

Non-commutative crystalline comparison theorem

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1 Introduction

1.1 Setting

K : finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$: integral ring of K , k : residue field, $\mathcal{C} := \widehat{\widehat{K}}$, $W := W(k)$

1.2 p -adic Hodge theory

In algebraic geometry, the concept of *period* refers to an entry of the matrix of the de Rham isomorphism:

$$H_{\text{dR}}^*(M) \simeq H_{\text{Sing}}^*(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

for a complex algebraic variety M defined over a number field K . To study the p -adic analogue of the notion of periods, Fontaine introduces the p -adic period ring B_{crys} . Building on Fontaine's work, Tsuji and Faltings proved the following.

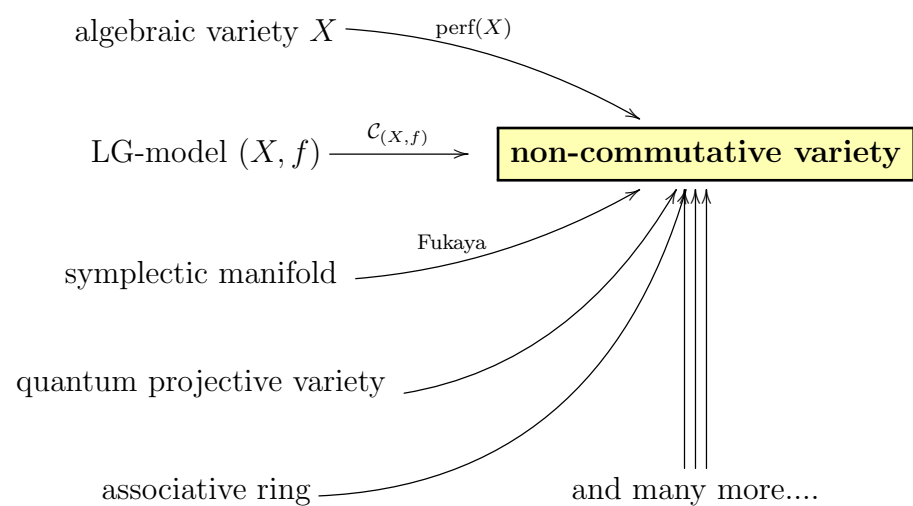
Theorem 1.1 (Tsuji, Faltings). *Let X be a smooth proper variety $\mathcal{O}_K$. There is an isomorphism*

$$H^i(X_k/W) \otimes_W B_{\text{crys}} \simeq H^i(X_{\mathcal{C}}; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{crys}}$$

which is compatible with the G_K -actions, the Frobenius endmorphisms and the filtration. In particular, $G_K \curvearrowright H^i(X_{\mathcal{C}}; \mathbb{Q}_p)$ is a crystalline representation.

1.3 Non-commutative variety

Let R be a commutative ring. Orlov and Kontsevich introduced *non-commutative algebraic geometry* in which an R -linear, idempotent-complete, small stable ∞ -category is studied as a *non-commutative variety over R* . Non-commutative algebraic geometry has been widely used in various areas of physics and mathematics.



Definition 1.2 (Orlov). A non-commutative variety $\mathcal{T}$ over R admits a *geometric realization* if there are a derived scheme $\mathcal{X}$ over R s.t. $\pi_0(\mathcal{X})$ is separated scheme of finite type over R and there is a fully faithful admissible inclusion $\mathcal{T} \hookrightarrow \text{perf}(\mathcal{X})$.

In many cases, a smooth proper non-commutative variety is known (or expected) to admit a geometric realization.

1.4 Non-commutative cohomology

Tsygan and Connes defined cyclic periodic homology $\text{HP}(A)$ for an associative ring A which can be regarded as the *non-commutative de Rham cohomology*. Similarly, there are non-commutative versions of Betti, étale, crystalline and Hodge cohomology groups for non-commutative variety $\mathcal{T}$ over R :

variety X over R	NC variety $\mathcal{T}$ over R	
$H_{\text{dR}}^*(X/R)$	$\pi_* \text{HP}(\mathcal{T}/R)$	
$H_{\text{Sing}}^*(X, \mathbb{Z})$	$\pi_* K^{\text{top}}(\mathcal{T})$	$R = \mathbb{C}$
$H_{\text{Zar}}^*(X, \Omega_{X/R}^*)$	$\pi_* \text{HH}(\mathcal{T}/k)$	
$H_{\text{ét}}^*(X, \mathbb{Z}_p)$	$\pi_* L_{K(1)} K(\mathcal{T})$	$(\text{ch}(R), p) = 1$
$H_{\text{crys}}^*(X/W(R))$	$\pi_* \text{TP}(\mathcal{T}; \mathbb{Z}_p)$	$R = \mathbb{F}_{p^n}$

1.5 Comparison theorems

Some of comparison theorems between non-commutative cohomologies are previously known or expected. For a smooth proper NC variety $\mathcal{T}$ over $\mathbb{C}$, Kontsevich expected that there is the *non-commutative Hodge decomposition*:

$$\pi_i \text{HP}(\mathcal{T}/\mathbb{C}) \simeq \bigoplus_{n \in \mathbb{Z}} \pi_{i+2n} \text{HH}(\mathcal{T}/\mathbb{C}),$$

and Kaledin proved it. For a smooth proper NC variety $\mathcal{T}$ over $W(\mathbb{F}_{p^n})$, Scholze announced that there is the *non-commutative Berthelot-Ogus isomorphism*:

$$\pi_i \text{TP}(\mathcal{T}_{\mathbb{F}_{p^n}}; \mathbb{Z}_p) \simeq \pi_i \text{HP}(\mathcal{T}/W(\mathbb{F}_{p^n})).$$

2 Non-commutative p -adic Hodge

Inspired by Kaledin and Scholze's results, we expected the following comparison theorem, which can be regarded as a non-commutative analogue of Theorem 1.1.

Conjecture 2.1 (M, 2022). *Let $\mathcal{T}$ be a smooth proper NC variety over $\mathcal{O}_K$. Then there is an isomorphism*

$$\pi_i \text{TP}(\mathcal{T}_k; \mathbb{Z}_p) \otimes_W B_{\text{crys}} \simeq \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} B_{\text{crys}}$$

which is compatible with the G_K -actions, the Frobenius endmorphisms. In particular, $G_K \curvearrowright \pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a crystalline representation.

Example 2.2. If $\mathcal{T} = \text{perf}(X)$ for a smooth proper variety X over $\mathcal{O}_K$, then the conjecture holds for $\mathcal{T}$ by Theorem 1.1.

$\therefore$ There are the following canonical isomorphisms due to Thomason and Bhatt-Morrow-Scholze:

$$\pi_i L_{K(1)} K(\mathcal{T}_{\mathcal{C}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \bigoplus_{n \in \mathbb{Z}} H_{\text{ét}}^{i+2n}(X_{\mathcal{C}}, \mathbb{Q}_p(n))$$

$$\pi_i \text{TP}(\mathcal{T}_k; \mathbb{Z}_p) \left[\frac{1}{p} \right] \simeq \bigoplus_{n \in \mathbb{Z}} H_{\text{crys}}^{i+2n}(X_k/W \left[\frac{1}{p} \right])(n)$$

3 Main result

We can prove a non-commutative version of Bhatt-Morrow-Scholze's comparison theorem. By using it and Du-Liu's prismatic $(\phi, \hat{G})$ -mod theory, we have the following result.

Theorem 3.1 (M, 2023). *Let $\mathcal{T}$ be a smooth proper NC variety over $\mathcal{O}_K$. We assume that $\mathcal{T}_{\mathcal{C}}$ admits a geometric realization, then Conjecture 2.1 holds for $\mathcal{T}$.*