup_{n \rightarrow \infty} \frac{v(t_n)^2}{4u(t_n)}; \{t_n\} \text{ satisfies (10)} \right\}, & \text{if } t \in T_0, \\ 0 & \text{if } u(t) = v(t) = 0 \text{ and } t \notin T_0, \\ -\infty & \text{otherwise,} \end{cases} \quad (9)$$

$$T_0 := \left\{ t \in T; \exists t_n \rightarrow t \text{ s.t. } u(t_n) > 0, -\frac{v(t_n)}{u(t_n)} \rightarrow +\infty \right\}. \quad (10)$$

THEOREM 3 *Let x and y be arbitrary functions in $C(T)^n$. Then it holds that*

$$\bar{S}''(x; y) = \max \left\{ \frac{y(t)^T G_{xx}(x(t), t)y(t)}{2} + E(t); t \in T(x; y) \right\}, \quad (11)$$

where $T(x; y) := \{t \in T(x); S'(x; y) = G_x(x(t), t)y(t)\}$ and $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x(t), t), \quad v(t) = S'(x; y) - G_x(x(t), t)y(t). \quad (12)$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 3, we get the following formula due to [9].

COROLLARY 2 *Let x and y be arbitrary points in R^n . Then it holds that*

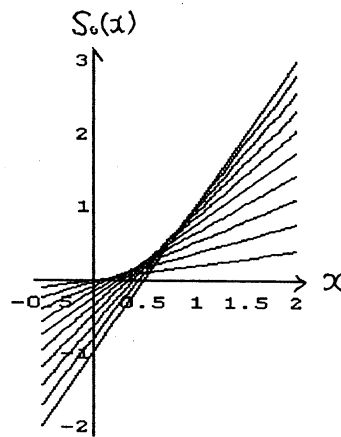
$$\bar{S}''(x; y) = \max \left\{ \frac{y^T G_{xx}(x, t)y}{2} + E(t) ; t \in T(x; y) \right\}, \quad (13)$$

where $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x, t), \quad v(t) = S'(x; y) - G_x(x, t)y. \quad (14)$$

We proved in [9] [10] that an envelope is formed from $G(x, t)$ when $E(t) > 0$ at some point $t \in T(x; y)$.

EXAMPLE 1 *Let us consider a family of straight lines $f(x, t) = 2tx - t^2$, $t \in [0, 1]$, $x \in R$. It is evident that it forms an envelope $S_0(x) = x^2$, $(0 \leq x \leq 1)$.*



Hence $S_0''(0; y) = y^2$ for any $y \geq 0$ and $f_{xx}(0, t) \equiv 0$. Thus there is a gap between the second-order directional derivatives of the max-type function $S_0(x)$ and its constituent functions $f(x, t)$. On the other hand, it is directly computed from the definition that $E(0) = y^2$ for every $y > 0$, which fills the gap.

$$\begin{aligned} \bar{S}_0''(0; y) &= \max \left\{ \frac{1}{2} y^T f_{xx}(0, t)y + E(t) ; t \in T(0; y) \right\} \\ &= E(0) = y^2 \end{aligned}$$

It is reasonable to guess that an envelope is formed from the phase constraint $x(t) \geq a(t)$ when the function $E(t)$ is positive at some $t \in T(x; y)$ as well as the max-type function $S_0(x)$. However this is not a proof but a guess. So we next give a proof that the one-sided phase constraint $x(t) \geq a(t)$ certainly forms an envelope for a certain direction $y(t)$ except two trivial cases.

THEOREM 4 *Let T be a connected compact metric space. Assume that $\bar{x}(t)$ does not satisfy neither*

$$x(t) \equiv a(t), \quad (15)$$

nor

$$x(t) > a(t) \text{ for every } t. \quad (16)$$

Then there exists a function $y \in C(T)$ such that the one-sided phase constraint $x(t) \geq a(t)$ forms an envelope in the direction y .

Proof. Let $y(t) := -2\sqrt{x(t) - a(t)}$ and put for $\xi \in R$

$$\begin{aligned} s(\xi) &:= S(\bar{x} + \xi y) = \max_t \{a(t) - \bar{x}(t) - \xi y(t)\} \\ &= \max_t \{a(t) - \bar{x}(t) + 2\xi \sqrt{\bar{x}(t) - a(t)}\} \end{aligned}$$

Then $s(\xi)$ becomes a standard max-function.

$$s(\xi) = \max_{\tau \in T'} \{2\xi\tau - \tau^2\}$$

Furthermore, from the assumption, the image of T by the continuous function $\sqrt{x(t) - a(t)}$ is a compact interval $T' := [0, t_1]$ with $t_1 > 0$. Hence $s(\xi)$ is same with Example 1, so that an envelope is formed.

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