

A generalization class of certain subclasses of P -valently analytic functions with negative coefficients*

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Abstract

Recently we [5] have discussed a new generalization class $A(n, \alpha, \beta)$ of certain subclasses of analytic functions with negative coefficients in the unit disk and have proved some properties of functions belonging to the class $A(n, \alpha, \beta)$. In the present paper we introduce a new generalization class $A_p(n, \alpha, \beta)$ of certain subclasses of p -valently analytic functions with negative coefficients in the unit disk and discuss some properties of functions belonging to the class $A_p(n, \alpha, \beta)$.

1. Introduction

Let p be a positive integer, and let $A_p(n)$ denote the class of functions of the form

$$(1.1) \quad f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0, n \in N = \{1, 2, 3, \dots\}),$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z)$ in the class $A_p(n)$ is said to be a member of the class $R_p(n, \alpha)$ if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{pf(z)}{z^p} \right\} > \alpha \quad (z \in U)$$

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for some $\alpha(0 \leq \alpha < p)$. Further, a function $f(z)$ in the class $A_p(n)$ is said to be in the class $P_p(n, \alpha)$ if it satisfies

$$(1.3) \quad \operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in U)$$

for some $\alpha(0 \leq \alpha < p)$.

By generalization of some results due to Sarangi and Uralegaddi [2], we see that

LEMMA A. A function $f(z) \in A_p(n)$ is in the class $R_p(n, \alpha)$ if and only if

$$(1.4) \quad \sum_{k=n+p}^{\infty} \frac{p}{p-\alpha} a_k \leq 1.$$

LEMMA B. A function $f(z) \in A_p(n)$ is in the class $P_p(n, \alpha)$ if and only if

$$(1.5) \quad \sum_{k=n+p}^{\infty} \frac{k}{p-\alpha} a_k \leq 1.$$

Now, we define

DEFINITION. Suppose that $f(z) \in A_p(n)$, $0 \leq \alpha < p$ and $\beta \geq 0$. Then the function $f(z)$ is said to be a member of the class $A_p(n, \alpha, \beta)$ if it satisfies

$$(1.6) \quad \operatorname{Re} \left\{ (1-\beta) \frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in U).$$

We note that $A_p(n, \alpha, 0) = R_p(n, \alpha)$ and $A_p(n, \alpha, 1) = P_p(n, \alpha)$. We have

LEMMA 1. Suppose that $f(z) \in A_p(n)$, $0 \leq \alpha < p$ and $\beta \geq 0$. Then the function $f(z)$ is in the class $A_p(n, \alpha, \beta)$ if and only if

$$(1.7) \quad \sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha} \right\} a_k \leq 1.$$

PROOF: Let $f(z) \in A_p(n, \alpha, \beta)$. Then we have, by (1.6),

$$(1.8) \quad \begin{aligned} & \operatorname{Re} \left\{ (1-\beta) \frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} \\ &= \operatorname{Re} \left\{ p - \sum_{k=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_k z^{k-p} \right\} \\ &> \alpha \quad (z \in U). \end{aligned}$$

Letting $z \rightarrow 1$ through real values, we obtain (1.7). Conversely, let $f(z) \in A_p(n)$ satisfy inequality (1.7). Then we have

$$(1.9) \quad \begin{aligned} & \left| \left\{ (1-\beta) \frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} - p \right| \\ &= \left| \sum_{k=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_k z^{k-p} \right| \\ &\leq \sum_{k=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_k |z|^{k-p} \\ &< p - \alpha \quad (z \in U). \end{aligned}$$

This proves that inequality (1.6) holds true. ■

The class $A_1(n, \alpha, \beta)$ is a special case $\left(B_k = \frac{1+(k-1)\beta}{1-\alpha} \right)$ of the class $A(n, B_k)$ introduced by Sekine [3].

2. Distortion Theorem

THEOREM 1. If $f(z) \in A_p(n, \alpha, \beta)$ for $0 \leq \alpha < p$ and $\beta \geq 0$, then

$$(2.1) \quad |z|^p - \frac{p-\alpha}{p+n\beta}|z|^{n+p} \leq |f(z)| \leq |z|^p + \frac{p-\alpha}{p+n\beta}|z|^{n+p} \quad (z \in U)$$

for $\beta \geq 0$, and

$$(2.2) \quad \begin{aligned} |f'(z)| &\leq p|z|^{p-1} + \frac{(p-\alpha)(n+p)}{p+n\beta}|z|^{n+p-1} & (z \in U) \\ |f'(z)| &\geq p|z|^{p-1} - \frac{(p-\alpha)(n+p)}{p+n\beta}|z|^{n+p-1} & (z \in U) \end{aligned}$$

for $\beta \geq 1$. The equalities in (2.1) and (2.2) are attained for the function

$$(2.3) \quad f(z) = z^p - \frac{p-\alpha}{p+n\beta}z^{n+p}.$$

PROOF: Note that

$$(2.4) \quad \sum_{k=n+p}^{\infty} a_k \leq \frac{p-\alpha}{p+n\beta} \quad (\beta \geq 0)$$

and

$$(2.5) \quad \frac{p+n\beta}{n+p} \sum_{k=n+p}^{\infty} ka_k \leq \sum_{k=n+p}^{\infty} \{(1-\beta)p + \beta k\}a_k \leq p-\alpha \quad (\beta \geq 1)$$

for $f(z) \in A_p(n, \alpha, \beta)$. Therefore, we have (2.1) and (2.2). ■

Remark. Putting $p = 1$ in Theorem 1, we have the corresponding result due to Yaguchi, Sekine, Saitoh, Owa, Nunokawa and Fukui [5].

3. Inclusion Relations

THEOREM 2. If

$$(3.1) \quad \begin{aligned} 0 &\leq \alpha_1 < p, \quad 0 \leq \alpha_2 < p, \\ 0 &\leq \beta_1, \quad 0 \leq \beta_2, \quad p(\beta_1 - \beta_2) < \alpha_2\beta_1 - \alpha_1\beta_2, \\ p\{\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)n\} &\leq n(\alpha_2\beta_1 - \alpha_1\beta_2), \end{aligned}$$

then we have

$$(3.2) \quad A_p(n, \alpha_2, \beta_2) \subsetneq A_p(n, \alpha_1, \beta_1).$$

PROOF: Suppose $f(z) \in A_p(n, \alpha_2, \beta_2)$. Since by Lemma 1

$$(3.3) \quad \sum_{k=n+p}^{\infty} \frac{(1-\beta_2)p + k\beta_2}{p - \alpha_2} a_k \leq 1,$$

we have only to prove the inequality

$$(3.4) \quad \frac{(1-\beta_1)p + k\beta_1}{p - \alpha_1} \leq \frac{(1-\beta_2)p + k\beta_2}{p - \alpha_2} \quad (k \geq n+p),$$

which is equivalent to the inequality

$$(3.5) \quad k \geq \frac{p\{(\beta_2 - \beta_1)p + \alpha_1 - \alpha_2 + \alpha_2\beta_1 - \alpha_1\beta_2\}}{(\beta_2 - \beta_1)p + \alpha_2\beta_1 - \alpha_1\beta_2} \quad (k \geq n+p).$$

But conditions (3.1) lead to the inequality

$$(3.6) \quad \frac{p\{(\beta_2 - \beta_1)p + \alpha_1 - \alpha_2 + \alpha_2\beta_1 - \alpha_1\beta_2\}}{(\beta_2 - \beta_1)p + \alpha_2\beta_1 - \alpha_1\beta_2} \leq n+p,$$

which proves (3.5). The function $f_0(z)$ defined by

$$(3.7) \quad f_0(z) = z^p - \frac{p - \alpha_1}{p + (n+1)\beta_1} z^{p+n+1}$$

belongs to the class $A_p(n, \alpha_1, \beta_1) - A_p(n, \alpha_2, \beta_2)$, which proves

$$(3.8) \quad A_p(n, \alpha_1, \beta_1) \neq A_p(n, \alpha_2, \beta_2). \quad \blacksquare$$

COROLLARY 1. If

$$(3.9) \quad 0 \leq \alpha_1 \leq \alpha_2 < p, \quad 0 \leq \beta_1 \leq \beta_2, \quad (\beta_2 - \beta_1) + (\alpha_2 - \alpha_1) > 0,$$

then we have

$$(3.10) \quad A_p(n, \alpha_2, \beta_2) \subsetneq A_p(n, \alpha_1, \beta_1)$$

PROOF: By Theorem 2, we have

$$(3.11) \quad \begin{aligned} A_p(n, \alpha_2, \beta_1) &\subsetneq A_p(n, \alpha_1, \beta_1) & (0 \leq \alpha_1 < \alpha_2 < p), \\ A_p(n, \alpha_2, \beta_2) &\subsetneq A_p(n, \alpha_2, \beta_1) & (0 \leq \beta_1 < \beta_2), \end{aligned}$$

which prove Corollary 1. ■

COROLLARY 2. If $0 < \beta_1 < 1 < \beta_2$, then

$$(3.12) \quad A_p(n, \alpha, \beta_2) \subsetneq P_p(n, \alpha) \subsetneq A_p(n, \alpha, \beta_1) \subsetneq R_p(n, \alpha).$$

4. Starlikeness

A function $f(z)$ in the class $A_p(n)$ is said to be p -valently starlike of order α if it satisfies

$$(4.1) \quad \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in U)$$

for some α ($0 \leq \alpha < p$). We need the following lemma which is a generalization of a result due to Chatterjea [1] (also Srivastava, Owa and Chatterjea [4]).

LEMMA C. A function $f(z) \in A_p(n)$ is p -valently starlike of order γ if and only if

$$(4.2) \quad \sum_{k=n+p}^{\infty} \frac{k-\gamma}{p-\gamma} a_k \leq 1$$

for some γ ($0 \leq \gamma < p$).

Lemma C is proved by using the similar method as in Chatterjea [1]. Using Lemma C, we have

THEOREM 3. If $f(z) \in A_p(n, \alpha, \beta)$ for $0 \leq \alpha < p$ and $\beta \geq 1$, then $f(z)$ is starlike of order $(1 - \frac{1}{\beta})p$.

PROOF: It follows from $f(z) \in A_p(n, \alpha, \beta)$ that

$$(4.3) \quad \sum_{k=n+p}^{\infty} \{k - (1 - \frac{1}{\beta})p\} a_k \leq \frac{p - \alpha}{\beta} \leq p - (1 - \frac{1}{\beta})p.$$

Therefore, by Lemma C, we have the assertion of Theorem 3. ■

5. Quadi-Hadamard product

For functions $f_1(z)$ and $f_2(z)$ defined by

$$(5.1) \quad f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{j,k} z^k \quad (a_{j,k} \geq 0, n \in N, j = 1, 2)$$

in the class $A_p(n)$, we denote by $f_1 * f_2(z)$ the quasi-Hadamard product of functions $f_1(z)$ and $f_2(z)$, that is,

$$(5.2) \quad f_1 * f_2(z) = z^p - \sum_{k=n+p}^{\infty} a_{1,k} a_{2,k} z^k.$$

THEOREM 4. If $f_j(z) \in A_p(n, \alpha_j, \beta)$ for $0 \leq \alpha_j < p, \beta \geq 0$ and $j = 1, 2$, then $f_1 * f_2(z) \in A_p(n, \alpha, \beta)$, where

$$(5.3) \quad \alpha = p - \frac{(p - \alpha_1)(p - \alpha_2)}{p + \beta n}.$$

The result is sharp for functions $f_1(z)$ and $f_2(z)$ defined by

$$(5.4) \quad f_j(z) = z^p - \frac{p - \alpha_j}{p + \beta n} z^{n+p} \quad (j = 1, 2).$$

PROOF: We have to find the largest α such that

$$(5.5) \quad \sum_{k=n+p}^{\infty} \frac{(1 - \beta)p + \beta k}{p - \alpha} a_{1,k} a_{2,k} \leq 1.$$

For functions $f_j(z) \in A_p(n, \alpha_j, \beta)$, we have

$$(5.6) \quad \sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha} \right\} a_{j,k} \leq 1 \quad (j = 1, 2).$$

By the Cauchy-Schwarz inequality, inequality (5.6) lead to the inequality

$$(5.7) \quad \sum_{k=n+p}^{\infty} \frac{(1-\beta)p + \beta k}{\sqrt{(p - \alpha_1)(p - \alpha_2)}} \sqrt{a_{1,k} a_{2,k}} \leq 1.$$

Therefore, it is sufficient to prove that

$$(5.8) \quad \begin{aligned} & \frac{(1-\beta)p + \beta k}{p - \alpha} a_{1,k} a_{2,k} \\ & \leq \frac{(1-\beta)p + \beta k}{\sqrt{(p - \alpha_1)(p - \alpha_2)}} \sqrt{a_{1,k} a_{2,k}} \quad (k \geq n + p), \end{aligned}$$

that is, that

$$(5.9) \quad \sqrt{a_{1,k} a_{2,k}} \leq \frac{p - \alpha}{\sqrt{(p - \alpha_1)(p - \alpha_2)}} \quad (k \geq n + p).$$

From (5.7), we need to show that

$$(5.10) \quad \frac{\sqrt{(p - \alpha_1)(p - \alpha_2)}}{(1 - \beta)p + \beta k} \leq \frac{p - \alpha}{\sqrt{(p - \alpha_1)(p - \alpha_2)}} \quad (k \geq n + p)$$

or

$$(5.11) \quad \alpha \leq p - \frac{(p - \alpha_1)(p - \alpha_2)}{(1 - \beta)p + \beta k} \quad (k \geq n + p).$$

Noting that the function

$$(5.12) \quad \phi(k) = p - \frac{(p - \alpha_1)(p - \alpha_2)}{(1 - \beta)p + \beta k} \quad (k \geq n + p)$$

is increasing on k , we have

$$(5.13) \quad \alpha \leq \phi(n + p) = p - \frac{(p - \alpha_1)(p - \alpha_2)}{p + \beta n}. \quad \blacksquare$$

Finally, we derive

THEOREM 5. Let $f_j(z)$ ($j = 1, 2$) define by (5.1). If $f_j(z) \in A_p(n, \alpha_j, \beta)$ ($j = 1, 2$), then the function

$$(5.14) \quad f(z) = z^p - \sum_{k=n+p}^{\infty} \left\{ (a_{1,k})^2 + (a_{2,k})^2 \right\} z^k$$

is in the class $A_p(n, \alpha, \beta)$, where

$$(5.15) \quad \alpha = p - \frac{2(p - \alpha_0)^2}{p + \beta n} \quad (\alpha_0 = \min\{\alpha_1, \alpha_2\}).$$

The result is sharp for the function $f(z)$ defined by

$$(5.16) \quad f_j(z) = z^p - \frac{p - \alpha_0}{p + \beta n} z^{n+p} \quad (j = 1, 2),$$

when $\alpha_0 = \alpha_1 = \alpha_2$.

PROOF: Since

$$(5.17) \quad \sum_{k=n+p}^{\infty} \left\{ \frac{(1 - \beta)p + \beta k}{p - \alpha_j} a_{j,k} \right\}^2 \leq \left\{ \sum_{k=n+p}^{\infty} \frac{(1 - \beta)p + \beta k}{p - \alpha_j} a_{j,k} \right\}^2 \leq 1 \quad (j = 1, 2),$$

we obtain that

$$(5.18) \quad \begin{aligned} & \sum_{k=n+p}^{\infty} \left\{ \frac{(1 - \beta)p + \beta k}{p - \alpha_0} \right\}^2 \left\{ (a_{1,k})^2 + (a_{2,k})^2 \right\} \\ & \leq \sum_{k=n+p}^{\infty} \left\{ \frac{(1 - \beta)p + \beta k}{p - \alpha_1} a_{1,k} \right\}^2 + \sum_{k=n+p}^{\infty} \left\{ \frac{(1 - \beta)p + \beta k}{p - \alpha_2} a_{2,k} \right\}^2 \\ & \leq 2, \end{aligned}$$

where α_0 is defined by (5.15). This implies that we only find the largest α such that

$$(5.19) \quad \frac{(1-\beta)p + \beta k}{p - \alpha} \leq \frac{1}{2} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_0} \right\}^2 \quad (k \geq n + p)$$

or

$$(5.20) \quad \alpha \leq p - \frac{2(p - \alpha_0)^2}{(1-\beta)p + \beta k} \quad (k \geq n + p).$$

Since the function

$$(5.21) \quad \phi(k) = p - \frac{2(p - \alpha_0)^2}{(1-\beta)p + \beta k} \quad (k \geq n + p).$$

is increasing on k , we have

$$(5.22) \quad \alpha \leq \phi(n + p) = p - \frac{2(p - \alpha_0)^2}{p + \beta n}. \quad \blacksquare$$

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