

Cartesian product of a homotopy 4-sphere with E^1

By

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§0. In this paper we will show that $H^4 \times E^1$ is PL homeomorphic to $S^4 \times E^1$ where H^4 is a homotopy 4-sphere which is a PL manifold and E^1 is an 1-dim. euclidean space. It is an alternating proof of [9. Th. 6], [10. p. 67]. Throughout this paper we consider PL category of polyhedra and piecewise linear maps (see [8]) if otherwise is stated. E^n , S^n , D^n always mean n -dimensional euclidean space, n -dim. PL sphere and PL ball.

§1.

Proposition 1. Let Σ^4 be a PL 4-sphere which is locally flat PL embedded in S^5 . Then M , the closure of one of the complement of Σ^4 in S^5 , is a PL 5-ball.

Proof. Since Σ^4 is PL locally flat embedded in S^5 , M is a PL manifold which is (TOP) homeomorphic to D^5 [1], [2]. And $\partial M = \Sigma^4$ is a (standard) PL 4-sphere. So $(p^*\partial M) \cup M$ is a PL manifold which is homeomorphic to S^5 . Then by the uniqueness of PL structure on S^5 [4], $(p^*\partial M) \cup M$ is a PL 5-sphere and hence $M \underset{PL}{\cong} S^5 - \text{Int st}(v, S^5)$ is a (standard) PL 5-ball.

Proposition 2 [3, p 89]. Let K be a closed PL subspace in

the interior of a PL manifold M . Then there exists a regular neighborhood of K in M which is unique up to ambient isotopy keeping K fixed.

Lemma 1. Let $f : S^3 \times E^1 \rightarrow E^5$ be a locally flat PL embedding satisfying the following condition; for any 5-ball $B^5 \subset E^5$ containing $f(S^3 \times \{0\})$ in its interior there is a positive number $s = s(B)$ such that $f(S^3 \times ((-\infty, -s] \cup [s, \infty))) \cap B^5 = \emptyset$. Then $(f(S^3 \times \{0\}) \subset E^5)$ is a PL trivial knot. And there is a locally flat PL embedding $g : D^4 \rightarrow E^5$ of 4-ball D^4 such that $g(\partial D^4) = f(S^3 \times \{0\})$, $g(\text{Int } D^4) \cap f(S^3 \times E^1) = \emptyset$.

Proof. Let $B^5 \subset E^5$ be a 5-ball with $\text{Int } B^5 \supset f(S^3 \times \{0\})$. Then by the assumption there is a $s = s(B^5) > 0$ such that

$$f(S^3 \times ((-\infty, -s] \cup [s, \infty))) \cap B^5 = \emptyset.$$

So $f(\{x\} \times [0, s]) \cap \partial B^5 = f(\{x\} \times \{s_1\}) \cup \dots \cup f(\{x\} \times \{s_m\})$ where $x \in S^3$, $0 < s_1 < \dots < s_m < s$ and $m = 2p + 1$. Now if B_1^5 is a 5-ball in E^5 with $\text{Int } B_1^5 \supset B^5 \cup f(S^3 \times [0, s])$, by the assumption there is a $t = t(B_1^5) > 0$ such that

$$f(S^3 \times ((-\infty, -t] \cup [t, \infty))) \cap B_1^5 = \emptyset.$$

Since $f(\{x\} \times (s_{2r-1}, s_{2r})) \cap B^5 = \emptyset$, $1 \leq r \leq p$, we take a simple arc γ_r on ∂B^5 joining $f(\{x\} \times \{s_{2r-1}\})$ with $f(\{x\} \times \{s_{2r}\})$ where $\gamma_i \cap \gamma_j = \emptyset$ ($i \neq j$). Then the simple closed curve $f(\{x\} \times [s_{2r-1}, s_{2r}]) \cup \gamma_r$ is homotopic to constant in $B_1^5 - \text{Int } B^5 \cong S^4 \times I$. Then using general position technique there

are non-singular 2-balls δ_r ($1 \leq r \leq p$) such that

- ① $\text{Int } \delta_r \subset B_1^5 - \text{Int } B^5$
- ② $\partial \delta_r = f(\{x\} \times [s_{2r-1}, s_{2r}]) \cup \gamma_r$
- ③ $\delta_r \cap B^5 = \gamma_r$
- ④ $\delta_i \cap \delta_j = \emptyset$ ($i \neq j$).

Using δ_r ($1 \leq r \leq p$) we can engulf $f(\{x\} \times [s_{2r-1}, s_{2r}])$ into B^5 by an ambient isotopy i.e. there is a level preserving PL homeomorphism $F : E^5 \times I \rightarrow E^5 \times I$ such that $F|(E^5 - B_1^5) \times I = \text{id.}$, $F_0 = \text{id.}$ and $F_1 f(\{x\} \times [s_{2r-1}, s_{2r}]) \subset \text{Int } B^5$ ($1 \leq r \leq p$).

Then

$$\begin{aligned} F_1 f(\{x\} \times [0, s]) \cap \partial B^5 &= F_1 f(\{x\} \times [0, t]) \cap \partial B^5 \\ &= F_1 f(\{x\} \times \{s_m\}). \end{aligned}$$

Let

$$\begin{aligned} &F_1 f(S^3 \times \{0\}) \cup F_1 f(N(x) \times [0, t]) \cup F_1 f(S^3 \times \{t\}) \\ &\quad - F_1 f(\text{Int } N(x) \times [0, t]) = \Sigma^3. \end{aligned}$$

Then $(\Sigma^3 \subset E^5)$ is a knot which is the sum of the knots $(f(S^3 \times \{0\}) \subset E^5)$ and $(f(S^3 \times \{t\}) \subset E^5)$ using ∂B^5 and it is trivial because Σ^3 bounds a 4-ball $f((S^3 - \text{Int } N(x)) \times [0, t])$ in E^5 . So $(f(S^3 \times \{0\}) \subset E^5)$, $(f(S^3 \times \{t\}) \subset E^5)$ are both topologically trivial by [5] and then piecewise linearly trivial by [7].

Now we define an embedding $g : D^4 \rightarrow E^5$ satisfying $g(\partial D^4) = f(S^3 \times \{0\})$ and $g(\text{Int } D^4) \cap f(S^3 \times E^1) = \emptyset$. Since $(f(S^3 \times \{0\}) \subset E^5)$ is trivial, $f(S^3 \times \{0\})$ bounds a locally flat 4-ball B_0^4

in E^5 and $(f(S^3 \times \{t\}) \subset E^5)$ is trivial for any t by using the infinite cylinder $f(S^3 \times E^1)$. Furthermore $(f(S^3 \times \{0\}) \cup f(S^3 \times \{t\}) \subset E^5)$ is a split link by the assumption for f . So $(f(S^3 \times \{0\}) \cup f(S^3 \times \{t\}) \subset E^5)$ is a trivial link for any $t \in E^1$ and there is $\epsilon > 0$ such that $f(S^3 \times [-\epsilon, \epsilon]) \cap \text{Int } B_0^4 = \emptyset$. And hence for any $t > 0$ there is a 4-ball B_t^4 in E^5 such that $\partial B_t^4 = f(S^3 \times \{0\})$ and $\text{Int } B_t^4 \cap f(S^3 \times [-t, t]) = \emptyset$. So there is a 4-ball B^4 in E^5 satisfying $\partial B^4 = f(S^3 \times \{0\})$, $\text{Int } B^4 \cap f(S^3 \times E^1) = \emptyset$. We may define $g : D^4 \rightarrow E^5$ by $g(D^4) = B^4$.

Let H^4 be a homotopy 4-sphere which is a PL manifold and $V^4 = H^4 - \text{Int } \sigma^4$ where σ^4 is a 4-simplex.

Lemma 2. If $f : S^3 \times E^1 \rightarrow \partial V^4 \times E^1$ is a PL homeomorphism, there is a PL homeomorphism $g : D^4 \times E^1 \rightarrow V^4 \times E^1$ which is an extension of f .

Proof. Let $\tilde{c}_1 : \partial D^4 \times I \rightarrow D^4$, $\tilde{c}_2 : \partial V^4 \times I \rightarrow V^4$ ($I = [0, 1]$) be boundary collars i.e. $\tilde{c}_1, \tilde{c}_2$ are embeddings such that $\tilde{c}_1(x, 0) = x$ ($x \in \partial D^4$) and $\tilde{c}_2(y, 0) = y$ ($y \in \partial V^4$). And let $c_1 : \partial D^4 \times I \times E^1 \rightarrow D^4 \times E^1$, $c_2 : \partial V^4 \times I \times E^1 \rightarrow V^4 \times E^1$ be $c_1(x, s, t) = (\tilde{c}_1(x, s), t)$, $c_2(y, s, t) = (\tilde{c}_2(y, s), t)$. Let $f_1 : c_1(\partial D^4 \times I \times E^1) \rightarrow c_2(\partial V^4 \times I \times E^1)$ be $f_1 c_1(p, s, t) = c_2(p', s, t')$ where $f_1 c_1(p, 0, t) = f c_1(p, 0, t) = c_2(p', 0, t')$. Since $\text{Int } V^4 \times E^1 \cong E^5$ by [6], let $\mathcal{J} : \text{Int } V^4 \times E^1 \rightarrow E^5$ be a PL homeomorphism. Then $\mathcal{J} f_1 c_1|_{\partial D^4 \times \{1\} \times E^1} : \partial D^4 \times \{1\} \times E^1$

$\rightarrow E^5$ satisfies the condition for f of Lemma 1, i.e. for any 5-ball $B^5 \subset E^5$ containing $\mathcal{Y}f_1c_1(\partial D^4 \times \{1\} \times \{0\})$ in its interior there is a positive number $s = s(B)$ such that $\mathcal{Y}f_1c_1(\partial D^4 \times ((-\infty, -s] \cup [s, \infty))) \cap B^5 = \emptyset$. Because $\mathcal{Y}^{-1}(B^5) \subset \text{Int } V^4 \times (-s', s')$ for some $s' > 0$ and $\mathcal{Y}^{-1}(B^5) \cap c_2(\partial V^4 \times \{1\} \times ((-\infty, -s'] \cup [s', \infty))) = \emptyset$. Hence there is a number $s > 0$ such that

$$\mathcal{Y}^{-1}(B^5) \cap f_1c_1(\partial D^4 \times \{1\} \times ((-\infty, -s] \cup [s, \infty))) = \emptyset$$

and so

$$B^5 \cap \mathcal{Y}f_1c_1(\partial D^4 \times \{1\} \times ((-\infty, -s] \cup [s, \infty))) = \emptyset.$$

So by Lemma 1 ($\mathcal{Y}f_1c_1(\partial D^4 \times \{1\} \times \{0\}) \subset E^5$) is a trivial knot and it bounds a locally flat 4-ball $\tilde{B}_0^4$ with $\text{Int } \tilde{B}_0^4 \cap \mathcal{Y}f_1c_1(\partial D^4 \times \{1\} \times E^1) = \emptyset$. So $f_1c_1(\partial D^4 \times \{1\} \times \{0\})$ bounds a locally flat PL 4-ball $B_0^4 = \mathcal{Y}^{-1}(\tilde{B}_0^4)$ such that

$$\text{Int } B_0^4 \cap f_1c_1(\partial D^4 \times \{1\} \times E^1) = \text{Int } B_0^4 \cap c_2(\partial V^4 \times \{1\} \times E^1) = \emptyset.$$

Similary we may assume there are 4-balls B_t^4 ($t \in \mathbb{Z}$: integer) such that $\partial B_t^4 = f_1c_1(\partial D^4 \times \{1\} \times \{t\})$ and $\text{Int } B_t^4 \cap c_2(\partial V^4 \times \{1\} \times E^1) = \emptyset$. And we may assume $B_t^4 \cap B_{t+1}^4 = \emptyset$ ($t \in \mathbb{Z}$). So we can extend f_1 to

$$\begin{aligned} f_2 : c_1(\partial D^4 \times I \times E^1) \cup (D^4 \times \mathbb{Z}) \\ \longrightarrow c_2(\partial V^4 \times I \times E^1) \cup \bigcup_{t \in \mathbb{Z}} B_t^4 \end{aligned}$$

by a cone extension. Since

$f_1c_1(\partial D^4 \times \{1\} \times [t, t+1]) \cup B_t^4 \cup B_{t+1}^4$ ($t \in \mathbb{Z}$) is a PL 4-sphere which is locally flat embedded in $\text{Int } V^4 \times E^1 \cong E^5$, it bounds

a PL 5-ball B_t^5 by Proposition 1. So we can extend f_2 to a required PL homeomorphism

$$g : D^4 \times E^1 \rightarrow c_2(\partial V^4 \times I \times E^1) \cup \bigcup_{t \in \mathbb{Z}} B_t^5 (= V^4 \times E^1)$$

by a cone extension.

Theorem. $H^4 \times E^1$ is PL homeomorphic to $S^4 \times E^1$ where H^4 is a homotopy 4-sphere which is a PL manifold.

Proof. Since any regular neighborhood N of $p \times E^1$ in $H^4 \times E^1$ is PL homeomorphic to $D^4 \times E^1$, we identify $N = D^4 \times E^1 \subset H^4 \times E^1$ using Proposition 2. So $H^4 \times E^1 - \text{Int } N$ is PL homeomorphic to $V^4 \times E^1$. Let $\mathcal{J}_1 : D^4 \times E^1 \rightarrow N$ be a PL homeomorphism. We may consider $S^4 \times E^1 = (D^4 \times E^1) \cup_{\partial} (D^4 \times E^1)$ where the one of $D^4 \times E^1$ is a regular neighborhood of $q \times E^1$ for some $q \in S^4$. Then by Lemma 2 we can extend $\partial \mathcal{J}_1 = (\mathcal{J}_1|_{\partial D^4 \times E^1})$ to another $D^4 \times E^1$ and so we can get a PL homeomorphism $\mathcal{J} : S^4 \times E^1 \rightarrow H^4 \times E^1$.

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