

ESCAPING DYNAMICS OF ENTIRE FUNCTIONS

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1. DEFINITIONS AND MOTIVATION

In the following, $f : \mathbb{C} \rightarrow \mathbb{C}$ will be a transcendental entire function. We are interested in studying the *escaping set*,

$$I(f) := \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} f^n(z) = \infty\}.$$

In contrast to the polynomial case, where $I(f)$ is a completely invariant component of the Fatou set of f (the *basin of infinity*), the set $I(f)$ is never open for a transcendental entire function, and always intersects the Julia set. In fact, in the cases we will mostly be considering, we always have $I(f) \subset J(f)$.

The escaping set made its first appearance in the very first study of transcendental dynamics, namely Fatou's *Acta* paper from 1926 [F]. Among many other things, Fatou observed that, for the functions

$$f(z) = \lambda \sin z \quad (\lambda \in \mathbb{R}),$$

the escaping set is contained in the Julia set, and furthermore contains infinitely many arcs to ∞ . Indeed, for $t \gg 0$, we have

$$\sin(i \cdot t) = \frac{i}{2}(\exp(t) - \exp(-t)) \approx i \cdot \frac{\exp(t)}{2},$$

so the arc $i[T, \infty)$ belongs to the escaping set for T large enough, as do all its preimages. (See Figure 1(a).)

Fatou then remarks

*"Il serait intéressant de rechercher si cette propriété n'appartiendrait pas à des substitutions beaucoup plus générales."*¹

In 1989, Eremenko [E] was the first to comprehensively study the escaping set of a general entire transcendental function. He proved that always $I(f) \neq \emptyset$, and deduced from this that every component of $\overline{I(f)}$ is unbounded.

Furthermore, he asks the following questions:

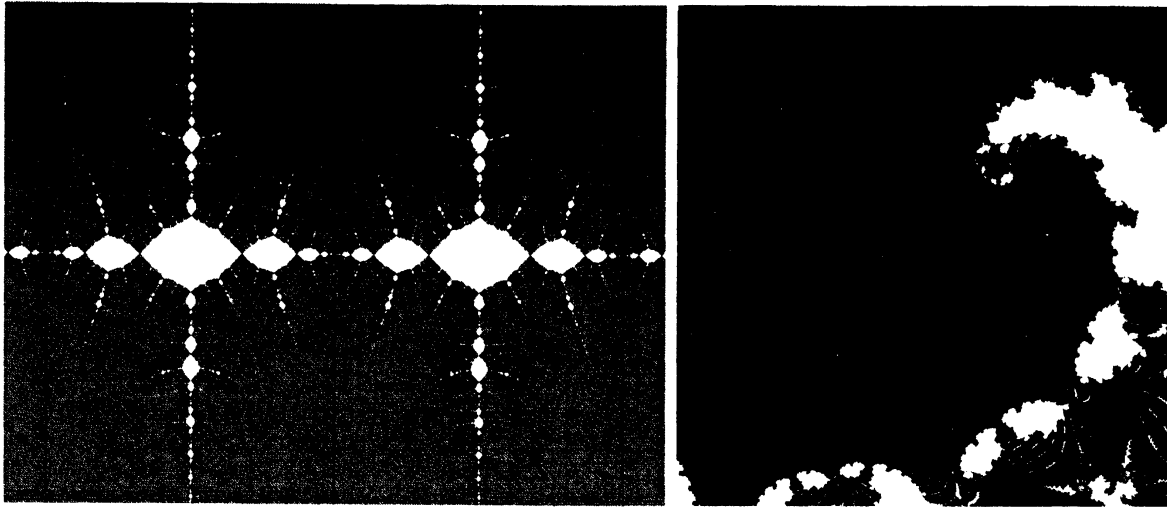
- Is every component of $I(f)$ unbounded?
- Can every point of $I(f)$ be connected to infinity by a curve of escaping points?

Date October 2006.

The work described in this note was supported in part by a postdoctoral fellowship of the German Academic Exchange Service (DAAD), by the Engineering and Physical Sciences Research Council (EPSRC), and by the Research Support Fund (RDF) at the University of Liverpool.

¹"It would be interesting to investigate whether this property might not hold for much more general functions."

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(a) Fatou's example

(b) The Julia set of a linearizer

FIGURE 1. Julia sets of some Eremenko-Lyubich type functions

These questions are of particular interest since such curves in $I(f)$ can be seen as analogs of “dynamic rays”, which are the basis for the success of polynomial dynamics. In particular, they are used in the celebrated *Yoccoz puzzle*.

The Eremenko-Lyubich class. We denote by $\text{sing}(f^{-1})$ the set of all values in which some branch of f^{-1} cannot be defined; that is, the set of all *critical* and *asymptotic values* of f . (Recall that a is an asymptotic value if there is a curve $\gamma : (0, 1] \rightarrow \mathbb{C}$ such that $\lim_{t \rightarrow 0} |\gamma(t)| = \infty$ and $\lim_{t \rightarrow 0} f(\gamma(t)) = a$.)

A class of entire functions which is particularly interesting for our considerations was introduced by Eremenko and Lyubich [EL] in 1986:

$$\mathcal{B} := \{f \text{ transcendental, entire} : \text{sing}(f^{-1}) \text{ is bounded}\}.$$

$\mathcal{B}$ contains the important *Speiser class*

$$\mathcal{S} := \{f \text{ transcendental, entire} : \text{sing}(f^{-1}) \text{ is finite}\}.$$

The importance of this class lies in the fact that functions in $\mathcal{B}$ have some expanding properties near ∞ ; similar to the way in which the derivative of the exponential map is large when the exponential itself is large. We will see a more precise statement later; for now let us just note the following result of Eremenko and Lyubich, which follows from these expansion properties:

Lemma. *If $f \in \mathcal{B}$, then $I(f) \subset J(f)$.*

We will see later that, in many respects, the Eremenko-Lyubich class is in fact a natural restriction for the study of the escaping set. For now, let us consider some examples for the kind of functions that do belong in this class.

First note that we will lose all functions which have a Baker domain (an invariant Fatou component in which the iterates tend to ∞), a multiply connected Fatou component, or

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any other wandering domain in which the iterates tend to infinity. Also, a function of order less than $1/2$ cannot belong to $\mathcal{B}$ (recall that the *order* of f is defined to be

$$\rho := \limsup_{z \rightarrow \infty} \frac{\log \log |f(z)|}{\log |z|},$$

which is a number in $[0, \infty]$).

On the other hand, exponential and trigonometric functions belong to class $\mathcal{S}$. A simple function which belongs to $\mathcal{B} \setminus \mathcal{S}$ is given by $\sin(z)/z$.

Slightly more complicated examples can be obtained as follows. Let p be a polynomial of degree ≥ 2 with a repelling fixed point at 0; that is, $p(0) = 0$ and $\mu := p'(0) \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

Then the fixed point is linearizable, and there is an entire function $\Psi : \mathbb{C} \rightarrow \mathbb{C}$ with $\Psi'(0) \neq 0$ such that

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto \mu z} & \mathbb{C} \\ \Psi \downarrow & & \downarrow \Psi \\ \mathbb{C} & \xrightarrow{p} & \mathbb{C}. \end{array}$$

This function Ψ has finite order, which follows easily from the functional equation. Now the singular values of Ψ are, in fact, the postcritical points of p , so if p has connected Julia set, then $\Psi \in \mathcal{B}$. If p is postcritically finite, then $\Psi \in \mathcal{S}$. Unless μ is real, the function Ψ will display a spiralling behavior near ∞ , unlike the simple functions we considered previously (see Figure 1(b)). However, as we will see, these functions are still among the tame functions in class $\mathcal{B}$! So this is, in fact, a very large class of functions, despite the restriction on $\text{sing}(f^{-1})$.

In the following, when discussing Fatou's and Eremenko's questions, let us adopt the following terminology: f satisfies

- the **Fatou Property (FP)** if $J(f) \cap I(f)$ contains a curve to infinity;
- the **Eremenko Property (EP)** if every component of $I(f)$ is unbounded;
- the **Strong Eremenko Property (SEP)** if every point of $I(f)$ can be connected to infinity by a curve in $I(f)$.

This note describes some recent results which shed light on these problems, and resolve most of them.

2. HISTORY

In the early 1980s, Devaney established the Strong Eremenko Property for exponential maps with an attracting fixed point [DK]. Following up on this, the existence of rays (that is, the Fatou Property) was established for arbitrary exponential maps $z \mapsto \exp(z) + \kappa$ by Devaney, Goldberg and Hubbard in 1986 [DGH]. Devaney and Tangerman [DT] generalized the latter result to a (fairly large) subclass of $\mathcal{B}$, essentially consisting of functions whose behavior at ∞ is similar to that of exponential maps. More recently, in 1999, Schleicher and Zimmer [SZ] proved the SEP for all exponential maps, and Rottenfu er and Schleicher [RoS] adapted this to the family $z \mapsto a \exp(z) + b \exp(-z)$ of cosine maps in 2002.

An interesting case in which the Eremenko Property (but not the SEP) was established is due to Rippon and Stallard [RiS]. In 2003, they showed that, for *any* transcendental

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entire function f , the set $I(f)$ has an unbounded component. (In fact, they prove that the set $A(f) \subset I(f)$, introduced by Bergweiler and Hinkkanen, has only unbounded components.) In particular, they establish the EP for entire functions with multiply connected wandering domains, where $A(f)$ is connected.

3. RESULTS

Curves in the escaping set. The following theorem establishes the Strong Eremenko Property for a large and natural class of functions.

Theorem 1 ([R³S], 2005). *Let f be a finite-order function in the Eremenko-Lyubich class (or, more generally, a finite composition of such functions).*

Then f satisfies the Strong Eremenko Property.

Remark. Baranski [B] also has a proof of Theorem 1 in the case where f is of finite order and hyperbolic with a single attracting fixed point.

In a sense, this provides a very satisfactory positive answer to Fatou's question, namely whether there is a general class of functions for which $I(f) \cap J(f)$ contains curves to infinity.

On the other hand, the same paper shows that the answer to Eremenko's question is "no" in general:

Theorem 2 ([R³S], 2005). *There is a (hyperbolic) function $f \in \mathcal{B}$ such that every path-connected component of $J(f)$ is bounded.*

Thus, f does not satisfy the Strong Eremenko Property, or even the Fatou Property.

Remark. The technique used in the proof can in fact be used to produce

- a counterexample of lower order $1/2$ and with

$$\log \log |f(z)| = (\log |z|)^{1+o(1)}.$$

- a function (albeit of much faster growth) such that every path-connected component of $J(f)$ is a point.

Unboundedness of escaping components. With respect to the (weak) Eremenko property, recall that the *postsingular set* of f is

$$\mathcal{P}(f) := \overline{\bigcup_{j \geq 0} f^j(\text{sing}(f^{-1}))}$$

Theorem 3 ([R2], 2006). *Suppose that $f \in \mathcal{B}$ and that $\mathcal{P}(f)$ is bounded.*

Then every component of $I(f)$ is unbounded.

Remark. This establishes the EP without necessarily having curves in the Julia set, or even uniform escape on some set connecting the given escaping point to ∞ .

Parameter spaces and an analog of Böttcher's theorem. Theorem 2 (and the remark following it) should have convinced the reader that the behavior of functions in class $\mathcal{B}$ is very diverse, and can be very "pathological" and unruly. We now discuss a result which, in light of this fact, is rather surprising.

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To state it, let us introduce a little bit of notation. Loosely following Eremenko and Lyubich, we say that two functions $f, g \in \mathcal{B}$ are *quasiconformally equivalent near ∞* if there are quasiconformal maps $\varphi, \psi : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$(1) \quad \varphi(f(z)) = g(\psi(z))$$

whenever $\max(|g(z)|, |f(z)|)$ is large enough. When (1) holds on all of $\mathbb{C}$, the maps are simply called *quasiconformally equivalent*. (Quasiconformal equivalence classes are the natural complex parameter spaces of entire functions.)

Recall that Böttcher's theorem implies that any two polynomials of the same degree ≥ 2 are conformally conjugate near ∞ . The following provides an analog of this theorem for class $\mathcal{B}$:

Theorem 4 ([R1], 2006). *Let $f, g \in \mathcal{B}$ be quasiconformally equivalent near infinity. Then there exists a quasiconformal map $\vartheta : \mathbb{C} \rightarrow \mathbb{C}$ such that $\vartheta \circ f = g \circ \vartheta$ on*

$$A_R := \{z : |f^n(z)| \geq R \text{ for all } n \geq 1\}$$

Furthermore, ϑ has zero dilatation on $\{z \in A_R : |f^n(z)| \rightarrow \infty\}$.

Thus, even though the structure of the escaping set can change dramatically within class $\mathcal{B}$, it will stay constant within any given parameter space. This is all the more surprising as other properties — the Hausdorff dimension of the escaping set, and the order, for example — *do* change within such families.

This result is also useful for proving rigidity, and hence density of hyperbolicity, for certain families of real entire functions in class $\mathcal{S}$ (current joint work with Sebastian van Stien).

It should be noted that the above theorem does not hold for general entire functions (outside class $\mathcal{B}$). Indeed, the map $z \mapsto z - 1 + \exp(z)$ has a Baker domain containing a left half-plane, while $z \mapsto z + 1 + \exp(z)$ has no Baker domains at all.

Finally, let us note another result, which shows that the structure of the escaping set actually remains constant for *hyperbolic* functions in the same parameter space:

Theorem 5 ([R1], 2006). *Suppose $f, g \in \mathcal{B}$ are quasiconformally equivalent, and furthermore that f and g are hyperbolic. Then $f|_{I(f)}$ and $g|_{I(g)}$ are topologically conjugate.*

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